Third sound and superflow on a striped substrate

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The propagation of third sound on a one-dimensional periodic substrate is studied for the case when the helium film is subjected to a uniform flow field. The modifications introduced by the flow in the band structure depend on the relative directions of the flow and the wave propagation. A simple derivation of the effects of the flow on the location of the transmission resonances corresponding to a disordered substrate is also presented. These results are relevant to the problem of third-sound Anderson localization.

There has been much activity recently in the study of Anderson localization phenomena in nonelectronic systems.[1–12] In the realm of acoustics, the use of third-sound waves as the excitations to be localized is particularly promising, both in one- and two-dimensional configurations.[5,6]

In the one-dimensional problem, it was suggested that the randomness required to obtain localization could be introduced by preparing a random array of parallel strips of a second substrate.[6] If we call α_i the van der Waals constants corresponding to the first and the second substrates, respectively, the ratio of the film thicknesses h_1 and h_2 on both regions is, to a good approximation, $h_2/h_1 = (\alpha_2/\alpha_1)^{1/3}$. The speed of third sound, $c=\alpha_1 h_1^{-3} = \alpha_2 h_2^{-3}$ is the same on both regions.

On the other hand, third-sound experiments in moving superfluid films have been performed for about 20 years,[13–17] partially with the goal of investigating the Bernoulli thinning predicted by Kontorovich.[18] Of special interest are the studies on the stability of persistent currents due to Hallock and co-workers.[15–17] It has been pointed out that the addition of a uniform supercurrent to the experiments described in Refs. 5 and 6 would open interesting possibilities:[19,20] Since the supercurrent destroys the time-reversal invariance, it might be expected to have substantial effects on the localization of third-sound waves. This turns out to be the case for the two-dimensional problem,[19] where the "strong" localization length $\xi \sim \exp(\omega_0/\omega)^2$ is replaced by the weaker form $\xi \sim \exp(\omega_0/\omega)^4$. Here ω is the excitation frequency and ω_0 a constant. In contrast, a detailed calculation shows that the one-dimensional localization does not suffer any important modification, in spite of the absence of time-reversal symmetry.[20]

An analysis has also been carried out of the propagation of third-sound waves on periodic substrates.[21] The bands of allowed wave numbers k and the corresponding gaps were studied for this system, which is an experimentally realizable analog of the electronic Kronig-Penney model.

It is the purpose of this paper to discuss the propagation of third sound on a flowing superfluid helium film located on a periodic substrate. Using simple physical arguments I will also draw some conclusions pertinent to the related problem of localization on a random substrate.

A portion of the substrate is shown in Fig. 1. The velocity potential $\phi(x)$ satisfies the equation[20]

$$\frac{\partial^2\phi}{\partial t^2} - c^2\frac{\partial^2\phi}{\partial x^2} + 2v_i\frac{\partial}{\partial x}\frac{\partial\phi}{\partial t} + v_i^2\frac{\partial^2\phi}{\partial x^2} = 0 \quad (i=1,2) , \qquad (1)$$

where v_1 and v_2 are the superflow velocities on the first and second substrates, respectively. The second substrate is formed by parallel strips of width $2a$. To the boundary conditions at the edges of the strips discussed in Ref. 21 we must add the mass conservation condition $v_1 h_1 = v_2 h_2$. The superflow will always be assumed to be parallel to the positive x axis.

The transmission coefficient for a single strip can be calculated easily. I obtain[22]

$$T_1 = \frac{4(h_1h_2k_1k_2)^2}{4(h_1h_2k_1k_2)^2 + [(h_2k_2)^2 - (h_1k_1)^2]^2 \sin^2(2k_2 a)} , \qquad (2)$$

where the wave vectors $k_i = \omega/(c+v_i)$. It is also convenient to define k_{i+} and k_{i-} as the wave vectors corre-

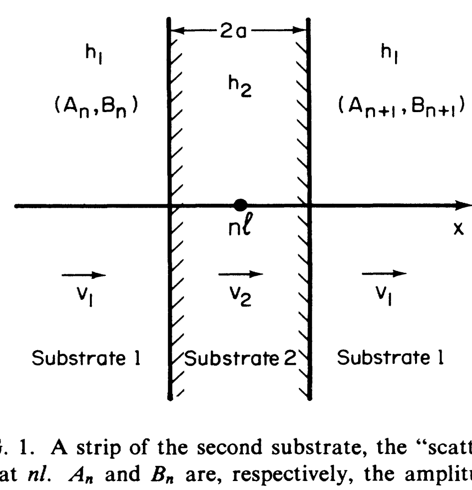

FIG. 1. A strip of the second substrate, the "scatterer," centered at nl. A_n and B_n are, respectively, the amplitudes of the waves traveling to the right and to the left in the region immediately to the left of the scatterer. The film thicknesses on the two substrates are h_1 and h_2 and the superflow velocities are v_1 and v_2.

sponding to waves propagating parallel (P) and antiparallel (A) to the direction of the flow, respectively. We also write $u = v_1/c$ and $y = h_2/h_1$. Note that Eq. (2) yields perfect transmission for all frequencies where $y = 1$, as expected, and also for A waves when the condition $u = y/(1+y)$ is fulfilled. For arbitrary values of u and y there is perfect transmission for the following wave numbers:

$$\tilde{k}_{1\pm} = \pm j\pi(y \pm u)[2ay(1\pm u)]^{-1}, \quad j = 1,2,\ldots \quad (3)$$

The case $u = 0$ was discussed in Ref. 6. There we considered the localization of the third-sound excitations by a random distribution of identical, parallel strips of the second substrate. It was found that the set of measure zero of extended states corresponds to the transmission resonances of the isolated scatterers. These are located at the frequencies $\omega_j = j\pi c/(2a)$. Around each of these resonances there is a region of localized states with long localization lengths, which in practice corresponds to a "pass band."[6] From Eq. (3) we see that these resonances will be shifted by the flow. The value of this shift is given by

$$\Delta\omega_{j\pm} = \pm j\pi cu/2ay \ , \quad (4)$$

where the + (−) sign corresponds to propagation parallel (antiparallel) to the flow. The magnitude of the shift is proportional to the order of the resonance and can become quite large as j is increased.

Due to the sign difference between $\Delta\omega_{j+}$ and $\Delta\omega_{j-}$, if we create, for example, an excitation with frequency $\omega_j + \Delta\omega_{j+}$, it will propagate to the right but not to the left; after successive partial reflections in the scatterers at the left, the excitation will leak out to the right. Let us now assume that the $u = 0$ localization lengths for states in the "pass bands" of Ref. 6 satisfy $\xi(\omega) > \Lambda$, where Λ is the total length of the random region. When we turn on the flow, the single "pass band" around ω_j is replaced by two contiguous bands; while in the upper band transmission can occur only in the direction parallel to the flow, in the lower one transmission can occur only in the opposite direction.

I next turn to the infinite periodic substrate, assuming that the length of the period is l. The elementary techniques used in Ref. 21 can be extended immediately to the case $u \neq 0$. A calculation of the eigenvalues $q_\pm$ of the unimodular matrix Q, which connects the wave amplitudes to the right of a scatterer $[(A_{n+1}, B_{n+1})$ in Fig. 1] to those to its left $[(A_n, B_n)$ in Fig. 1], yields

$$q_\pm = H(k_1) \pm [H^2(k_1) - 1]^{1/2} \ , \quad (5)$$

where the explicit form for $H(k_1)$ will be given below. [In this equation the pair of signs (±) is not related to the direction of propagation.] It is clear that values of k_1 such that $|H(k_1)| < 1$ correspond to the interior of an allowed band, while $|H(k_1)| > 1$ corresponds to values of k_1 inside a forbidden gap. The band edges are defined by $|H(k_1)| = 1$.

To write an explicit expression for $H(k_1)$, I define

$$\tau_\pm = [4y^2(y\pm u)(1\pm u)]^{-1}, \quad \theta_\pm = \gamma y(1\pm u)[2(y\pm u)]^{-1} \ ,$$

$$v_\pm = [y^2 + y \pm u(y^2+1)]^2, \quad \mu_\pm = [y^2 - y \pm u(y^2-1)]^2 \ ,$$

$$\phi_{1\pm} = \pm \gamma u(y-1)/[2(y\pm u)], \quad \phi_{2\pm} = \gamma[2y \pm u(1+y)]/[2(y\pm u)] \ ,$$

with $\gamma = 4a/l$. I obtain

$$H(k_{1\pm}) = \tau_\pm \{v_\pm \cos[(1+\phi_{1\pm})k_{1\pm}l] - \mu_\pm \cos[(1-\phi_{2\pm})k_{1\pm}l]\} \ . \quad (6)$$

The upper and lower signs again correspond to P and A waves, respectively.

Since both source and detector are located in the lab frame, it is convenient to use as a variable either the frequency ω they register or the dimensionless frequency $W = \omega l/c$. In Fig. 2, I have plotted the lowest bands for the case $u = 0.1$ and $\gamma = 1$, with y varying over the physical range. (The ratio between two van der Waals constants seems to be generally smaller than 4.[23–25]) Because the bands are modified differently for P and A propagation, some frequencies will be allowed for propagation in one direction but not in the opposite one. In particular, we note that (i) the transmission gaps are narrowed (widened) for A (P) propagation, (ii) they are shifted downward (upward) for A (P) propagation, (iii) a new, thin gap appears at a frequency of about twice that of the first gap. (This second gap is also present when $\gamma \neq 1$ even if $u = 0$. See Ref. 21 and Fig. 3.)

These features can be seen from a different viewpoint in Fig. 3, where I fixed $y = 1.4$ and let u vary. The tilting of the gaps becomes more pronounced for the higher order gaps. It is also clear that the flow tends to favor (oppose) A (P) propagation.

Finally, I present the results for the transmission coefficient T_N corresponding to an ordered array of N identical scatterers. The calculation is again a straightforward extension of that in Ref. 21 and yields

$$T_N(k_{1\pm}) = \{1 + F(\gamma, l, \pm u; W)\sin^2[NK(k_{1\pm})l]\}^{-1} \ , \quad (7)$$

for wave vectors corresponding to the bands of the infinite system and

$$T_N(k_{1\pm}) = \{1 - F(\gamma, l, \pm u; W)\sinh^2[N\rho(k_{1\pm})l]\}^{-1} \ , \quad (8)$$

for wave vectors in the gaps of the infinite system. We have written $q_\pm = \exp[\pm iK(k_1)]$ in Eq. (7) and $q_\pm = \varepsilon\exp[\pm\rho(k_1)l]$ in Eq. (8), with $\varepsilon = \pm 1$. The function $F(\gamma, l, \pm u; W)$ is

$$F(\gamma, l, \pm u; W) = \frac{4v_\pm \mu_\pm \sin^2(\theta_\pm k_{1\pm} l)}{\tau_\pm^{-2} - \{v_\pm \cos[(1+\phi_{1\pm})k_{1\pm}l] - \mu_\pm \cos[(1-\phi_{2\pm})k_{1\pm}l]\}^2} \ , \quad (9)$$

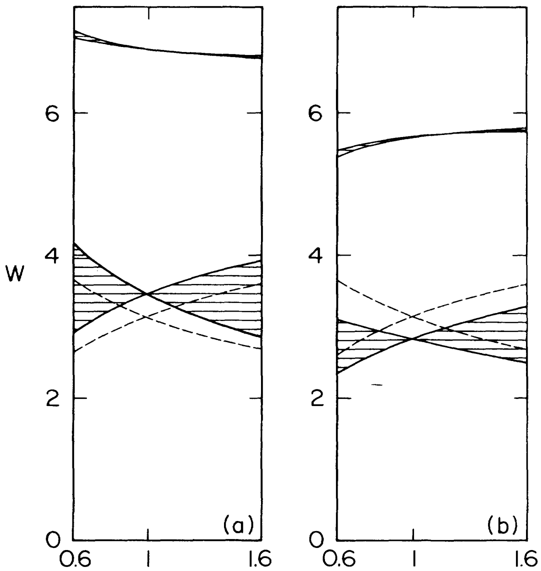

FIG. 2. Allowed and forbidden regions for the frequency W as functions of thickness ratio for $\gamma = 1$ and $u = 0.1$. The shaded areas are the transmission gaps for (a) propagation parallel to the flow, and (b) antiparallel to the flow. The dashed lines are the gap edges when $u = 0$.

where

$$k_{1\pm} l = \pm W/(1 \pm u) \ .$$

As usual, the upper (lower) sign corresponds to P (A) propagation.

The amplitude of the transmitted wave is an oscillating function of the length of the strip array if $k_{1\pm}$ is in one of the bands, and it is attenuated when $k_{1\pm}$ is in one of the gaps. In this case, we can write, for N large, $T_N(k_{1\pm}) \sim \exp(-x/L_\pm)$, where $x = Nl$ and $L_\pm = |2\rho(k_{1\pm})|^{-1}$ is the anisotropic penetration length.

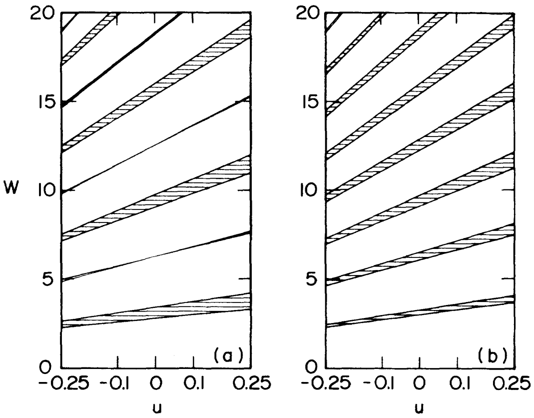

FIG. 3. Forbidden gaps (shaded areas) for $y = 1.4$: (a) $\gamma = 1$, (b) $\gamma = 0.25$. Positive (negative) values of u correspond to P (A) propagation.

Although all the calculations presented here can also be carried out for the "index of refraction" scatterers discussed in Ref. 6, I believe that, in that case, the substrate roughness may substantially modify the properties of the flow field.

In conclusion, the calculations of Ref. 21 have been extended to analyze the effects of a uniform flow field on the characteristics of third sound propagating on periodic substrates. It has also been shown how the transmission resonances corresponding to the disordered problem are modified by the flow.

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