Quantized Hall effect and geometric localization of electrons on lattices

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The electronic properties of the tight-binding model on a two-dimensional regular lattice and of the Weaire-Thorpe network model in a uniform magnetic field are studied. The quantization of the Hall conductance is shown. The effects of the geometrically localized states of the Weaire-Thorpe model are examined.

I. INTRODUCTION

The quantized Hall effect has been one of the most active fields in condensed-matter physics since its discovery by von Klitzing, Dorda, and Pepper[1] in 1980. Recent research activities are mainly focused on the fractional quantized Hall effect of Tsui, Stormer, and Gossard.[2] However, the normal quantized Hall effect has not been completely understood yet in spite of some heuristic arguments.[3]

The case of a weak periodic potential in two dimensions was studied by Thouless, Kohmoto, Nightingale, and den Nijs[4] (TKNN) and by Streda.[5] In this case, the Landau level is split into many subbands which have amazingly rich structures[6] including scaling.[7–9] However, it was shown that the Hall conductance is still quantized as long as the Fermi energy lies in a gap of the subband structure.

Later the topological aspects of the problem were discussed by Avron, Seiler, and Simon[10] and by Kohmoto.[11] In particular, the contribution of the Hall conductance from a magnetic subband is identified as a Chern class associated with a fiber bundle defined on a magnetic Brillouin zone.[11] Thus the quantization of the Hall conductance is well established for the periodic systems. Applications of this type of geometrical idea to more general situations are attempted by several authors.[12–14]

In this paper, we study discrete systems where electrons subject to a magnetic field hop from a site to a neighboring site on a regular lattice. The following two models are studied: (i) Each site has a single electronic state whose energy is constant through the lattice. This is the simplest version of the tight-binding model (one-orbital model). (ii) Each site has γ electronic states, where γ is the coordination number of the lattice. (For a two-dimensional regular lattice studied here, γ is constant and equals 4.) These states are directed towards γ bonds, respectively. An electron transfers among γ states within a site. It also can hop from a state of a site to that of the neighboring site which shares the same bond with the original state. This model is originally studied by Weaire and Thorpe[15] to account for band gaps in topologically disordered solids such as amorphous Si and Ge. In both models, the electron-electron interactions are not taken into account. Also the electron spin is neglected for the sake of simplicity.

In Sec. II the two-dimensional one-orbital model in a uniform magnetic field is studied and the Hall conductance is shown to be topologically invariant and quantized. In Sec. III we study the Weaire-Thorpe model in arbitrary dimension. We show that the localized ringlike states do not contribute to the current and that the current can be calculated in terms of that of an equivalent one-orbital problem. As an intermediate step we prove a Streda-like[5] formula for the current.

II. TIGHT-BINDING MODEL ON A TWO-DIMENSIONAL REGULAR LATTICE WITH A UNIFORM MAGNETIC FIELD

Let us consider an electron which hops among sites in the two-dimensional regular lattice with a uniform magnetic field applied in the perpendicular direction to the plane. The Hamiltonian may be written as

$$H = A\sum_{m,n}(a^{\dagger}_{m+1,n}a_{mn}+a^{\dagger}_{mn}a_{m+1,n}) + B\sum_{m,n}(a^{\dagger}_{m,n+1}e^{+i2\pi\phi m}a_{mn}+a^{\dagger}_{mn}e^{-i2\pi\phi m}a_{m,n+1}]\,, \tag{2.1}$$

where a site on the lattice is labeled by two integers m and n (see Fig. 1), and a_{mn} and $a^{\dagger}_{mn}$ are the usual fermionic annihilation and creation operators at a site (m,n). The hopping matrix elements along bonds in the x and y directions are chosen to be real and are denoted by A and B, respectively. The magnetic field forces the coupling constants to be complex. Specifically, we have chosen a gauge in which the coupling constants along the bonds in the x direction are real, the coupling constants along the y direction are imaginary, and ϕ represents a magnetic flux within a plaquette in units of hc/e; it is called the Landau gauge or the axial gauge.

Define a Fourier transform of a_{mn} by

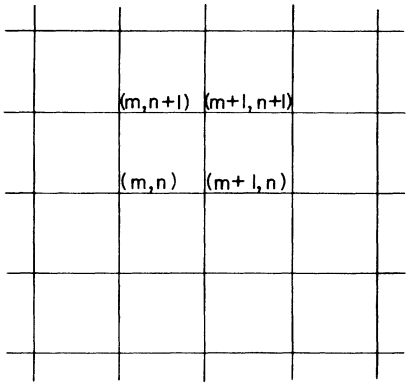

FIG. 1. Square lattice for the one-orbital model.

$$a_{nn}=\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}dk_x\int_{-\pi}^{\pi}dk_y\exp[i(k_xm+k_yn)]a(k_x,k_y)\ , \tag{2.2}$$

where the lattice spacing is taken to be unity and the first Brillouin zone for the regular lattice is given by $-\pi\le k_x\le\pi$ and $-\pi\le k_y\le\pi$. The operators $a(\mathbf{k})=a(k_x,k_y)$ satisfy the anticommutation relation

$$\begin{aligned}&\{a(\mathbf{k}),a^\dagger(\mathbf{k}')\}=\delta(\mathbf{k}-\mathbf{k}')\ ,\\&\{a(\mathbf{k}),a(\mathbf{k}')\}=0\ ,\\&\{a^\dagger(\mathbf{k}),a^\dagger(\mathbf{k}')\}=0\ ,\end{aligned} \tag{2.3}$$

where $\{\ ,\ \}$ is an anticommutator, and $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$ are required to be periodic, i.e.,

$$\begin{aligned}&a(k_x+2\pi j,k_y+2\pi l)=a(k_x,k_y)\ ,\\&\text{and}\\&a^\dagger(k_x+2\pi j,k_y+2\pi l)=a^\dagger(k_x,k_y)\ ,\end{aligned} \tag{2.4}$$

for arbitrary integers j and l.

In terms of $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$ the Hamiltonian (2.1) is written as

$$H=\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}dk_x\int_{-\pi}^{\pi}dk_y\{A\,2(\cos k_x)a^\dagger(k_x,k_y)a(k_x,k_y)$$
$$+B[e^{-ik_y}a^\dagger(k_x+2\pi\phi,k_y)a(k_x,k_y)+e^{+ik_y}a^\dagger(k_x-2\pi\phi,k_y)a(k_x,k_y)]\}\ , \tag{2.5}$$

and the Schrödinger equation is given by

$$H\,|\psi\rangle=E\,|\psi\rangle\ . \tag{2.6}$$

Here notice that the Hamiltonian mixes a state having wave number (k_x,k_y) only with states having $(k_x\pm2\pi\phi,k_y)$. Let us consider a magnetic field whose flux ϕ within a unit cell is a rational number, then one may write

$$\phi=p/q\ , \tag{2.7}$$

where p and q are integers which are relatively prime. In this case one can rewrite the Hamiltonian (2.5) as

$$H=\frac{q}{(2\pi)^2}\int_{-\pi/q}^{\pi/q}dk_x^0\int_{-\pi}^{\pi}dk_y\,\hat{H}(k_x^0,k_y)\ , \tag{2.8}$$

where

$$\hat{H}(k_x^0,k_y)=\frac{1}{q}\sum_{n=0}^{q-1}\{2A\cos(k_x^0+2\pi\phi n)a^\dagger(k_x^0+2\pi\phi n,k_y)a(k_x^0+2\pi\phi n,k_y)$$
$$+B[e^{-ik_y}a^\dagger(k_x^0+2\pi\phi(n+1),k_y)a(k_x^0+2\pi\phi n,k_y)$$
$$+e^{+ik_y}a^\dagger(k_x^0+2\pi\phi(n-1),k_y)a(k_x^0+2\pi\phi n,k_y)]\}\ . \tag{2.9}$$

The integral variable k_x in (2.5) has been replaced by $k_x=k_x^0+2\pi\phi n$ and we have an integral over the first *magnetic* Brillouin zone,

$$-\pi/q\le k_x^0<\pi/q,\quad -\pi\le k_y<\pi\ . \tag{2.10}$$

In direct space, we need q plaquettes to have an integer magnetic flux unit with p. Therefore, the magnetic unit cell is formed by q plaquettes. In fact, the reciprocal space for the lattice in which q plaquettes form a unit cell is the magnetic Brillouin zone given by (2.10). The Schrödinger equation (2.6) is decoupled to sectors, each of which is a point in the magnetic Brillouin zone; hence states with different wave numbers (k_x^0,k_y) do not mix. The reduced Schrödinger equation is

$$\hat{H}(k_x^0,k_y)\,|\psi\rangle=E(k_x^0,k_y)\,|\psi\rangle\ . \tag{2.11}$$

The Hamiltonian $\hat{H}(k_x^0,k_y)$ for fixed values of k_x^0 and k_y represents a tight-binding model on a one-dimensional lattice chain $k_x^0+2\pi\phi n$ $(n=0,1,2,\ldots,q-1)$. The eigenstate $|\psi\rangle$ is expanded with respect to the states on these lattice points as

$$|\psi\rangle=\sum_{n=0}^{q-1}c_na^\dagger(k_x^0+2\pi\phi n,k_y)\,|0\rangle\ , \tag{2.12}$$

where $|0\rangle$ is the vacuum state. The coefficients c_n satisfy

$$2A\cos(k_x^0+2\pi\phi n)c_n+B(e^{-ik_y}c_{n-1}+e^{-ik_y}c_{n+1}) =E(k_x^0,k_y)c_n \quad (2.13)$$

($n=0,1,2,\ldots,q-1$; $c_q=c_0$). This equation can be regarded as a (discrete) Schrödinger equation or a tight-binding equation in one dimension. Especially when ϕ is an irrational number, this equation is quasiperiodic and has attracted a lot of attention with its unusual properties of an energy spectrum and wave functions.[16,17]

Here we consider only the case with a rational $\phi=p/q$. For fixed values of k_x^0 and k_y, there are q eigenvalues (except for degeneracies which could occur in special cases). The eigenvalues change continuously as k_x^0 and k_y are varied, and form the magnetic subbands. Thus a single band for the two-dimensional tight-binding model is split into q magnetic subbands upon an application of a uniform magnetic field whose flux unit within a unit cell of the lattice is $\phi=p/q$.

In the case of two-dimensional electrons in a strong magnetic field with a weak sinusoidal potential, one also gets magnetic subbands.[4] The strong magnetic field first gives equally spaced Landau levels which are infinitely degenerate. A perturbation potential of the form $A\cos(2\pi x)+B\cos(2\pi y)$ removes the degeneracy, and the secular equation of the perturbation theory has a form similar to (2.13) with an important difference: The magnetic flux unit per unit area enters in the argument of cosine in (2.8) as $2\pi n/\phi$, in this case, instead of $2\pi n\phi$. Therefore, each Landau level splits into p subbands instead of q subbands, which are for the tight-binding case. The tight-binding model (strong-potential case) and the weak-potential case both have the magnetic subband structure given by the same equation, but the magnetic flux ϕ must be replaced by $1/\phi$ to have analogous subband structures in the two cases.

The Hall conductance of the system is given by the Kubo formula as

$$\sigma_{xy}=\frac{e^2}{\hbar}\sum_{\substack{\alpha,\beta\\(E_\alpha<E_F<E_\beta)}}\hbar^2\frac{(v_y)_{\alpha\beta}(v_x)_{\beta\alpha}-(v_x)_{\alpha\beta}(v_y)_{\beta\alpha}}{(E_\alpha-E_\beta)^2}, \quad (2.14)$$

where E_α is the energy level of state α which is below the Fermi energy E_F, and E_β is the energy level of state β which is above the Fermi energy. The velocity operators in the x and y directions are v_x and v_y, respectively, and in the present case they are explicitly written as

$$v_x=A\frac{1}{i\hbar}\sum_{m,n}(a^\dagger_{m+1,n}a_{mn}-a^\dagger_{mn}a_{m+1,n}),$$
$$v_y=B\frac{1}{i\hbar}\sum_{m,n}(a^\dagger_{m,n+1}e^{i2\pi\phi m}a_{mn}-a^\dagger_{mn}e^{-i2\pi\phi m}a_{m,n+1}). \quad (2.15)$$

In terms of $\hat{H}(k_x^0,k_y)$, the velocity operators are written as

$$v_x=\frac{1}{\hbar}\frac{q}{(2\pi)^2}\int_{-\pi/q}^{\pi/q}dk_x^0\int_{-\pi}^{\pi}dk_y\frac{\partial\hat{H}(k_x^0,k_y)}{\partial k_x^0},$$

and

$$v_y=\frac{1}{\hbar}\frac{q}{(2\pi)^2}\int_{-\pi/q}^{\pi/q}dk_x^0\int_{-\pi}^{\pi}dk_y\frac{\partial\hat{H}(k_x^0,k_y)}{\partial k_y}. \quad (2.16)$$

In order to obtain the matrix elements of v_x and v_y, the following identity is useful:

$$\left\langle\beta\left|\frac{\partial\hat{H}}{\partial k_i}\right|\alpha\right\rangle=(E_\alpha-E_\beta)\left\langle\beta\left|\frac{\partial}{\partial k_i}\right|\alpha\right\rangle, \quad (2.17)$$

which is obtained by taking derivative with respect to k_i of the equation

$$\hat{H}(k_x^0,k_y)\,|\,\alpha\rangle=E_\alpha\,|\,\alpha\rangle.$$

After some manipulation, (2.14) with (2.17) gives

$$\sigma_{xy}=\frac{e^2}{\hbar}\sum_\alpha\left[\left(\frac{\partial}{\partial k_y}\langle\alpha|\right)\left(\frac{\partial}{\partial k_x^0}|\alpha\rangle\right)-\left(\frac{\partial}{\partial k_x^0}\langle\alpha|\right)\left(\frac{\partial}{\partial k_y}|\alpha\rangle\right)\right], \quad (2.18)$$

where the summation is over the all occupied states. If we consider a contribution from a filled magnetic subband, the summation is replaced by an integral over the magnetic Brillouin zone. As explained in Ref. 11, the integral is the Chern number of a fiber bundle whose base space is the magnetic Brillouin zone and the fiber is the wave function. Since the Chern number is always an integer, the contribution of the Hall conductance from a filled subband is always an integer times $e^2/\hbar$.

III. HALL CURRENTS IN THE WEAIRE-THORPE MODEL

Weaire and Thorpe[15] (WT) introduced a simplified model describing the physics of systems such as silicon and germanium. The main idea is to consider a lattice, which is regular or is topologically disordered with coordination number γ, and a set of γ electronic states at each site in the form of orbitals oriented along lattice links. They considered a nearest-neighbor model in which the single-particle Hamiltonian has the form, in Straley's[18] version,

$$H=\gamma V_1\hat{U}+V_2\hat{T}, \quad (3.1)$$

where $\hat{U}$ is a projection operator that describes the mixing of orbitals at a site and $\hat{T}$ is an exchange-type operator describing the mixing of orbitals at nearest-neighboring sites pointing towards each other. The parameters V_1 and V_2 are the site and bond energies. From their definition $\hat{U}$ and $\hat{T}$ satisfy[18]

$$\hat{U}^2=\hat{U}, \quad (3.2a)$$
$$\hat{T}^2=\hat{I}, \quad (3.2b)$$

where $\hat{I}$ is the unit operator. Let A and B be a pair of nearest-neighboring sites of the Bravais lattice with coor-

dinates $\mathbf{r}_A$ and $\mathbf{r}_B$, respectively. Let $\{\mathbf{d}_i^a\}$ be a set of γ unit vectors at site A and $\{\mathbf{d}_i^B\}$ a set of γ unit vectors at site B. There, $\mathbf{r}_A$ and $\mathbf{r}_B$ must be such that $\mathbf{r}_A+\mathbf{d}_i^A$ belong to sublattice B and $\mathbf{r}_B+\mathbf{d}_i^B$ belong to sublattice A. Let us label the states at site A by $|\mathbf{r}_A,\mathbf{d}_i^A\rangle$, indicating an orbital at site A pointing towards $\mathbf{r}_A+\mathbf{d}_i^A$. Similarly, at site B we have $|\mathbf{r}_B,\mathbf{d}_i^B\rangle$. We choose the index i so that $\mathbf{d}_i^A$, and $\mathbf{d}_i^B$ are on the same link, i.e.,

$$\mathbf{d}_i^A+\mathbf{d}_i^B=\mathbf{0}\ . \tag{3.3}$$

With this notation we can write

$$\hat{U}=\frac{1}{\gamma}\sum_{i,j=1}^{\gamma}\left[\sum_{\mathbf{r}_A}(|\mathbf{r}_A,\mathbf{d}_i^A\rangle\langle\mathbf{r}_A,\mathbf{d}_j^A|)+\sum_{\mathbf{r}_B}(|\mathbf{r}_B,\mathbf{d}_i^B\rangle\langle\mathbf{r}_B,\mathbf{d}_j^B|)\right], \tag{3.4a}$$

and

$$\hat{T}=\sum_{(\mathbf{r}_A,\mathbf{r}_B),\{\mathbf{d}_i^A\}}(|\mathbf{r}_A,\mathbf{d}_i^A\rangle\langle\mathbf{r}_B,\mathbf{d}_i^B|+|\mathbf{r}_B,\mathbf{d}_i^B\rangle\langle\mathbf{r}_A,\mathbf{d}_j^A|)\ , \tag{3.4b}$$

where the summation is over pairs of sites (or links) with

$$\mathbf{r}_B=\mathbf{r}_A+\mathbf{d}_i^A\ , \tag{3.5a}$$

$$\mathbf{r}_A=\mathbf{r}_B+\mathbf{d}_i^B\ , \tag{3.5b}$$

and

$$\mathbf{d}_i^A+\mathbf{d}_i^B=\mathbf{0}\ . \tag{3.5c}$$

It is now easy to prove[18] that the operators $\hat{U}$ and $\hat{T}$ thus defined do satisfy the conditions (3.2a) and (3.2b), i.e.,

$$\hat{U}^2=\hat{U},\quad \hat{T}^2=\hat{I}\ .$$

External electromagnetic fields can easily be incorporated into the WT model. We work within the gauge $A_0=0$ in which the electrostatic potential is zero. Thus electric fields are to be calculated from a time-dependent vector potential

$$E=\frac{\partial A}{\partial t}\ . \tag{3.6}$$

The standard prescription for incorporating a vector potential in a lattice system is to multiply all bond amplitudes by a phase $e^{i\varphi_{\text{bond}}}$, where φ_{bond} is the line integral

$$\varphi_{\text{bond}}=\int_{\text{bond}} d\mathbf{l}\cdot A\ . \tag{3.7}$$

Thus we end up with a generalized WT model, with the same site potential and bond amplitudes of the form $V_2 e^{iA(\mathbf{r}_A,\mathbf{r}_B,t)}$ for the bond $(\mathbf{r}_A,\mathbf{r}_B)$. In order to incorporate both electric and magnetic field effects, we will take

$$A(\mathbf{r}_A,\mathbf{r}_B,t)=A(\mathbf{r}_A,\mathbf{r}_B)+\frac{e}{c\hbar}E(\mathbf{r}_A,\mathbf{r}_B)t\ , \tag{3.8}$$

where the lattice constant is taken to be unity, the time-independent piece $A(\mathbf{r}_A,\mathbf{r}_B)$ represents the magnetic part of the vector potential, and the time-dependent piece represents the projection of the local electric field onto the bond $(\mathbf{r}_A,\mathbf{r}_B)$. The WT model in the presence of an electromagnetic field has the property that the operators $\hat{U}[A]$ and $\hat{T}[A]$ still satisfy the conditions (3.2a) and (3.2b). Thus, as was shown by Straley[18] for the standard WT model, the spectrum and wave functions of the system can be calculated in terms of the spectrum and wave functions of a one-orbital model on the same Bravais lattice and a set of nonpropagating states. In particular, we show below that the currents on the WT model are the same as the currents of the one-orbital model and that, within the WT model, the nonpropagating states do not contribute to the electromagnetic currents.[19] This is not to say that the wave functions are the same. As a matter of fact, the wave functions have some very interesting geometrical properties in the presence of the field.

Let us now review, briefly, Straley's method[18] as applied to the present problem. The argument that Eqs. (3.2a) and (3.2b) are satisfied in the presence of a nonvanishing vector potential follows from the observation that both $A(\mathbf{r}_A,\mathbf{r}_B)$ and $E(\mathbf{r}_A,\mathbf{r}_B)$ are components of a vector and thus satisfy

$$A(\mathbf{r}_A,\mathbf{r}_B,t)+A(\mathbf{r}_B,\mathbf{r}_A,t)=0\ . \tag{3.9}$$

The site potential is unaffected by the field, so (3.2a) is automatically valid. In computing the square of the kinetic energy operator $\hat{T}[A]$, we have to evaluate products of the form

$$e^{iA(\mathbf{r}_A,\mathbf{r}_B,t)}e^{iA(\mathbf{r}_B,\mathbf{r}_A,t)}=1\ , \tag{3.10}$$

which leads to (3.2b).

A. Eigenstates of the WT model in an electromagnetic field

Within the framework of the adiabatic approximation we can now calculate the eigenstates and eigenvalues at time t if we assume that the electric field E is very weak and we restrict ourselves to time scales τ, such that $\tau E a_0 \ll \hbar$, where a_0 is the lattice constant ($a_0=1$). Straley showed that if Eqs. (3.2a) and (3.2b) are true, then there are two types of states in the spectrum of H.

1. Propagating states

Propagating states are states which are *not* in the null space of the projection operator $\hat{U}$. The operator $\hat{U}$ and $\hat{U}\hat{T}\hat{U}$ commute. Thus we can find states of the form

$$\hat{U}\hat{T}\hat{U}\Psi_\varepsilon=\frac{\varepsilon}{\gamma}\Psi_\varepsilon\ , \tag{3.11a}$$

$$\hat{U}\Psi_\varepsilon=\Psi_\varepsilon\ . \tag{3.11b}$$

Straley then shows that Ψ_ε further satisfies

$$(H^2-\gamma V_1 H)\Psi_\varepsilon=(V_2^2+V_1V_2\varepsilon)\Psi_\varepsilon\ . \tag{3.12}$$

Thus, if E is the eigenvalue of H, i.e., $H\Psi_E=E\Psi_E$, then

$$(H^2-\gamma V_1 H)\Psi_E=(E^2-\gamma V_1 E)\Psi_E\ . \tag{3.13}$$

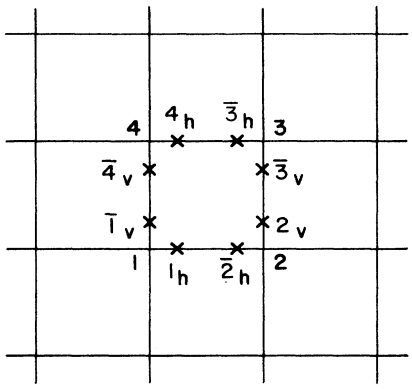

FIG. 2. Ring state in zero field.

The relation between Ψ_E and Ψ_ε is provided by the operator $\hat{P}_\varepsilon$

$$\Psi_{E_\pm}=P_\varepsilon^\pm\Psi_\varepsilon\ , \tag{3.14}$$

with

$$\hat{P}_\varepsilon^\pm=\left[\left(\frac{\gamma V_1}{2}\pm Q_\varepsilon\right)\hat{U}+V_2\hat{T}\right]\ , \tag{3.15a}$$

and

$$Q_\varepsilon^2=\tfrac{1}{4}\gamma^2V_1^2+V_2^2+V_1V_2\varepsilon\ . \tag{3.15b}$$

The eigenvalues are

$$E_\pm=\tfrac{1}{2}\gamma V_1\pm Q_\varepsilon\ . \tag{3.15c}$$

But Ψ_ε is the eigenstate of $\hat{U}\hat{T}\hat{U}$, with eigenvalue ε/γ, which is not in the null space of $\hat{U}$. Let us denote by $\{|\mathbf{r}_A\rangle\}$ the Hilbert space of states $|\mathbf{r}_A\rangle=(1/\sqrt{\gamma})\sum_{i=1}^{\gamma}|\mathbf{r}_A,\mathbf{d}_i^A\rangle$, which spans the states outside the null space of $\hat{U}$. In this basis the operator $\hat{U}\hat{T}\hat{U}$ is

$$\hat{U}\hat{T}\hat{U}=\frac{1}{\gamma}\sum_{(\mathbf{r}_A,\mathbf{r}_B)}|\mathbf{r}_A\rangle\langle\mathbf{r}_B|e^{iA(\mathbf{r}_A,\mathbf{r}_B,t)}+\text{H.c.} \tag{3.16}$$

Thus the wave functions Ψ_ε are those of a one-orbital problem on a tight-binding lattice in an external field. Notice though that the mapping discussed by Straley is true in all dimensions and for all Bravais lattices.

Thus if we know the eigenstates and eigenvalues of the one-orbital Hamiltonian $\hat{U}\hat{T}\hat{U}$ we can calculate the eigenvalues and eigenstates for the network lattice using (3.14) and (3.15). Thus the Landau levels of the one-orbital problem determine the Landau levels on the network lattice through (3.15).

Thorpe and Weaire used a mapping of this sort to calculate the density of states (DOS) for the diamond lattice in terms of the DOS for the fcc lattice.

2. *Nonpropagating states*

Nonpropagating states are states in the null space of $\hat{U}$, i.e., $\hat{U}\Psi=0$. Since $\hat{T}$ is an exchange operator ($\hat{T}^2=\hat{I}$) its eigenvalues are $S=\pm1$. The nonpropagating states are eigenstates of $\hat{T}$ projected into the null space of $\hat{U}$. Their energy eigenvalues are

$$H\Psi=(\gamma V_1\hat{U}+V_2\hat{T}(A))\Psi=V_2\hat{T}(A)\Psi=\pm V_2\Psi\ , \tag{3.17}$$

since $T^2=I$ is valid even with $A\neq0$. This set has a huge degeneracy, which varies with the Bravais lattice, but it is always extensive. (In fact, there are two sets of nonpropagating states with energies $\pm V_2$.) This in turn implies that wave packets constructed out of states in this degenerate set do not propagate, i.e., the group velocity vanishes. Furthermore, (3.17) indicates that their eigenvalues are always $\pm V_2$ *for all values of the vector potential.* We will show below that this fact implies that the nonpropagating states do not contribute to the currents. Of course, the wave functions *do* change with the vector potential. Let us comment on passing that such "ring" states are also present in other lattices, such as the example worked out by Sutherland,[20] and on a Penrose lattice.[21] For instance, in Fig. 2 we see a nonpropagating state $|\eta\rangle_\pm$ (without a field) with energy $\pm V_2$ for the case of a two-dimensional square lattice (a "ring state") by constructing linear combinations of symmetric and antisymmetric states on

$$\begin{aligned}|\eta_\pm\rangle=&(|1_h\rangle\pm|\bar{2}_h\rangle)-(|2_v\rangle\pm|\bar{3}_v\rangle)\\&+(|\bar{3}_h\rangle\pm|4_h\rangle)-(|\bar{4}_v\rangle\pm|1_v\rangle)\ .\end{aligned} \tag{3.18}$$

Clearly,

$$\hat{T}|\eta_\pm\rangle=\pm V_2|\eta_\pm\rangle,\quad \hat{U}|\eta_\pm\rangle=0\ . \tag{3.19}$$

It is clear that one could construct such ring states around any plaquette (or group of plaquettes). Such a set is not orthonormal. It is possible to construct an orthonormal basis for the space of eigenvalue $+V_2$ (or $-V_2$) but they are clearly delocalized. If an external field is now turned on, we face the problem of matching phases. Now the eigenstates of $\hat{T}(A)$ on a link (say, the link 1-2 of the Fig. 2) are

$$|\eta_\pm\rangle_{12}=e^{iA_{12}/2}|1_h\rangle\pm e^{-iA_{12}/2}|\bar{2}_h\rangle\ . \tag{3.20}$$

We now seek linear combinations of link eigenstates which are in the null space of $\hat{U}$. The states are now of the form of a sum of links, of a path Γ of the lattice

$$|\eta_\pm\rangle=\sum_{\text{links}(\Gamma)}e^{i\phi_{\text{link}}/2}|\eta_\pm\rangle_{\text{link}}\ . \tag{3.21}$$

The phases ϕ_{link} are to be determined by the consistency condition $\hat{U}|\eta_\pm\rangle=0$. It is easy to show that the consistency condition cannot be satisfied if Γ is an open path (or the union of open paths). Thus Γ is either a closed path (or the union of closed paths) or a path that closes through the boundaries. Furthermore, the only admissible paths must be such that the oriented sum of the vector potentials around the path is zero mod(2π)

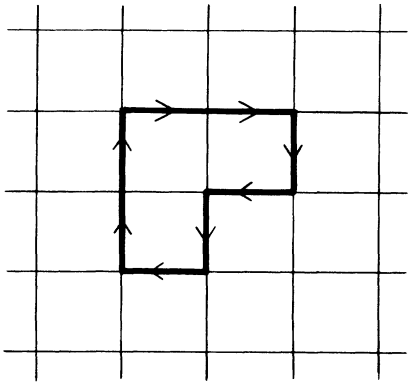

FIG. 3. Ring state for field $\phi=\frac{1}{3}$. The arrows indicate the direction in which the phase increases.

$$\sum_{\Gamma} A_{\text{link}}=2\pi j \quad (j=0,\pm 1,\pm 2,\ldots) . \tag{3.22}$$

Note that the electric field term is curl free and does not affect this condition. Using a lattice version of Stokes theorem we can state (in units in which the lattice constant is one),

$$\phi N_{\Gamma}=j , \tag{3.23}$$

where ϕ is the magnetic flux within a plaquette, $\phi=B/2\pi$. Hence the construction only applies for rational magnetic fields of the form $\phi=p/q$, with p and q relatively prime integers. We then conclude that *the number of plaquettes N_{Γ} enclosed by the path must be a multiple integer of q.* Under these conditions it is possible to find a consistent set of matching phases such that $(p/q)N_{\Gamma}=j$. See Fig. 3 for an example.

B. Currents in the WT model

We are interested in computing the expectation values of the current operators for eigenstates of energy E. In particular, we will want to know about both ohmic and Hall currents and what is the contribution of the nonpropagating states to them.

The *gauge-invariant current operator* for the bond with end points at $\mathbf{r}_A$ and $\mathbf{r}_A+\mathbf{d}_i^A$ is

$$J_i(\mathbf{r}_A,\mathbf{r}_B)=\frac{1}{2i}(\,|\,\mathbf{r}_A,\mathbf{d}_i^A\rangle e^{iA(\mathbf{r}_A,\mathbf{r}_A+\mathbf{d}_i^A)}\langle\mathbf{r}_B,\mathbf{d}_i^B\,|-\text{H.c.}) \tag{3.24}$$

Consider the quantity $D_E[A]$ given by following determinant

$$D_E[A]=\det(\gamma V_1\hat{U}+V_2\hat{T}[A]-E\hat{I}) . \tag{3.25}$$

The expectation value of the operator $J_i(\mathbf{r}_A,\mathbf{r}_B)$ for an eigenstate of energy E is

$$\langle J_i(\mathbf{r}_A,\mathbf{r}_B)\rangle=\sum_n \delta(E-E_n)\langle n\,|\,J_i\,|\,n\rangle , \tag{3.26}$$

which can be written in the form

$$\begin{aligned}\langle J_i(\mathbf{r}_A,\mathbf{r}_B)\rangle&=\frac{1}{\pi V_2}\,\text{tr}\,\text{Im}\frac{1}{H(A)-E}\frac{\delta H(A)}{\delta A(\mathbf{r}_A,\mathbf{r}_B)}\\&=\frac{1}{\pi V_2}\,\text{Im}\frac{\delta}{\delta A(\mathbf{r}_A,\mathbf{r}_B)}\,\text{tr}\ln[H(A)-E]\\&=\frac{1}{\pi V_2}\,\text{Im}\frac{\delta}{\delta A(\mathbf{r}_A,\mathbf{r}_B)}\ln\det[H(A)-E] .\end{aligned} \tag{3.27}$$

In (3.27) an $i\varepsilon$ prescription is assumed.

The determinant $D_E[A]$ can be written formally in terms of the eigenvalues of H

$$D_E[A]=(V_2-E)^{n_+}(-V_2-E)^{n_-}\prod_l (E_l-E) , \tag{3.28}$$

where $n_+(n_-)$ is the degeneracy of the eigenvalue $E=+V_2(-V_2)$ and E_l are the eigenvalues of the propagating modes, (3.15). Here we consider problems in which $n_+=n_-=n$. Using Eq. (3.15) we can write

$$\begin{aligned}D_E[A]&=(E^2-V_2^2)^n\prod_l\left[\frac{\gamma V_1}{2}-Q[A]-E\right]_l\left[\frac{\gamma V_1}{2}+Q[A]-E\right]_l\\&=(E^2-V_2^2)^n\prod_l\left\{\left[\frac{\gamma V_1}{2}-E\right]^2-\left[\left[\frac{\gamma V_1}{2}\right]^2+V_2^2+V_1V_2\varepsilon_l(A)\right]\right\} .\end{aligned} \tag{3.29}$$

From (3.29) we learn that (i) *the nonpropagating modes do not contribute to the current since their eigenvalues are independent of the vector potential,* and (ii) $D_E(A)$ depends on the vector potential only through the eigenvalues ε_l of the one-orbital operator $\hat{U}\hat{T}\hat{U}$. This latter observation implies that there must be a relationship between the gauge invariant current operator of the WT model and that of the one-orbital problem. More specifically, we find

$$\langle J_i \rangle = \frac{1}{\pi V_2} \mathrm{Im} \frac{\delta \ln D_E(A)}{\delta A_i(\mathbf{r}_A, \mathbf{r}_B)}$$
$$\equiv \frac{1}{\pi V_2} \mathrm{Im} \frac{\delta}{\delta A_i(\mathbf{r}_A, \mathbf{r}_B)} \sum_l \ln \left\{ \left[\frac{\gamma V_1}{2} - E \right]^2 - \left[\left[\frac{\gamma V_1}{2} \right]^2 + V_2^2 + V_1 V_2 \varepsilon_l(A) \right] \right\}$$
$$= \frac{1}{\pi V_2} \mathrm{Im}\, \mathrm{tr} \frac{1}{(\gamma V_1/2)^2 + V_2^2 + V_1 V_2 \tilde{H}[A] - [(\gamma V_1/2) - E]^2} (-V_1 V_2) \frac{\delta}{\delta A_i} \tilde{H} , \tag{3.30}$$

where $\tilde{H}$ is the one-orbital Hamiltonian

$$\tilde{H} = \gamma \hat{U} \hat{T}[A] \hat{U} , \tag{3.31}$$

on the same Bravais lattice. The gauge-invariant current operator for $\tilde{H}$ is just

$$\tilde{J}_i = \frac{\delta \tilde{H}}{\delta A_i} . \tag{3.32}$$

Hence, we have

$$\langle J_i \rangle_E = -\frac{V_1}{\pi} \langle \tilde{J}_i \rangle_{\tilde{\varepsilon}} , \tag{3.33}$$

where the energies E and $\tilde{\varepsilon}$ are related by

$$\tilde{\varepsilon} = \frac{1}{V_1 V_2} (E^2 - \gamma V_1 E - V_2^2) . \tag{3.34}$$

Thus we have shown that the currents of the network lattice and that of a single-orbital Hamiltonian on the *same* Bravais lattice are essentially identical (up to a sign) if the energies of the eigenstates are related by (3.34). Equation (3.30) also shows that only the propagating states contribute to the *gauge-invariant current.* We finally note that (3.27) with (3.28) is a Streda-like[5] formula since they show that the only eigenvalues that contribute to the current are those which change with the vector potential.

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