$1/N$ expansion for the transport coefficients of the single-impurity Anderson model

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In this paper the properties of the single-impurity Anderson model are studied via a loop expansion using the functional-integral method developed by Read and Newns, referred to here as I. First, the low-temperature equilibrium properties were considered and it was shown that, to two-loop order after zero modes are properly treated, for physical properties (free energy and f-level occupancy) the loop expansion is one-to-one with the $1/N$ expansion and to order $1/N$ the results first derived by Read, referred to here as II, were recovered. For the transport properties we present the first calculations of the low-temperature conductivity, thermopower, and thermal conductivity to order $1/N$. For the conductivity, our result for the T^2 coefficient in the Kondo limit when $N=6$ is $(1-8/3N)\pi^2=5\pi^2/9=5.6$ which is to be compared with the result 5, derived previously from the "noncrossing" approximation. In the case of the thermal power S, our result agreed to order $1/N$ with the exact result derived here for the first time. In the Kondo regime, this implies that the thermopower is reduced by a factor of $(1-1/N)$ relative to the mean-field result, and that this is an *exact* ratio, analogous to the well-known χ/γ ratio: We may write $(S/T\chi_0)=[2\pi^2k_B^2/g^2j(j+1)\mu_B^2e](1-1/N)(\pi/N)\cot(\pi/N)$, which is a universal result in the Kondo regime.

I. INTRODUCTION

Recent theoretical work on the valence fluctuation, Kondo lattice, and heavy-fermion problems has tended to center around the Anderson Hamiltonian in either its single-impurity or lattice forms, usually with very strong on-site f repulsion ($U=\infty$).[1,2] A powerful method capable of handling low-temperature Kondo-like effects in both single-impurity and lattice systems has emerged—the $1/N$ expansion.[3] Here N is the degeneracy of the lowest spin-orbit multiplet of the f shell, $N=2j+1$. In the standard model, the infinite f-f repulsion U allows only zero or one f electrons to occupy the f shell at each site, and the spin-orbit coupling is also taken to be infinite, so that the angular momentum j of the $n_f=1$ state is $j=\frac{5}{2}$ (cerium) or $j=\frac{7}{2}$ (ytterbium, for which one treats holes in the f shell instead of electrons). Though initially formulated in several ways by different groups,[4–8] the most flexible and elegant approach appears to be the functional-integral method.[5,6] Originally, Read and Newns[5] treated the Coqblin-Schrieffer[9] [or SU(N) Kondo] Hamiltonian using a Hubbard-Stratonovich decoupling of the spin-flip interaction term, and an integral over another field λ to ensure the constraint $n_f=1$. Following Coleman's introduction[10] of a "slave boson" version of the $U=\infty$ Anderson model, Read and Newns[6,11] noted its formal similarity to the decoupled version of the Coqblin-Schrieffer model (which in fact is a limiting case of the model, via a Schrieffer-Wolff transformation[9]) and thus extended their results to incorporate the $U=\infty$ Anderson impurity model. The same authors also discussed the lattice versions of these models.[12]

The physical content of this method is as follows. The lowest level of approximation is a mean-field theory,[5,6] which at low temperatures describes a noninteracting Fermi gas with a resonant density of states[13] (impurity case) or a "renormalized band structure" (Ref. 12) (lattice case). The f density of states (or effective mass) at the Fermi level is large, giving rise to the observed large magnetic susceptibility and linear coefficient γ of the specific heat. The charge response (compressibility), on the other hand, has to be calculated from the mean field "equations of state" and is small[5,6,12] especially when the f occupation n_f tends to 1. Inclusion of fluctuations around the mean field theory[5,6,11] leads to a feedback of the suppression of charge fluctuations into the specific heat, giving a reduction of γ, and consequently an enhancement of the ratio $\chi(T=0)/\gamma$ above its value for the noninteracting Fermi gas (which is normalized to 1), in accordance with the Fermi liquid arguments[13,14] for the $n_f\rightarrow 1$ (Kondo) limit.

In this paper our goal is a similar calculation for low-temperature transport properties of the Anderson impurity. We next review this transport problem and summarize existing results.

A. Transport properties

In the relaxation time approximation to the Boltzmann equation, the electrical conductivity σ is obtained from the energy- and temperature-dependent transport relaxation time τ_1 (Ref. 15) as

$$\sigma=\frac{\rho e^2 v_f^2}{3}\int d\varepsilon_k\left[\left.\frac{-df}{d\varepsilon}\right|_{\varepsilon=\varepsilon_k}\right]\tau_1(\varepsilon_k,T)\,, \tag{1.1}$$

where ρ is the density of states in the conduction band at the Fermi surface, and v_F is the Fermi velocity. In general, the transport time τ_1 differs from the usual relaxation time τ, the time between collisions, but as can be seen from p. 596 of Ref. 15, when all the phase shifts of the impurity vanish except for one value of angular momentum, these quantities become proportional. This is the case in our model, in which only conduction electrons of total angular momentum j about the impurity site can hop into the f shell (conserving angular momentum). Explicitly, the Anderson Hamiltonian is

$$H=\sum_{k,m}\varepsilon_k c_{km}^\dagger c_{km}+E_0\sum_m F_m^\dagger F_m+V\sum_{k,m}(F_m^\dagger c_{km}+c_{km}^\dagger F_m)+\tfrac{1}{2}U\sum_{\substack{m,m'\\ m'\neq m}}F_{m'}^\dagger F_m^\dagger F_m F_{m'} . \tag{1.2}$$

Here the conduction electron operators c_{km} destroy electrons in a partial wave state of angular momentum j and radial momentum k; m runs from $-j$ to j. E_0 is the energy of the f level (all one-electron energies are measured relative to the chemical potential, which is thereby set equal to zero). The hybridization matrix element has been taken to be a constant, and we assume a rectangular band of width $2D$ and density of states ρ. For the Anderson model, the T matrix is proportional to the F Green's function, evaluated on energy shell:[16,17]

$$T=V^2G_F(\varepsilon_k+i\delta)\times(\text{angle- and spin-dependent factors}) , \tag{1.3}$$

and hence through the optical theorem, the inverse relaxation time $1/\tau(\varepsilon_k,T)$ is proportional to the F spectral function $\rho_F=(-1/\pi)\mathrm{Im}G_F(\varepsilon_k+i\delta)$

$$\tau^{-1}(\varepsilon_k,T)=-2\mathrm{Im}T_{k\sigma k\sigma}\propto V^2\rho_F(\varepsilon_k,T) . \tag{1.4}$$

Similarly, if we define moments of τ,

$$L^n=\int d\varepsilon\left[\frac{-df}{d\varepsilon}\right]\tau(\varepsilon,T)\varepsilon^n , \tag{1.5}$$

one has for the thermopower S and Lorentz ratio $L=\kappa/\sigma T$, here κ is the thermal conductivity,

$$S(T)=-\frac{1}{eT}\frac{L^1}{L^0} , \tag{1.6a}$$

$$\frac{L(T)}{L_0}=\frac{3}{\pi^2(k_BT)^2}\left[\frac{L^2}{L^0}-\left[\frac{L^1}{L^0}\right]^2\right] , \tag{1.6b}$$

where

$$L_0=\frac{\pi^2}{3}\left[\frac{k_B}{e}\right]^2 . \tag{1.6c}$$

Therefore the central object of study is the F spectral function, which is obtained in the usual way from the imaginary-time Green's function:

$$G_F(\tau)=-\langle\mathcal{T}F_m(\tau)F_m^\dagger(0)\rangle . \tag{1.7}$$

We will calculate $\tau(\varepsilon,T)$ to $O(\varepsilon^2)$ and $O(T^2)$, and hence $\sigma(T)/\sigma(0)$ and $L(T)$ to $O(T^2)$ and $S(T)$ to order T, all to order $1/N$ in the $1/N$ expansion (i.e., the first corrections to the leading terms).

We now discuss the existing results for these quantities. Since the low-temperature, low-excitation-energy properties of the system are those of a Fermi liquid, it is natural to ask whether any *exact* results are available from Fermi liquid relations. The resistivity of the Kondo[18] and Anderson[17] models has been investigated. In the next subsection we derive a new exact result for the thermopower. At $T=0$, the T-matrix formulas referred to above give[15]

$$1/\sigma(0)\propto\sum_m\sin^2\delta(0) , \tag{1.8}$$

where $\delta(0)$ is the phase shift at the Fermi energy, and is related to the f occupation via the Friedel sum rule:[16]

$$\sum_m\frac{\delta(0)}{\pi}=n_f ; \tag{1.9}$$

in fact one has the exact result[16] (at $T=0$) for the F spectral function:

$$\rho_F(0,0)=\frac{1}{\pi\Delta_0}\sin^2\left[\frac{\pi n_f}{N}\right] , \tag{1.10}$$

where $\Delta_0=\pi\rho V^2$ is the hybridization width of the Anderson model. However, at $T\neq0$, the only exact results[17,18] are for the coefficient of T^2 in the Kondo limit under the extra assumption of particle-hole symmetry. For our model (with $U\to\infty$) this holds only for $N=2$, i.e., the spin-$\frac{1}{2}$ case.

Far more extensive results have recently been published,[19] based on approximate numerical calculations. Several authors[20,7,10(a)] arrived at identical approximate self-consistent equations governing ρ_F from different formalism. The approximation used is widely termed the "noncrossing approximation" (NCA),[20,7,10(a)] and amounts to calculate of self-energy diagrams with neglect of all vertex corrections (the vertex in question being *not* the U interaction of (1.2), since $U=\infty$, but the hybridization event, as in the formalism[5,6,10,11] we describe in the next subsection). The method is related to the $1/N$ expansion and is sometimes erroneously equated with it, since the diagrams assumed coincide through order $1/N$; however the NCA includes diagrams of all orders in $1/N$ and is therefore an attempt at a partial resummation of the $1/N$ expansion. Detailed calculations[21] showed, however, that at low temperatures and frequencies a spurious feature appears in ρ_F, whose value as $T,\varepsilon\to0$ is independent of n_f, in contradiction with (1.10). In contrast, as we show in this paper, calculation strictly in powers of $1/N$ leads to agreement with (1.10) and no low T, low ε pathologies, which are therefore an artifact of the resummation in the NCA. It has been shown elsewhere[22] that the NCA incorrectly treats the cancellations of infrared divergences[11] in the $1/N$ expansion, so that the effective value of n_f (which appears as a coupling constant at low frequencies) is renormalized towards a fixed point value at $n_f\simeq1$ in the large-N limit; this "spurious scaling" of n_f then accounts for the independence of $\rho_F(0,0)$ of the true value of n_f, and for the power law[21(b)] approach to this value. Also accounted for[22] is the narrowness of the spurious peak in ρ_F as $n_f\to0$;[21(a)] significant deviations from the correct behavior set in at frequencies of order

$$N\Delta_0 n_f^{-1}\exp(-N/n_f) , \tag{1.11}$$

and below, as the true $n_f\to0$. The conflict between the argument that crossing diagrams are of order $1/N^2$ and so negligible, and the result that ρ_F deviates from the correct result already at leading order is thus resolved: some of the omitted terms are needed to cancel infrared divergences in the terms retained; the divergence of the coefficients of $1/N^2$, $1/N^3$, etc. means that these terms are not negligible at all. The scaling approach[22] enables one to understand and sum these terms, and the result is a leading order effect at sufficiently low T and ε.

In any case, the spurious features appear only at very low temperatures, and the NCA provides smooth and reasonable-looking results for all other frequency and temperature ranges. Bickers, Cox, and Wilkins[19] used the approximation to study transport and dynamical properties of Ce compounds, and found that for low temperature transport properties, little deviation from the universal behavior at $n_f=1$ occurs even for n_f as small as 0.7. Coefficients for the low temperature expansion in powers of T were not reported, except for the result $\sigma(T)/\sigma(0)\simeq 1+5(T/T_0)^2$ (T_0 is the Kondo temperature, defined more carefully below). The numerical nature of this study, and the possibility that the results are affected by the spurious features inherent in the method, motivates the present work.

B. Exact result for the thermopower

Here we will briefly derive an exact formula connecting the linear term in the thermopower S with the specific heat coefficient γ and the f occupation n_f, based on results to all orders in perturbation theory in U for the Hamiltonian (1.2). This has not, to our knowledge, appeared in the literature previously.

When the F Green's function (1.7) is expanded in powers of U, one obtains

$$G_F(i\omega)=(i\omega-E_0-\Sigma(i\omega)+i\Delta_0\mathrm{sgn}\omega)^{-1} . \tag{1.12}$$

Here we have let $D\to\infty$, which is permissible since U is finite.[23] γ can be obtained from G_F at $T=0$, the result is

$$\gamma=\frac{\pi^2}{3}k_B^2N\rho_F(0)\left[1-\left.\frac{\partial\Sigma(\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0}\right] . \tag{1.13}$$

The thermopower involves the ratio of moments of τ, L^1/L^0, and so we will drop the proportionality constant in $\tau^{-1}\propto\rho_F$. Then

$$\begin{aligned} L^1&=\int d\varepsilon\left[\frac{-df}{d\varepsilon}\right]\frac{\varepsilon}{\rho_F(\varepsilon)}\\ &=-\frac{\pi^2}{3}(k_BT)^2\frac{1}{[\rho_F(0)]^2}\left.\frac{d\rho_F}{d\varepsilon}\right|_{\varepsilon=0}+O(T^4)\\ &=-\frac{\pi^2}{3}\frac{(k_BT)^2}{[\rho_F(0)]^2}\frac{2}{\pi}\frac{\Delta_0[E_0+\Sigma(0)]}{\{[E_0+\Sigma(0)]^2+\Delta_0^2\}^2}\\ &\quad\times\left[1-\left.\frac{d\Sigma}{d\varepsilon}\right|_{\varepsilon=0}\right] , \end{aligned} \tag{1.14a}$$

$$L^0=1/\rho_F(0)+O(T^2) . \tag{1.14b}$$

Now using the Friedel sum rule[16] (true at $T=0$)

$$\frac{\Delta_0}{E_0+\Sigma(0)}=\tan\left[\frac{\pi n_F}{N}\right] \tag{1.15}$$

and

$$\rho_F(0)=\frac{\Delta_0/\pi}{[E_0+\Sigma(0)]^2+\Delta_0^2} , \tag{1.16}$$

we find our exact result

$$S(T)=\frac{-1}{eT}\frac{L^1}{L^0}=\frac{2\pi}{eN}\gamma T\cot\left[\frac{\pi n_f}{N}\right] . \tag{1.17}$$

Comparison of our $1/N$ expansion result with (1.17) will provide an extra consistency check on our calculation. We note that in the numerical results of Bickers *et al.*,[19] close to $n_f=1$, the observed "universality" holds only for the linear part of the $S(T)$ curve and is merely a consequence of (1.17).

C. Formalism

We now discuss the slave boson[10] based functional integral[5,6,11] approach to the $1/N$ expansion, with emphasis on the new technical aspects encountered in the present calculation.

Coleman[10] writes the Anderson Hamiltonian (1.2) for $U=\infty$ as

$$\begin{aligned} H=&\sum_{k,m}\varepsilon_k c_{km}^\dagger c_{km}+E_0\sum_m f_m^\dagger f_m\\ &+V\sum_{k,m}(f_m^\dagger c_{km}b^\dagger+c_{km}^\dagger f_m b) , \end{aligned} \tag{1.18}$$

where the Fermion operators $f,f_m^\dagger$ have a meaning somewhat different from $F_m,F_n^\dagger$ and $b,b^\dagger$ are boson operators. The Hamiltonian (1.18) commutes with the operator

$$Q=\sum_m f_m^\dagger f_m+b^\dagger b , \tag{1.19}$$

so that H can be diagonalized for each (integer) value of Q. The eigenspace with $Q=1$ represents the $U=\infty$ Anderson model since the f occupation

$$n_f=\sum_m f_m^\dagger f_m$$

takes only the values 0 or 1. The operator $f_m^\dagger b$ commutes with Q and raises n_f, and so is the analogue of $F_m^\dagger$; thus the F Green's function (1.7) is now[10]

$$G_F(\tau)=-\langle Tb^\dagger(\tau)f_m(\tau)f_m^\dagger(0)b(0)\rangle , \tag{1.20}$$

where the brackets denote an average over states restricted to $Q=1$.

Read and Newns[5,6] introduced a functional integral formalism for the partition function in which the constraint $Q=1$ is imposed through an integral over a τ-independent field λ:

$$Z=\int_{-\pi/\beta}^{\pi/\beta}d\lambda\frac{\beta}{2\pi}\iint Df\,Df^\dagger Dc\,Dc^\dagger Db\,Db^\dagger\exp\left[-\int_0^\beta L(\tau)d\tau\right] , \tag{1.21a}$$

$$L(\tau)=\sum_m f_m^\dagger\frac{d}{d\tau}f_m+\sum_{k,m}c_{km}^\dagger\frac{d}{d\tau}c_{km}+b^\dagger\frac{d}{d\tau}b+H+i\lambda(Q-1) , \tag{1.21b}$$

where $f,f^\dagger$, $c,c^\dagger$ are Grassmann numbers antiperiodic under $\tau\to\tau+\beta$, and $b,b^\dagger$ are c numbers, periodic under $\tau\to\tau+\beta$. The Green's function G_F is now obtained by calculating the average (1.20) with the same functional measure, (1.21), divided by Z. Q generates a gauge symmetry of the system, under which

$$f(\tau)\to f(\tau)e^{i\theta(\tau)}\ ,$$
$$b(\tau)\to b(\tau)e^{i\theta(\tau)}\ , \tag{1.22}$$

and λ is the corresponding gauge field; in (1.12) a particular gauge, specified by $\lambda=0$ has already been chosen. Other gauge choices are possible[5,6,24] but here we will continue to use this "cartesian" gauge in which b is a complex, τ-dependent field. With this choice, our formulas have a close correspondence with those that would be obtained in other formalisms;[4,7,9,20,10(a)] of course, all *physical* results should be independent of the gauge.[24]

The $1/N$ expansion is obtained from (1.21) by first integrating out the Fermi fields. By absorbing V into b to form a new field $\sigma,\sigma=Vb$, one sees that if NV^2 is held fixed as $N\to\infty$, the effective action for the remaining $\sigma,\sigma^\dagger$ and λ integrations is proportional to N except for the last term, in which λ multiplies the eigenvalue of Q, which is 1. This comes about because the fermions have N components, while the remaining σ terms have coefficients $1/V^2\sim N$. The basic idea of the $1/N$ expansion is to evaluate the functional integral by a steepest descent-like expansion, in which the first term in Z comes from the value of the integrand maximized with respect to $\sigma,\sigma^\dagger,\lambda$, multiplied by the determinant arising from performing the Gaussian integral in the deviations in these variables from the saddle point. Higher terms follow from higher order expansion of the effective action, and correspond to diagrams with 2 or more loops in the bose-like variables $\sigma,\sigma^\dagger,\lambda$. Thus in the free energy, if the whole of the effective action were proportional to N, the leading (saddle point or mean field) term would be of order N, the Gaussian fluctuations of order 1, and in general diagrams containing r-independent boson loops (fermion loops being counted as point vertices) would be of order N^{1-r}.

At first sight, as noted in Ref. 5 (to be referred to as I), the desired value of Q being 1, not of order N, seems to preclude such an expansion; there appears to be no reason why successive terms should be smaller by factors of $1/N$. But with Q (or n_f for the Coqblin-Schrieffer model) fixed at 1,[5,6] it was found that although the first term in the free energy at low temperature (which is in fact the mean field theory referred to above) was of order 1, which can be traced to the value of Q (or n_f), Gaussian fluctuations gave corrections to physical quantities of order $1/N$. Therefore it was asserted that the $1/N$ expansion nevertheless exists, at least at low temperatures. This is also the result of Refs. 4, 7, 8, 10(a) and 20, where essentially the same limit is taken. On the other hand, Coleman[25] has proposed that (1.18) be studied for general $Q=q_0N$ of order N, so that if it is desired to study a model with $Q=1$ and $N=N_0$, say, these values may be reached in the $1/N$ expansion by setting $q_0=1/N_0$ and expanding in powers of $1/N$ (with N_0 fixed). Clearly this leads immediately to a $1/N$ expansion for all temperatures, by the above argument. He further asserts[25,26] that with $Q=1$, a $1/N$ expansion "certainly did not exist," quoting the fact that a phase transition (at which $\langle\sigma\rangle\to 0$) occurs at a temperature[27] of order $1/\ln(N/Q)$, which tends to zero as $N\to\infty$ if $Q=1$. Then it would appear that the small but nonzero temperature region cannot be accessed, in contradiction with the low-temperature results of Refs. 5–8.

How can one resolve this paradox? In particular, in what sense, if any, does a $1/N$ expansion at low temperatures exist if $Q=1$? Clearly this is important, since most other calculations[19,20,7,10(a)] have been performed in this limit. In this paper, we again use this method of taking $N\to\infty$, rather than Coleman's,[25] in order to facilitate comparisons with these works (it also gives rise to simpler expressions). Let us now attempt to answer these questions, and outline one or two new features of the detailed calculations presented in this paper.

First we consider the mean field approximation[5,6] to (1.18), which as explained above is the first step in our approximation scheme. One replaces $\sigma=Vb$ by its expectation value s_0 [chosen to be real, positive by use of a τ-independent gauge rotation (1.22)] and includes the constraint $Q=1$ by a Lagrange multiplier ε_f-E_0, thus adding $(\varepsilon_f-E_0)[n_f+(\sigma^\dagger\sigma/V^2)-1]$ to (1.18). In terms of the functional integral (1.21), one sets the mean field value of $i\lambda=\varepsilon_f-E_0$. We then have the mean-field Hamiltonian

$$H_{mf}=\sum_{k,m}\varepsilon_k c^\dagger_{km}c_{km}+\varepsilon_f\sum_m f^\dagger_m f_m+s_0\sum_{k,m}(f^\dagger_m c_{km}+c^\dagger_{km}f_m)+(\varepsilon_f-E_0)\left[\frac{s_0^2}{V^2}-1\right] \tag{1.23}$$

The parameters ε_f and s_0 are found by minimizing the free energy corresponding to (1.23) with respect to ε_f and s_0; the result as shown in Refs. 5 and 6 is a pair of equations (here at $T=0$):

$$n_f=\frac{N}{\pi}\tan^{-1}\left[\frac{\Delta}{\varepsilon_f}\right]=\left[1-\frac{s_0^2}{V^2}\right]\ , \tag{1.24a}$$

$$\varepsilon_f-E_0+\frac{N\Delta_0}{\pi}\ln\left[\frac{(\varepsilon_f^2+\Delta^2)^{1/2}}{D}\right]=0\ . \tag{1.24b}$$

(Here $\Delta=\pi\rho s_0^2=|\langle b\rangle|^2\Delta_0$.) Then in the Kondo regime $n_f\to 1$ ($E_0\to-\infty$) at large N we have

$$\varepsilon_f\simeq\frac{N\Delta}{\pi}\simeq D\exp\left[\frac{\pi E_0}{N\Delta_0}\right]=T_K\ ; \tag{1.25}$$

T_K is the correct Kondo temperature for large N.[5,6] The Hamiltonian (1.23) thus describes, in addition to the broad, flat conduction band, a "Kondo resonance"[4] which is a Lorentzian,[13] centered at $\varepsilon_f\simeq T_K$, with width $\Delta\simeq\pi T_K/N$. This resonance obviously gets narrower and taller as $N\to\infty$. At the Fermi level, the f density of states (per angular momentum channel) is of order $1/N$. This is why thermodynamic results[5,6] such as χ (magnetic susceptibility) and γ (linear coefficient of specific heat) at $T=0$ are of order 1, not of order N as would be expected from the general large N analysis described above, or as would be obtained from Coleman's way of taking the large-N limit. If we consider the full temperature dependence of, e.g., $\chi(T)$ or $C_V(T)$, we see immediately that the density of states peak at $\varepsilon \simeq T_K$ contributes at order 1, not $1/N$, per channel, but this contribution vanishes exponentially as $T \to 0$. This behavior provides a prototype for the behavior we expect at all orders; if we first expand in powers of T (throwing away exponentially small terms; we call this the Sommerfeld expansion) then a $1/N$ expansion for these coefficients apparently does exist, which resolves the paradox discussed above. Note that the temperature at which the excitations to the peak at T_K are as large as the density of states at the Fermi level is given by $e^{-T_K/T} \simeq 1/N$, giving $T \simeq T_K/\ln N$, the same as the mean-field transition temperature discussed earlier. One may also understand the existence of a $1/N$ expansion for the Sommerfeld coefficients by saying that $T_c \to 0$ only logarithmically, which is too slow to affect the expansion in powers of $1/N$.

To underline this picture and to see why fluctuation corrections are not of the same order as the mean field terms in this low-T regime, let us consider the effects of fluctuations of σ, σ^* at the Gaussian level.[5,6,11] At low frequency (and $T=0$) the boson propagator $D_{\sigma\sigma^*}$ takes the form for large N:[11]

$$D_{\sigma\sigma^*}(\omega) \simeq V^2 \left[i\omega \left[1+\frac{N\Delta_0}{\pi\varepsilon_f} \right] + i\omega \frac{N\Delta_0}{\pi\varepsilon_f} i\Delta \operatorname{sgn}\omega \right]^{-1} . \tag{1.26}$$

The spectral function at real $\varepsilon = i\omega$ is therefore

$$\rho(\varepsilon) \simeq \frac{V^2}{1+\dfrac{N\Delta_0}{\pi\varepsilon_f}} \delta(\varepsilon) + \frac{V^2}{\left[1+\dfrac{N\Delta_0}{\pi\varepsilon_f}\right]} \frac{N\Delta_0}{\pi\varepsilon_f} \frac{\Delta/\pi}{\varepsilon} \tag{1.27}$$

for small ε, to leading order in $1/N$ in each term. The second term in (1.27) represents a branch cut on the real axis, which arises from electron-hole pair excitations, in this case an f electron and a conduction hole, for $\varepsilon > 0$ or vice versa for $\varepsilon < 0$. At larger frequencies, $\varepsilon \simeq T_K$ or greater, this term is of order $1/N$, not $1/N^2$ as in (1.27) at small ε. This occurs because the f electron now occupies the peak at $\varepsilon \simeq T_K$ described above. As in the single f electron spectrum, discussed above, this region is not significantly thermally excited at low T and all contributions come from $\varepsilon \lesssim T$ or negative, where the second term is a factor of $1/\tilde{N}$ smaller than would be expected from the naive power counting. Thus the contribution of this term is indeed a $1/N$ correction to the thermodynamics, when expanded in powers of T (or magnetic field h) as in Ref. 11 (to be referred to as II). However, the first term in (1.27) is only of order $1/N$, which would appear to be a contribution of the same order as mean field theory. This part was omitted in II. In this paper, one of our major results is the proper treatment of this and related terms.

The δ-function term in (1.27) obviously arise from the vanishing of the denominator in (1.26), which is a consequence of the Goldstone-like nature of the fluctuations transverse to $\langle \sigma \rangle$ (angular fluctuations[11]). To see how these terms which we call "pole terms" contribute, let us examine the fluctuation correction to (1.24a) where they first arise. This equation is obtained by varying ε_f in the effective action and expresses the constraint

$$\langle Q \rangle = 1$$

which we rewrite as

$$\frac{|\langle \sigma \rangle|^2}{V^2} = 1 - n_f - \frac{1}{V^2} \langle (\sigma - \langle \sigma \rangle)(\sigma^* - \langle \sigma^* \rangle) \rangle . \tag{1.28}$$

We need not examine here the fluctuation corrections to n_f in (1.28) (we do not explicitly distinguish n_f from $\langle n_f \rangle$). At the Gaussian level, the last term in (1.28) is just the sum over frequency of the propagator for the fluctuations in σ, σ^* about their expectation values, given at low temperatures in (1.26).

The $\omega = 0$ term in the Matsubara sum over frequency in (1.26) is obviously meaningless. Indeed, since the $\omega = 0$ mode is a Goldstone-like mode, symmetry demands that it not appear in the effective action to any order in $1/N$, and so this divergence always occurs. The solution is to factor out this zero mode from the functional integral, and do the integral over the angular part of σ at zero frequency exactly, giving a constant. This we do in the next section. The sum over frequency then contains no $\omega = 0$ piece, except for the radial fluctuations in σ, which we handle separately. The last term in (1.28) is then of the form

$$-\frac{1}{V^2} \langle (\sigma - \langle \sigma \rangle)(\sigma^* - \langle \sigma^* \rangle) \rangle = \frac{1}{\beta} \sum_{\omega_\nu \neq 0} \frac{1}{i\omega_\nu \left[1+\dfrac{N\Delta_0}{\pi\varepsilon_f} \right] + i\omega_\nu \dfrac{N\Delta_0}{\pi\varepsilon_f} i\Delta \operatorname{sgn}\omega_\nu} . \tag{1.29}$$

Converting the sum to a contour integral in the usual way we obtain, after bending the contour to surround the real axis,

$$\frac{1}{2\pi i} \left[\int_{-\infty+i\delta}^{\infty+i\delta} d\varepsilon - \int_{-\infty-i\delta}^{\infty-i\delta} d\varepsilon \right] b(\varepsilon) \frac{1}{\varepsilon \left[1+\dfrac{N\Delta_0}{\pi\varepsilon_f} \right] + \dfrac{N\Delta_0}{\pi} i\Delta\varepsilon \operatorname{sgn} \operatorname{Im}\varepsilon}$$
$$= \frac{1}{2\left[1+\dfrac{N\Delta_0}{\pi\varepsilon_f}\right]} + \frac{N\Delta_0}{2\pi\varepsilon_f} \frac{1}{\left[1+\dfrac{N\Delta_0}{\pi\varepsilon_f}\right]^2} \frac{T}{\varepsilon_f} + (\text{branch-cut part}) . \tag{1.30}$$

The last step requires careful consideration of the double pole inside the contour at $\varepsilon=0$. The second term in (1.30) is half cancelled by including the fluctuations in the radial part of σ at zero frequency, leaving

$$\frac{|\langle\sigma\rangle|}{V^2}=1-n_f+(\text{branch-cut term})+\frac{1}{2\left[1+\frac{N\Delta_0}{\pi\varepsilon_f}\right]}+\frac{N\Delta_0}{4\pi\varepsilon_f}\frac{T/\varepsilon_f}{\left[1+\frac{N\Delta_0}{\pi\varepsilon_f}\right]^2}\,. \tag{1.31}$$

(Here the two terms have been evaluated to lowest nontrivial order in T only; in general the coefficients of T^0 and T have temperature dependence in powers of T^2). The branch cut term is exactly that given in II, the other two correction terms are new. They are of the same order in $1/N$ as the mean-field terms (indeed the first of them is $\frac{1}{2}$ the mean field value of $1-n_f$).

The form of (1.31) suggests that the extra terms can be regarded simply as a renormalization of Δ/Δ_0. This would present no difficulty, as $|\langle\sigma\rangle|^2$ is not directly observable, since it is not gauge invariant.[11,24] To make this interpretation of (1.31) work, we must show that the other self-consistent equation, which does not involve Δ to $O(1/N)$,[11] has no pole term corrections, and that in all other physical results, pole terms appear wherever Δ appears, and so can be absorbed in the manner of (1.31). This is in fact precisely what happens, as we show in the following. In general, one finds various terms, many of which are cancelled when one includes also the (zero-frequency) fluctuations of λ, which were also omitted in II. The terms left over are cancelled when one uses the renormalization of the value of $|\langle\sigma\rangle|^2$ implied by (1.30). For the free energy at one loop order, we obtain the same result as II, a correction which is of order $(1/N)$, at two-loop order the correction to the free energy is at least of order $(1/N)^2$. Thus though we do not have a general proof we have established to two loop order that a $(1/N)$ expansion *does* exist for the coefficients of the Sommerfeld expansion, with terms containing r boson loops contributing to order N^{-r} at largest. If by extension it is assumed that all pole terms and λ fluctuations can be absorbed into $|\langle\sigma\rangle|^2$, or if these terms are ignored throughout, this result can be established in general.[28]

Finally, we summarize the layout of the remainder of this paper. In Sec. II, the thermodynamics at low T is rederived, including elimination of the zero mode and calculation of pole terms and λ fluctuations. Some new results at finite temperature are given. In Sec. III, the F spectral function is calculated at small ε and T; no new technical problems arise, other than the need to include the "anomalous" boson propagator $\langle\sigma\sigma\rangle$. Section IV then calculates the low-temperature transport properties to order $1/N$. The final section summarizes the results, and three appendices contain further details of the calculations.

II. THERMODYNAMICS AT LOW T

Following Read (II), the partition function is written as a functional integral

$$Z=\int_{-\pi/\beta}^{\pi/\beta}d\lambda\frac{\beta}{2\pi}\int\!\!\!\int Db\,Db^\dagger Df\,Df^\dagger Dc\,Dc^\dagger\exp\left[-\int_0^\beta L(\tau)d\tau\right], \tag{2.1}$$

where

$$L(\tau)=b^\dagger\frac{d}{d\tau}b+\sum_m f_m^\dagger\left[\frac{\partial}{\partial\tau}+E_0\right]f_m+\sum_{k,m}c_{km}^\dagger\left[\frac{\partial}{\partial\tau}+\varepsilon_k\right]c_{km}+V\sum_{k,m}(c_{km}^\dagger f_m b^\dagger+f_m^\dagger c_{km}b)+i\lambda(n_f+b^\dagger b-1)\,. \tag{2.2}$$

The integral over λ enforces the constraint $Q=n_f+b^\dagger b=1$ and so the model is equivalent to the $U=\infty$ Anderson model. The Fermi fields can now be integrated out to give

$$Z=Z_0\int_{-\pi/\beta}^{\pi/\beta}d\lambda\frac{\beta}{2\pi}\int\!\!\!\int D\sigma\,D\sigma^*\exp(-S)\,, \tag{2.3}$$

where the action S is a functional of the boson fields (σ,λ) only:

$$S[\sigma(\tau),\sigma^*(\tau),\lambda]=-N\,\mathrm{Tr}\ln\left[\frac{\partial}{\partial\tau}+E_0+i\lambda_0-\sigma\sum_k\left[\frac{\partial}{\partial\tau}+\varepsilon_k\right]^{-1}\sigma^*\right]+\int_0^\beta d\tau\frac{1}{V^2}\left[\sigma^*\left[\frac{\partial}{\partial\tau}+i\lambda\right]\sigma\right]-\int_0^\beta d\tau(i\lambda)\,, \tag{2.4}$$

Here Z_0 is the partition function of the noninteracting conduction band and the fields $\sigma=Vb$ have been introduced for convenience.

The functional integral is evaluated by first determining the saddle point of the action for τ independent $\sigma,\sigma^*=s_0$ and $\lambda=\lambda_0$ and then expanding in the fluctuations about these stationary values:

$$\begin{aligned}\sigma(\tau)-s_0&=\frac{1}{\beta}\sum_{\omega_\nu}\tilde\sigma(i\omega_\nu)e^{-i\omega_\nu\tau}\,,\\ \sigma^*(\tau)-s_0&=\frac{1}{\beta}\sum_{\omega_\nu}\tilde\sigma^*(i\omega_\nu)e^{-i\omega_\nu\tau}\,,\\ \lambda-\lambda_0&=\tilde\lambda/\beta\,.\end{aligned} \tag{2.5}$$

Our strategy will be to perform a loop expansion of (2.3) and (2.4) in terms of these fluctuations and then to show that to two loop order the fluctuation corrections to physical quantities at low T are $O(1/N)$ compared with the mean field results. To this order our expressions for physical quantities are the same as those first derived in II.

To proceed we notice that S is invariant under uniform rotations of the σ fields, $\sigma \to e^{i\phi}\sigma$, which in turn means that it is independent of the phase θ of $\sigma(\omega_\nu=0)$. Consequently one cannot do Gaussian integrals over θ [or $\sigma(\omega_\nu=0)-\sigma^*(\omega_\nu=0)$] since the propagator for this mode would be infinite. The correct procedure is to integrate out this variable and then carry out the saddle-point analysis. This is achieved by separating the zero-frequency and (nonzero) finite-frequency parts of the functional integral and then in the zero-frequency sector transforming to polar coordinates $\sigma(\omega_\nu=0)=se^{i\theta}$, in so doing, the integral over the phase θ may be carried out exactly. The result is

$$Z=Z_0 e^{-S(\lambda_0,s_0)-\delta S}\,, \tag{2.6}$$

here

$$S(\lambda_0,s_0)=-N\,\mathrm{Tr}\ln\left[\frac{\partial}{\partial\tau}+E_0+i\lambda_0-s_0\sum_k\left[\frac{\partial}{\partial\tau}+\varepsilon_k\right]^{-1}s_0\right]+i\frac{\beta}{V^2}s_0^2\lambda_0-\beta i\lambda_0 \tag{2.7}$$

and

$$\begin{aligned}
e^{-\delta S}=\int_{-\pi}^{\pi}d\tilde\lambda\int ds\frac{(s_0+s)}{\pi V^2\beta^2}\prod_{\omega_\nu(\neq 0)}\int d\tilde\sigma^*\frac{(i\omega_\nu)d\tilde\sigma(i\omega_\nu)}{2\pi i\beta^2V^2}&\exp-i\lambda\beta\left[\frac{N}{\beta}\sum_n g_f(i\omega_n)+\frac{s_0^2}{V^2}-1\right]\\
&\times\exp\frac{-2s_0s}{V^2}\left[i\lambda_0+\frac{NV^2}{\beta}\sum_n g_f(i\omega_n)g_0(i\omega_n)\right]\\
&\times\exp-\frac{1}{2\beta}\sum_{\omega_\nu(\neq 0)}[\tilde\sigma^*(-i\omega_\nu),(-i\omega_\nu)]\underline{D}^{-1}(i\omega_\nu)\begin{bmatrix}\tilde\sigma(i\omega_\nu)\\ \tilde\sigma^*(i\omega_\nu)\end{bmatrix}\\
&\times\exp\left\{-\frac{1}{2\beta}\left[(\tilde\lambda,s)\underline{D}^{-1}(0)\begin{bmatrix}\tilde\lambda\\ s\end{bmatrix}\right]\right\}e^{\text{higher-order terms}}\,.
\end{aligned} \tag{2.8}$$

Here the normalization of the functional integral has, for the first time, been shown explicitly. In (2.8) the f electron propagator is given by

$$g_f(i\omega_n)=\left[i\omega_n-E_0-i\lambda_0-s_0\sum_k(i\omega_n-\varepsilon_k)^{-1}s_0\right]^{-1}, \tag{2.9}$$

the conduction electron propagator by

$$g_0(i\omega_n)=\sum_k(i\omega_n-\varepsilon_k)^{-1}\cong -i\pi\rho\,\mathrm{sgn}\omega_n\,, \tag{2.10}$$

and the inverse boson propagators by, at finite frequency,

$$\underline{D}^{-1}(\omega_\nu)=\begin{bmatrix}\frac{1}{V^2}(-i\omega_\nu+i\lambda_0)+\Gamma_0(i\omega_\nu)+\Gamma_1(i\omega_\nu) & \Gamma_2(i\omega_\nu)\\ \Gamma_2(i\omega_\nu) & \frac{1}{V^2}(i\omega_\nu+i\lambda_0)+\Gamma_0(-i\omega_\nu)+\Gamma_1(-i\omega_\nu)\end{bmatrix}, \tag{2.11}$$

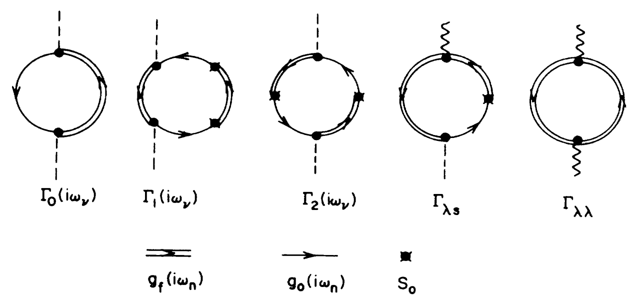

FIG. 1. Self-energies Γ.

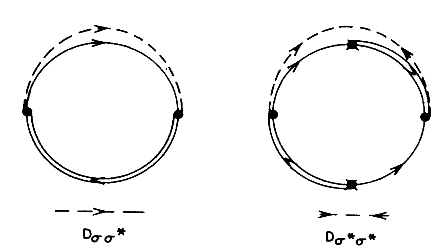

FIG. 2. Diagrams contributing to the action at one-loop order.

and at zero frequency

$$\underline{D}^{-1}(0)=\begin{bmatrix}\Gamma_{\bar{\lambda}\bar{\lambda}} & 2i\Gamma_{\bar{\lambda}_s}\\ 2i\Gamma_{\bar{\lambda}_s} & 2\Gamma_{ss}\end{bmatrix}. \tag{2.12}$$

The self-energies Γ which arise from the fermion loops shown in Fig. 1, see also I, are given explicitly in Appendix A.

Differentiating the action $S(s_0,\lambda_0)$ with respect to λ_0 and s_0 we find the mean field equations of state (stationary point conditions)

$$\frac{\varepsilon_f(T)-E_0}{V^2}+\frac{N}{\beta}\sum_{\omega_n}g_f(i\omega_n)g_0(i\omega_n)=0\ . \tag{2.13}$$

and

$$\frac{N}{\beta}\sum_{n}g_f(i\omega_n)+\frac{\Delta}{\Delta_0}-1=0 \tag{2.14}$$

here $\Delta=\pi\rho s_0^2$, $\Delta_0=\pi\rho V^2$, ρ is the constant density of states in the conduction band, and we have defined a renormalized f level energy by $\varepsilon_f(T)=E_0+i\lambda_0$. The free energy is given by

$$F=E_0-\varepsilon_f(T)+\frac{\Delta}{\Delta_0}\eta^2(T)\ , \tag{2.15}$$

where

$$\eta^2(T)=\varepsilon_f(T)-E_0+\frac{NV^2}{\beta}\sum_{n}g_f(i\omega_n)g_0(i\omega_n) \tag{2.16}$$

is equal to zero at this level of approximation because of the stationary point condition (2.13). Evaluating the Matsubara sum in (2.13) and keeping terms to order T^2, we find in the large-N limit, $N\to\infty$ with $N\Delta_0$ fixed,

$$\varepsilon_f(T)-E_0-\frac{N\Delta_0}{\pi}\ln\left[\frac{\varepsilon_f}{D}\right]-\frac{N\Delta_0}{\pi}\frac{\pi^2}{6}(k_BT)^2=0 \tag{2.17}$$

in the notation of II, $\varepsilon_f(0)=T_A$ and therefore

$$\varepsilon_f(T)=T_A\left[1+\frac{\mu}{1+\mu}\frac{\pi^2}{6}\left[\frac{T}{T_A}\right]^2\right], \tag{2.18}$$

where

$$\mu=\frac{N\Delta_0}{\pi T_A}\ . \tag{2.19}$$

The f level occupation can be found on using the relation

$$n_f=\frac{dF}{dE_0}\ , \tag{2.20}$$

hence

$$n_f=1-\frac{\partial\varepsilon_f(T)}{\partial E_0}=1-\frac{\Delta(T)}{\Delta_0}\ , \tag{2.21}$$

where $\Delta(T)$ is the solution of the equation of state (2.14)

$$\frac{\Delta(T)}{\Delta_0}=\frac{1}{1+\mu}\left[1+\left[\frac{1}{(1+\mu)^2}-1\right]\frac{\pi^2}{6}\left[\frac{T}{T_A}\right]^2\right]. \tag{2.22}$$

We will now demonstrate explicitly that the corrections to F at one loop order are in fact at most of order $(1/N)$, and at two-loop order at most of order $(1/N)^2$ as outlined in the Introduction. The one-loop contribution to the action is determined by computing the functional integral over the Gaussian fluctuations shown in (2.8), the corresponding Feynman diagrams are shown in Fig. 2. The contribution to S from the *finite frequency* fluctuations to order $(1/N)$, for details see Appendix B, is given by the diagram involving the boson propagation $D_{\sigma\sigma^*}$ which to this level of approximation is written

$$D_{\sigma\sigma^*}(i\omega)=\frac{V^2}{i\omega-(\varepsilon_f-E_0)-V^2\Gamma_0(i\omega,T)}\ . \tag{2.23}$$

The propagator $D_{\sigma\sigma^*}$ is explicitly temperature dependent through its self-energy $\Gamma_0(i\omega,T)$. (See Appendix A). The diagram involving the anomalous propagator

$$D_{\sigma\sigma}(i\omega)=D_{\sigma^*\sigma^*}(i\omega)=\frac{V^4\Gamma_2(i\omega)}{[-i\omega+(\varepsilon_f-E_0)+V^2\Gamma_0(i\omega,T)][i\omega+(\varepsilon_f-E_0)+V^2\Gamma_0(-i\omega,T)]} \tag{2.24}$$

only contributes to the action at order $1/N^2$ at low T. We find

$$\begin{aligned}\delta S^{(1)}_{\omega_\nu\neq 0}=-\beta a\Bigg[&\frac{1}{N}\frac{\Delta}{\Delta_0}\left[\frac{N\Delta_0}{\pi}\right]^2\int_{-D}^{D}d\varepsilon\, b(\varepsilon)\frac{\left[\dfrac{1}{\varepsilon-\varepsilon_f}+\dfrac{1}{\varepsilon_f}\right]+\dfrac{\pi^2}{3}(k_BT)^2\left[\dfrac{1}{\varepsilon_f^3}+\dfrac{1}{(\varepsilon-\varepsilon_f)^3}\right]}{\varepsilon-\eta^2-\dfrac{N\Delta_0}{\pi}\ln\left|\dfrac{\varepsilon-\varepsilon_f}{-\varepsilon_f}\right|-\dfrac{N\Delta_0}{\pi}\dfrac{\pi^2}{6}(k_BT)^2\left[\dfrac{1}{\varepsilon_f^2}-\dfrac{1}{(\varepsilon-\varepsilon_f)^2}\right]}\\ &+\frac{1}{\beta}\ln\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}+\frac{1}{2\beta}\frac{\dfrac{N\Delta_0}{\pi\varepsilon_f}\left[1+\pi^2\left[\dfrac{k_BT}{\varepsilon_f}\right]^2\right]}{\left\{1+\dfrac{N\Delta_0}{\pi\varepsilon_f}\left[1+\dfrac{\pi^2}{3}\left[\dfrac{k_BT}{\varepsilon_f}\right]^2\right]\right\}^2}\frac{\eta^2}{\varepsilon_f}\\ &+\frac{1}{2}\frac{1}{1+\dfrac{N\Delta_0}{\pi\varepsilon_f}\left[1+\dfrac{\pi^2}{3}\left[\dfrac{k_BT}{\varepsilon_f}\right]^2\right]}\eta^2+O\left[\frac{1}{N}\right]^2\Bigg]\ ,\end{aligned} \tag{2.25}$$

here $b(\varepsilon)$ is the Bose-Einstein distribution function and the parameter "a" counts the number of boson loops, in this case one, the correction to the action at two loops is of order a^2 etc. At the end of the calculation "a" is set equal to one. The first term in (2.25) which was given, at $T=0$, in II is of order $(1/N)$, however the remaining terms which stem from the fact that $D_{\sigma\sigma^*}$ has a pole at zero frequency contain terms that are not only $O(1)$ but would also give contributions to the free energy linear in T. These terms were omitted in II.

The contribution from the zero-frequency sector to this order is given by

$$S^{(1)}_{(\omega_\nu=0)}=a\left[\ln\Gamma_{\lambda s}+\frac{1}{4}\frac{\Gamma_{\lambda\lambda}\Gamma_{ss}}{\Gamma_{\lambda s}^2}-\ln\left[\frac{s_0}{V^2}\right]\right], \tag{2.26}$$

the last term in (2.26) comes from the Jacobian of the transformation to polar coordinates [see (2.8)]. On combining (2.25) and (2.26) and making use of Appendix A, the logarithms cancel exactly and the total action to one loop order is given by

$$S=\beta\left[E_0-\varepsilon_f+\frac{\Delta}{\Delta_0}\eta^2\right]+\delta S^{(1)}, \tag{2.27}$$

where

$$\delta S^{(1)}=-\beta a\left[\frac{1}{N}\frac{\Delta}{\Delta_0}\left[\frac{N\Delta_0}{\pi}\right]^2\int_{-D}^{D}d\varepsilon\, b(\varepsilon)\frac{\left[\frac{1}{\varepsilon_f}+\frac{1}{\varepsilon-\varepsilon_f}\right]+\frac{\pi^2}{3}(k_BT)^2\left[\frac{1}{\varepsilon_f^3}+\frac{1}{(\varepsilon-\varepsilon_f)^3}\right]}{\varepsilon-\eta^2-\frac{N\Delta_0}{\pi}\ln\left|\frac{\varepsilon-\varepsilon_f}{-\varepsilon_f}\right|-\frac{\pi^2}{6}(k_BT)^2\frac{N\Delta_0}{\pi}\left[\frac{1}{\varepsilon_f^2}-\frac{1}{(\varepsilon-\varepsilon_f)^2}\right]}\right.$$
$$\left.+\frac{1}{4\beta}\frac{\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\pi^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}^2}\frac{\eta^2}{\varepsilon_f}+\frac{1}{2}\frac{1}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}}\eta^2+O((1/N)^2)\right]. \tag{2.28}$$

Differentiating with respect to Δ and ε_f gives the one-loop corrected equations of state

$$\varepsilon_f-E_0+\frac{N\Delta_0}{\pi}\ln\left[\frac{\varepsilon_f}{D}\right]-\frac{N\Delta_0}{\pi}\frac{\pi^2}{6}\left[\frac{k_BT}{\varepsilon_f}\right]^2$$
$$-\frac{a}{N}\left[\frac{N\Delta_0}{\pi}\right]^2\int_{-D}^{D}d\varepsilon\, b(\varepsilon)\frac{\left[\frac{1}{\varepsilon_f}+\frac{1}{\varepsilon-\varepsilon_f}\right]+\frac{\pi^2}{3}(k_BT)^2\left[\frac{1}{\varepsilon_f^3}+\frac{1}{(\varepsilon-\varepsilon_f)^3}\right]}{\varepsilon-\frac{N\Delta_0}{\pi}\ln\left|\frac{\varepsilon-\varepsilon_f}{-\varepsilon_f}\right|-\frac{\pi^2}{6}(k_BT)^2\frac{N\Delta_0}{\pi}\left[\frac{1}{\varepsilon_f^2}-\frac{1}{(\varepsilon-\varepsilon_f)^2}\right]}=0 \tag{2.29}$$

and

$$-1+\frac{\Delta}{\Delta_0}\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_k}\right]^2\right]\right\}$$
$$-a\left[\frac{1}{N}\frac{\Delta}{\Delta_0}\left[\frac{N\Delta_0}{\pi}\right]^2\frac{\partial}{\partial\varepsilon_f}\left[\int_{-D}^{D}d\varepsilon b(\varepsilon)\frac{\left[\frac{1}{\varepsilon_f}+\frac{1}{\varepsilon-\varepsilon_f}\right]+\frac{\pi^2}{3}(k_BT)^2\left[\frac{1}{\varepsilon_f^3}+\frac{1}{(\varepsilon-\varepsilon_f)^3}\right]}{\varepsilon-\eta^2-\frac{N\Delta_0}{\pi}\ln\left|\frac{\varepsilon-\varepsilon_f}{-\varepsilon_f}\right|-\frac{\pi^2}{6}(k_BT)^2\frac{N\Delta_0}{\pi}\left[\frac{1}{\varepsilon_f^2}-\frac{1}{(\varepsilon-\varepsilon_f)^2}\right]}\right]\right.$$
$$\left.+\frac{1}{2}+\frac{1}{4\beta}\frac{\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\pi^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}}\frac{1}{\varepsilon_f}\right]=0. \tag{2.30}$$

Modulo, the T dependence the equation for ε_f, which is independent of Δ, was first derived in II, the equation for Δ

differs from the corresponding result of II in the appearance of a term, of $O(1)$ at order a. The free energy to order a is now obtained if in the mean-field action one-loop corrected equations of state are used, while the one-loop correction is evaluated at the mean-field level.

From (2.29), $\eta^2 \simeq O(a/N)$,

$$\eta^2 = aL(\varepsilon_f)\ , \tag{2.31}$$

where

$$L(\varepsilon_f) = \frac{1}{N}\left[\frac{N\Delta_0}{\pi}\right]^2 \int_{-D}^{D} d\varepsilon\, b(\varepsilon) \frac{\left[\dfrac{1}{\varepsilon-\varepsilon_f}+\dfrac{1}{\varepsilon_f}\right]+\dfrac{\pi^2}{3}(k_B T)^2\left[\dfrac{1}{\varepsilon_f^3}+\dfrac{1}{(\varepsilon-\varepsilon_f)^3}\right]}{\varepsilon-\dfrac{N\Delta_0}{\pi}\ln\left|\dfrac{\varepsilon-\varepsilon_f}{-\varepsilon_f}\right|-\dfrac{N\Delta_0}{\pi}\dfrac{\pi^2}{6}(k_B T)^2\left[\dfrac{1}{\varepsilon_f^2}-\dfrac{1}{(\varepsilon-\varepsilon_f)^2}\right]} \tag{2.32}$$

and therefore given the form of the action it suffices to compute (Δ/Δ_0) at the mean field level. For $\delta S^{(1)}$ at the one-loop level we can set $\eta^2=0$, hence,

$$\delta S^{(1)} = a\beta\frac{\Delta}{\Delta_0}L(\varepsilon_f)\ , \tag{2.33}$$

which exactly cancels the term proportional to η^2 in (2.27). The free energy is therefore given by

$$F = E_0 - \varepsilon_f(T)\ , \tag{2.34}$$

where ε_f is the solution of (2.29). The result obtained in I: After some algebra ε_f may be written explicitly as

$$\varepsilon_f(T) = T_A\left[1+\frac{a}{N}\frac{\mu^2}{1+\mu}L_1 + \frac{\mu}{1+\mu}\frac{\pi^2}{6}\left[\frac{T}{T_A}\right]^2\left[1+\frac{a}{N}\delta\right]\right]\ , \tag{2.35}$$

where

$$\delta = \frac{\mu^2}{1+\mu}\left[2\frac{(1+\mu)}{\mu}L_3 - L_2 - (1+\mu)K_{12} + \mu K_{11} - \left[2-\frac{\mu}{1+\mu}\right]L_1\right] + \left[\frac{1}{(1+\mu)^2}-1\right]\ ; \tag{2.36}$$

here

$$L_k = \int_0^{D/T_A} dx \frac{\left[1-\dfrac{1}{(1+x)^k}\right]}{x+\mu\ln(1+x)} \tag{2.37}$$

and

$$K_{pq} = \int_0^{D/T_A} dx \frac{\left[1-\dfrac{1}{(1+x)^p}\right]\left[1-\dfrac{1}{(1+x)^q}\right]}{[x+\mu\ln(1+x)]^2}\ . \tag{2.38}$$

Putting $a=1$ in (2.35) and differentiating the free energy (2.34) with respect to T we obtain for γ the coefficient of the linear term in the specific heat,

$$\gamma = \frac{\pi^2}{3}k_B^2\frac{\mu}{1+\mu}\frac{T}{T_A}\left\{1+\frac{\mu^2}{N(1+\mu)}\left[\frac{2(1+\mu)}{\mu}L_3 - L_2 - (1+\mu)K_{12}+\mu K_{11} - \left[2-\frac{\mu}{1+\mu}\right]L_1\right] + \frac{1}{N}\left[\frac{1}{(1+\mu)^2}-1\right]\right\}\ , \tag{2.39}$$

the result first derived in II.

Using (2.20), $1-n_f(T) = \partial\varepsilon_f/\partial E_0$, the equation of state (2.30) can be rewritten as

$$\frac{\Delta}{\Delta_0} = [1-n_f(T)]\left\{1+\frac{a}{N}\left[\frac{N\Delta_0}{\pi}\right]^2\int_{-D}^{D} d\varepsilon\, b(\varepsilon)\frac{\left[\dfrac{1}{\varepsilon_f}+\dfrac{1}{\varepsilon-\varepsilon_f}\right]+\dfrac{\pi^2}{3}(k_B T)^2\left[\dfrac{1}{\varepsilon_f^3}+\dfrac{1}{(\varepsilon-\varepsilon_f)^3}\right]}{\left[\varepsilon-\dfrac{N\Delta_0}{\pi}\ln\left|\dfrac{\varepsilon-\varepsilon_f}{-\varepsilon_f}\right|-\dfrac{N\Delta_0}{\pi}\dfrac{\pi^2}{6}(k_B T)^2\left[\dfrac{1}{\varepsilon_f^2}-\dfrac{1}{(\varepsilon-\varepsilon_f)^2}\right]\right]^2}\right.$$

$$\left. + a\left[\frac{1}{4\beta}\frac{\dfrac{N\Delta_0}{\pi\varepsilon_f}\left[1+\pi^2\left[\dfrac{k_B T}{\varepsilon_f}\right]^2\right]}{\left\{1+\dfrac{N\Delta_0}{\pi}\left[1+\dfrac{\pi^2}{3}\left[\dfrac{k_B T}{\varepsilon_f}\right]^2\right]\right\}^2}\frac{1}{\varepsilon_f}+\frac{1}{2}\frac{1}{1+\dfrac{N\Delta_0}{\pi}\left[1+\dfrac{\pi^2}{3}\left[\dfrac{k_B T}{\varepsilon_f}\right]^2\right]}\right]\right. \tag{2.40}$$

the result quoted in the Introduction, (1.31). The f level occupancy n_f is given explicitly by

$$n_f=\frac{\mu}{(1+\mu)}+\frac{\pi^2}{6}\left[\frac{T}{T_A}\right]^2\frac{\mu(2+\mu)}{(1+\mu)^3}$$
$$+\frac{a}{N}\left[\frac{\mu^2}{(1+\mu)^2}\left(L_2-\mu K_{11}-\frac{\mu}{1+\mu}L_1\right)\right.$$
$$-\frac{\pi^2}{6}\left[\frac{T}{T_A}\right]^2\left\{\frac{2\mu^3(2+\mu)}{(1+\mu)^4}\left(L_2-\mu K_{11}-\frac{\mu}{1+\mu}L_1\right)+\frac{\mu^2}{(1+\mu)^3}\left[\delta-\left[\frac{2\mu^2}{1+\mu}-6\mu\right]L_1\right]\right.$$
$$+\frac{\mu^3}{(1+\mu)^3}(2L_3-3\mu K_{12}+2\mu^2M_{111})+\frac{\mu^3}{(1+\mu)^2}(4K_{13}+K_{22}-2\mu M_{112})$$
$$\left.\left.-\frac{6\mu^2}{(1+\mu)^2}L_4+\frac{2\mu^2}{(1+\mu)^3}\left[\left[3-\frac{\mu}{1+\mu}\right]-\frac{\mu}{1+\mu}\left[2-\frac{\mu}{1+\mu}\right]\right]\right\}\right], \tag{2.41}$$

where

$$M_{ijk}=\int_0^{D/T_A}dx\frac{\left[1-\frac{1}{(1+x)^i}\right]\left[1-\frac{1}{(1+k)^j}\right]\left[1-\frac{1}{(1+x)^k}\right]}{[x+\mu\ln(1+x)]^3}\ . \tag{2.42}$$

Equation (2.38), which contains terms which diverge in the infrared (we will return to this point later), reduces to the result quoted in II with one important difference: the last term, a one loop correction, which is of $O(1)$. On the other hand, the physical quantity n_f is well defined and reduces in the zero temperature limit to that obtained in II.

The results (2.34) and (2.41) have so far been shown to be correct only to one-loop order. They contain $1/N$ corrections, and in principle higher-order terms in the loop expansion could contribute terms of $O(1)$ or $O(1/N)$ to F or n_f. We now show that at two-loop order such terms cancel either among themselves or when the one-loop corrected equations of state are used and hence do not contribute to physical observables such as F or n_f which are correctly given to this order by (2.34) and (2.41).

At two-loop order the contribution to the action from the finite frequency sector is given by the diagrams shown in Fig. 3, some of which contain zero frequency propagators. Diagrams involving the propagator $D_{\lambda\lambda}$ can be discarded at this order, because as

$$D_{\tilde{\lambda}\tilde{\lambda}}=\frac{1}{2\beta}\frac{\Delta_0}{\Delta}\frac{1}{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]}\eta^2 \tag{2.43}$$

and $\eta^2=O(a)$, from (2.29), they will only contribute to the free energy at order a^3, three loops. Pole term contributions proportional to η^2 can be ignored for the same reason. On neglecting these terms we find

$$\delta S^{(2)}_{(\omega_\nu\neq 0)}=a^2\left[\left[\frac{1}{4\varepsilon_f}\frac{N\Delta_0}{\pi\varepsilon_f}\frac{\left[1+\pi^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}^2}+\frac{1}{2}\frac{\beta}{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]}\right]\right.$$
$$\left.\times\left[\frac{N\Delta_0}{\pi}\right]^2\frac{1}{N}\int_{-D}^{D}d\varepsilon\, b(\varepsilon)\frac{\left[\frac{1}{\varepsilon_f}+\frac{1}{\varepsilon-\varepsilon_f}\right]+\frac{\pi^2}{3}(k_BT)^2\left[\frac{1}{\varepsilon_f^3}+\frac{1}{(\varepsilon-\varepsilon_f)^3}\right]}{\varepsilon-\frac{N\Delta_0}{\pi}\ln\left|\frac{\varepsilon-\varepsilon_f}{-\varepsilon_f}\right|-\frac{\pi^2}{6}(k_BT)^2\left[\frac{1}{\varepsilon_f^2}-\frac{1}{(\varepsilon-\varepsilon_f)^2}\right]}+O((1/N)^2)\right]\ . \tag{2.44}$$

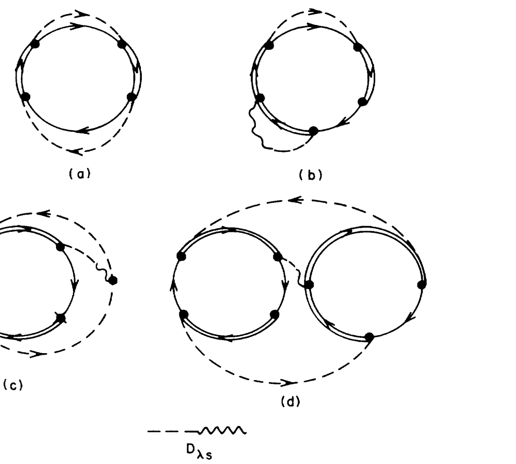

FIG. 3. Diagrams in the finite frequency sector which contribute to the action at two-loop order.

This contribution to the action contains terms of $O(1/N)$ in addition to the expected terms $O((1/N)^2)$. The contribution from the zero-frequency sector, again discarding terms proportional to $D_{\tilde{\lambda}\tilde{\lambda}}$ is given by the diagrams of Fig. 4:

$$\delta S^{(2)}_{(\omega_\nu=0)} = \frac{-a^2}{8}\frac{1}{\beta\varepsilon_f}\frac{\Delta_0}{\Delta} \times \frac{\dfrac{N\Delta_0}{\pi\varepsilon_f}\left[1+\pi^2\left[\dfrac{k_BT}{\varepsilon_f}\right]^2\right]}{\left\{1+\dfrac{N\Delta_0}{\pi\varepsilon_f}\left[1+\dfrac{\pi^2}{3}\left[\dfrac{k_BT}{\varepsilon_f}\right]^2\right]\right\}^2}, \tag{2.45}$$

which is $O(1)$; but this is exactly cancelled by the contribution of the Jacobian, which is expanded in powers of s and contracted using D_{ss}:

$$\delta S_J^{(2)} = -\frac{1}{2}\frac{1}{\beta^2 s_0^2}D_{ss} = \frac{1}{8}\frac{1}{\beta\varepsilon_f}\frac{\Delta_0}{\Delta}\frac{\dfrac{N\Delta_0}{\pi\varepsilon_f}\left[1+\pi^2\left[\dfrac{k_BT}{\varepsilon_f}\right]^2\right]}{\left\{1+\dfrac{N\Delta_0}{\pi\varepsilon_f}\left[1+\dfrac{\pi^2}{3}\left[\dfrac{k_BT}{\varepsilon_f}\right]^2\right]\right\}^2}. \tag{2.46}$$

Thus, the correction to the action at the two-loop order modulo terms of order $(1/N)^2$ is given by

$$\delta S^{(2)} = \beta a^2\left[\frac{1}{4\beta\varepsilon_f}\frac{\dfrac{N\Delta_0}{\pi\varepsilon_f}\left[1+\pi^2\left[\dfrac{k_BT}{\varepsilon_f}\right]^2\right]}{\left\{1+\dfrac{N\Delta_0}{\pi\varepsilon_f}\left[1+\dfrac{\pi^2}{3}\left[\dfrac{k_BT}{\varepsilon_f}\right]^2\right]\right\}^2} + \frac{1}{2}\frac{1}{\left\{1+\dfrac{N\Delta_0}{\pi\varepsilon_f}\left[1+\dfrac{\pi^2}{3}\left[\dfrac{k_BT}{\varepsilon_f}\right]^2\right]\right\}}\right]L(\varepsilon_f), \tag{2.47}$$

where $L(\varepsilon_f)$ was defined by (2.32). Notice that $\delta S^{(2)}$ is independent of Δ and therefore does not contribute to ε_f, for this stationary point Eq. (2.29) the naive power counting is valid, i.e., terms with r boson loops are automatically of order N^{-r} at most and hence we have $\eta^2=O(1/N)$. Furthermore, as we only need (Δ/Δ_0) to one-loop order the contribution of $\delta S^{(2)}$ to the stationary point conditions can be ignored. Returning to the free energy we see that when $\delta S^{(1)}$ (2.28) is corrected to order a^2 the terms proportional to η^2 exactly cancel $\delta S^{(2)}$, the correction to $\delta S^{(1)}$ on correcting (Δ/Δ_0) to one-loop order is cancelled by the correction to (Δ/Δ_0) in the mean-field term, giving

$$F=E_0-\varepsilon_f(T) \tag{2.48}$$

to two loops and $(1/N)$. This then confirms that the results (2.34) and (2.41) are correct to two-loop order as claimed and that for physical observables such as F and n_f naive power counting is valid. The results of II are retained apart from changes in Δ/Δ_0 which is unobservable.

If we now consider the b (or σ) correlation function

$$\langle \mathcal{T}\sigma(\tau)\sigma^*(0)\rangle , \tag{2.49}$$

which at lowest order $\sim |\langle b\rangle|^2 V^2$. Whereas explained above $|\langle b\rangle|^2$ at one loop contains terms of $O(1)$ as well as infrared divergencies. These terms, however, are required to cancel infrared divergencies and $O(1)$ zero frequency and pole terms which appear in the one loop correction to the correlation function. The result to $O(1/N)$ is, as $\tau\to\infty$,

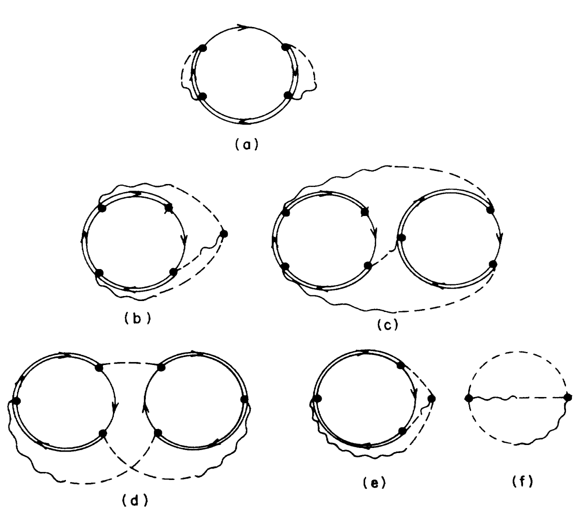

FIG. 4. Diagrams in the zero frequency sector which contribute to the action at two-loop order.

$$\langle \mathscr{T}\sigma(\tau)\sigma^*(0)\rangle \sim V^2(1-n_f)\left[1-\frac{n_f^2}{N}\ln T_A|\tau|\right]; \tag{2.50}$$

assuming the series exponentiates we have

$$\langle \mathscr{T}\sigma(\tau)\sigma^*(0)\rangle \sim (1-n_f)V^2(T_A|\tau|)^{-\alpha}, \tag{2.51}$$

where $\alpha=n_f^2/N$, the result found in II. In summary the additional terms in (Δ/Δ_0) are required to cancel order 1 corrections to the $(\sigma\sigma^*)$ propagator which as a result has power-law decay at large times as stated in II. This in turn implies that fluctuations drive Δ/Δ_0 to zero restoring the symmetry.

FIG. 5. Self-energy at mean-field level.

III. CONDUCTION-ELECTRON LIFETIME

The conduction-electron self-energy is given by

$$\Sigma_c(\tau)=-V^2\langle T_\tau b^\dagger(\tau)f_m(\tau)f_m^\dagger(0)b(0)\rangle, \tag{3.1}$$

where the statistical average is taken with respect to the Lagrangian (2.2). Expressing the boson operators in terms of the fluctuations about their stationary point values this can be rewritten as

$$\Sigma_c(\tau)=-s_0^2\langle \mathscr{T}_\tau f_m(\tau)f_m^\dagger(0)\rangle+s_0\langle \mathscr{T}_\tau\tilde{\sigma}^\dagger(\tau)f_m(\tau)f_m^\dagger(0)\rangle+\langle \mathscr{T}_\tau f_m(\tau)f_m^\dagger(0)\tilde{\sigma}(0)\rangle+\langle \mathscr{T}_\tau\tilde{\sigma}^\dagger(\tau)f_m(\tau)f_m^\dagger(0)\tilde{\sigma}(0)\rangle . \tag{3.2}$$

The conduction electron scattering rate $R(\omega,T)$ is obtained on computing the Fourier transform $\Sigma_c(i\omega_n)$ and then after analytically continuing $\omega_n\to\omega+i\delta$ taking the imaginary part. At the mean-field level the only diagram contributing to the scattering rate is shown in Fig. 5:

$$R_0(\omega,T)=\frac{2}{\pi\rho}\frac{\Delta_0^2}{[\varepsilon_f(T)-\omega]^2}\left[\frac{\Delta(T)}{\Delta_0}\right]^2 \tag{3.3}$$

as we calculate to one-loop order $\varepsilon_f(T)$ and $\Delta(T)/\Delta_0$ are given by the one loop stationary point conditions. Equation (2.40) can be used to rewrite (3.3) as

$$\begin{aligned}
R_0(\omega,T)=\frac{2}{\pi\rho}\frac{\Delta_0^2}{[\varepsilon_f(T)-\omega]^2}\Bigg\{&[1-n_f(T)]^2\Bigg\{1-\frac{2a}{N}\Bigg[\frac{\left[\frac{N\Delta_0}{\pi\varepsilon_f}\right]^2\left[1+\pi^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}^2}\int_{-D}^{D}d\varepsilon\frac{b(\varepsilon)}{\varepsilon}\\
&+\left[\frac{N\Delta_0}{\pi}\right]^2\int_{-D}^{0}d\varepsilon\Bigg[\frac{\left[\frac{1}{\varepsilon_f}+\frac{1}{\varepsilon-\varepsilon_f}\right]+\frac{\pi^2}{3}(k_BT)^2\left[\frac{1}{\varepsilon_f^3}+\frac{1}{(\varepsilon-\varepsilon_f)^3}\right]}{\left[\varepsilon-\frac{N\Delta_0}{\pi}\ln\left|\frac{\varepsilon-\varepsilon_f}{-\varepsilon_f}\right|-\frac{N\Delta_0}{\pi}\frac{\pi^2}{6}\left[\frac{1}{\varepsilon_f^2}-\frac{1}{(\varepsilon-\varepsilon_f)^2}\right]\right]^2}\\
&\qquad+\frac{1}{\varepsilon}\frac{\frac{1}{\varepsilon_f^2}\left[1+\pi^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}^2}\Bigg]\\
&+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\frac{\mu^2}{(1+\mu)^2}\left[1-\frac{5\mu}{3(1+\mu)}+\frac{3}{4}\frac{\mu}{1+\mu}\right]^2\Bigg]\Bigg\}\\
&+a\frac{\Delta(T)}{\Delta_0}\Bigg[\frac{1}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}}+\frac{1}{2}\frac{\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\pi^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}^2}\Bigg]\Bigg\},
\end{aligned} \tag{3.4}$$

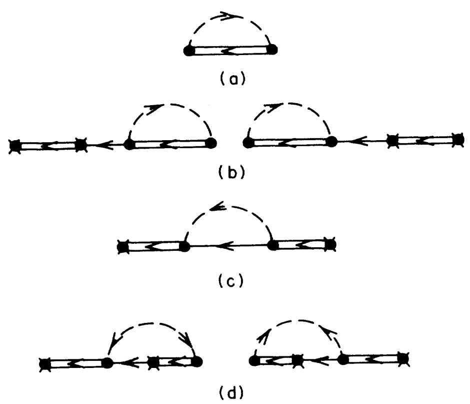

FIG. 6. Diagrams in the (nonzero) finite-frequency sector which contribute to the self-energy at one-loop order.

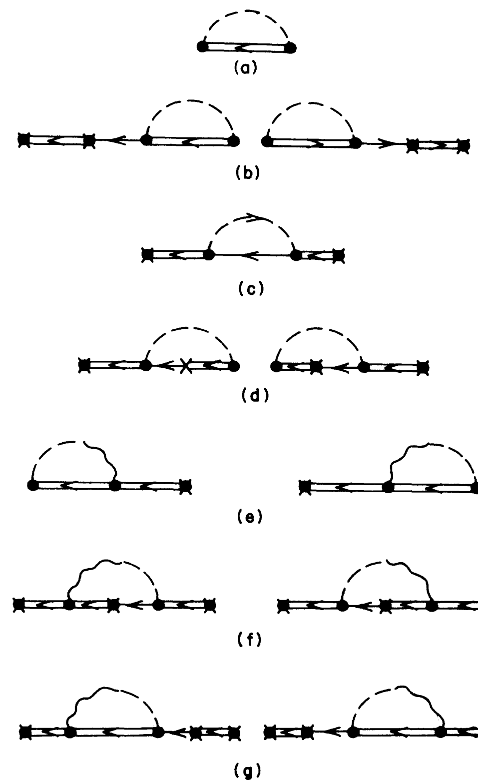

FIG. 7. Diagrams in the zero-frequency sector which contribute to the self-energy at one-loop order.

where $\varepsilon_f(T)$, $n_f(T)$, and $\Delta(T)$ are given by (2.35), (2.41), and (2.22), respectively. Equation (3.4) illustrates the difficulties inherent in the calculation, as the equation of state (2.40) contains terms which diverge in the infrared as well as one-loop corrections of $O(1)$ which originate from the pole of the boson propagator, as discussed in Sec. II, the equation for the relaxation rate $R_0(\omega,T)$ is fraught with similar difficulties.

To complete the calculation of the relaxation rate to one-loop order, as in Sec. II, we separate the contributions to the functional integral which must be computed in order to determine the statistical averages of (3.2) into a finite frequency sector and a zero-frequency sector. The Feynman diagrams contributing to the self-energy in the finite-frequency sector are given to one-loop order in Fig. 6. As an example, a detailed evaluation of the diagram of Fig. 6(a) is given in Appendix C where the contributions of all diagrams to the relaxation rate are listed. Each individual diagram is divergent in the infrared, however as we shall see this divergence exactly cancels the corresponding divergencies of (3.4) which originate in the stationary point condition, hence the relaxation rate is well defined. In addition each such one-loop diagram contains terms of $O(1)$ which originate from the pole of the boson propagator, we will find that these terms together with the "pole" term of (3.2) are exactly cancelled by contribution to the one-loop relaxation rate from the zero-frequency sector and hence the one-loop correction to the self-energy is of order $(1/N)$ with respect to the leading term. The diagrams contributing to the zero-frequency sector are shown in Fig. 7 and their contributions listed in Appendix C. Summarizing the information contained in this appendix we find

$$R^{(1)}_{\omega_\nu\neq 0}=-\frac{2a}{\pi\rho}\left[\frac{\Delta}{\Delta_0}\right]^2\left[-\frac{2\Delta_0^2}{(\varepsilon_f-\omega)^2}\left[\frac{N\Delta_0}{\pi\varepsilon_f}\right]^2\frac{1+\pi^2\left[\dfrac{k_BT}{\varepsilon_f}\right]^2}{\left\{1+\dfrac{N\Delta_0}{\pi\varepsilon_f}\left[1+\dfrac{\pi^2}{3}\left[\dfrac{k_BT}{\varepsilon_f}\right]^2\right]\right\}^2}\frac{1}{N}\int_{-D}^{D}d\varepsilon\,\frac{b(\varepsilon)}{\varepsilon}\right.$$

$$+\frac{2\Delta_0^2}{(\varepsilon_f-\omega)}\frac{1}{N}\left[\frac{N\Delta_0}{\pi}\right]^2\int_{-D}^{0}d\varepsilon\left\{\frac{\left[\left[\dfrac{1}{\varepsilon_f}+\dfrac{1}{\varepsilon-\varepsilon_f}\right]+\dfrac{\pi^2}{3}(k_BT)^2\left[\dfrac{1}{\varepsilon_f^3}+\dfrac{1}{(\varepsilon-\varepsilon_f)^3}\right]\right]\dfrac{1}{\varepsilon+\omega-\varepsilon_f}}{\left[\varepsilon-\dfrac{N\Delta_0}{\pi}\ln\left|\dfrac{\varepsilon-\varepsilon_f}{-\varepsilon_f}\right|-\dfrac{N\Delta_0}{\pi}\dfrac{\pi^2}{6}(k_BT)^2\left[\dfrac{1}{\varepsilon_f^2}-\dfrac{1}{(\varepsilon-\varepsilon_f)^2}\right]\right]^2}\right.$$

$$
\left.-\frac{1}{\varepsilon}\frac{\dfrac{1}{\varepsilon_f^2(\varepsilon_f-\omega)}\left[1+\pi^2\left(\dfrac{k_BT}{\varepsilon_f}\right)^2\right]}{\left\{1+\dfrac{N\Delta_0}{\pi\varepsilon_f}\left[1+\dfrac{\pi^2}{3}\left(\dfrac{k_BT}{\varepsilon_f}\right)^2\right]\right\}^2}\right]
$$

$$
+\frac{2\Delta_0^2}{\varepsilon_f-\omega}\frac{1}{N}\frac{N\Delta_0}{\pi}\int_{-D}^{0}d\varepsilon\,\frac{\left[\dfrac{1}{(\varepsilon_f-\omega)^2}-\dfrac{1}{(\varepsilon+\omega-\varepsilon_f)^2}\right]}{\varepsilon-\dfrac{N\Delta_0}{\pi}\ln\left|\dfrac{\varepsilon-\varepsilon_f}{-\varepsilon_f}\right|-\dfrac{N\Delta_0}{\pi}\dfrac{\pi^2}{6}(k_BT)^2\left[\dfrac{1}{\varepsilon_f^2}-\dfrac{1}{(\varepsilon-\varepsilon_f)^2}\right]}
$$

$$
-\frac{\pi^2}{3}\left(\frac{k_BT}{\varepsilon_f}\right)^2\left(\frac{\Delta_0}{\varepsilon_f}\right)^2\frac{1}{N}\frac{\mu}{1+\mu}\left[3-\frac{\mu}{1+\mu}\right]
$$

$$
-\frac{\pi^2}{2}\left(\frac{k_BT}{\varepsilon_f}\right)^2\left(\frac{\Delta_0}{\varepsilon_f}\right)^2\frac{1}{N}\left(\frac{\mu}{1+\mu}\right)^2\left[\frac{4}{1+\mu}+\left(\frac{\mu}{1+\mu}\right)^2\right]
$$

$$
-\frac{2\pi^2}{3}\left(\frac{k_BT}{\varepsilon_f}\right)^2\left(\frac{\Delta_0}{\varepsilon_f}\right)^2\frac{1}{N}\left(\frac{\mu}{1+\mu}\right)^2\left[2+\frac{1}{1+\mu}-\frac{5}{3}\frac{\mu}{1+\mu}+\frac{3}{4}\left(\frac{\mu}{1+\mu}\right)^2\right]
$$

$$
+2\left(\frac{\Delta_0}{\varepsilon_f}\right)^2\frac{1}{N}\left(\frac{\mu}{1+\mu}\right)\left[\frac{\omega}{\varepsilon_f}\left(2-\frac{\mu}{1+\mu}\right)+\left(\frac{\omega}{\varepsilon_f}\right)^2\left(\frac{13}{2}-\frac{5}{2}\frac{\mu}{1+\mu}\right)\right]
$$

$$
\left.-\frac{1}{2}\left(\frac{\Delta_0}{\varepsilon_f}\right)^2\frac{1}{N}\left(\frac{\mu}{1+\mu}\right)^2\left(\frac{\omega}{\varepsilon_f}\right)^2\left[\frac{4}{1+\mu}+\left(\frac{\mu}{1+\mu}\right)^2\right]\right]
$$

$$
-\frac{2a}{\pi\rho}\frac{\Delta}{\Delta_0}\left[\frac{\Delta_0^2}{(\varepsilon_f-\omega)^2}\frac{1}{\left\{1+\dfrac{N\Delta_0}{\pi\varepsilon_f}\left[1+\dfrac{\pi^2}{3}\left(\dfrac{k_BT}{\varepsilon_f}\right)^2\right]\right\}}+\frac{2\Delta_0^2}{(\varepsilon_f-\omega)^2}\frac{1}{\beta\varepsilon_f}\frac{\left(\dfrac{N\Delta_0}{\pi\varepsilon_f}\right)\left[1+\pi^2\left(\dfrac{k_BT}{\varepsilon_f}\right)^2\right]}{\left\{1+\dfrac{N\Delta_0}{\pi\varepsilon_f}\left[1+\dfrac{\pi^2}{3}\left(\dfrac{k_BT}{\varepsilon_f}\right)^2\right]\right\}^2}\right.
$$

$$
\left.-\frac{4\Delta_0^2}{(\varepsilon_f-\omega)^3}\frac{1}{\beta}\frac{1}{1+\dfrac{N\Delta_0}{\pi\varepsilon_f}\left[1+\dfrac{\pi^2}{3}\left(\dfrac{k_BT}{\varepsilon_f}\right)^2\right]}\right] \tag{3.5}
$$

and

$$
R^{(1)}_{(\omega_\nu=0)}=-\frac{2a}{\pi\rho}\frac{\Delta}{\Delta_0}\left[-\frac{3}{2}\frac{\Delta_0^2}{(\varepsilon_f-\omega)^2}\frac{1}{\beta\varepsilon_f}\frac{\dfrac{N\Delta_0}{\pi\varepsilon_f}\left[1+\pi^2\left(\dfrac{k_BT}{\varepsilon_f}\right)^2\right]}{\left\{1+\dfrac{N\Delta_0}{\pi\varepsilon_f}\left[1+\dfrac{\pi^2}{3}\left(\dfrac{k_BT}{\varepsilon_f}\right)^2\right]\right\}^2}+\frac{4\Delta_0^2}{(\varepsilon_f-\omega)^3}\frac{1}{\beta}\frac{1}{1+\dfrac{N\Delta_0}{\pi\varepsilon_f}\left[1+\dfrac{\pi^2}{3}\left(\dfrac{k_BT}{\varepsilon_f}\right)^2\right]}\right]. \tag{3.6}
$$

The last three terms of (3.5) all of $O(1)$ are "pole" terms on combining with the "pole" term of (3.4) which originates in the equation of state we see that they are exactly cancelled by the contribution of the zero-frequency sector. Thus the corrections to the relaxation rate at one-loop order are at most of order $(1/N)$. In the calculation of the relaxation rate as in the calculation of the free energy and therefore by extension any other physical quantity the contribution of the zero-frequency sector and the pole contribution to the finite-frequency sector exactly cancel once the constraint equations are used consistently. Further the first two terms of $R^1(\omega_\nu\neq 0)$ are infrared divergent and exactly cancel the corresponding infrared divergences of R^0 leaving us with a well-defined relaxation rate which could have been calculated directly from the finite-frequency sector if pole terms were ignored. Putting a equal to one,

$$
\begin{aligned}
R(\omega,T)=&\frac{2}{\pi\rho}[1-n_f(T)]^2\\
&\times\left[\frac{\Delta_0^2}{(\varepsilon_f-\omega)^2}-\frac{2\Delta_0^2}{(\varepsilon_f-\omega)}\frac{1}{N}\left[\frac{N\Delta_0}{\pi}\right]^2\right.\\
&\times\int_{-D}^{0}d\varepsilon\frac{\left[\left[\frac{1}{\varepsilon_f}+\frac{1}{\varepsilon-\varepsilon_f}\right]+\frac{\pi^2}{3}(k_BT)^2\left[\frac{1}{\varepsilon_f^3}+\frac{1}{(\varepsilon-\varepsilon_f)^3}\right]\right]\left[\frac{1}{\varepsilon_f-\omega}+\frac{1}{\varepsilon+\omega-\varepsilon_f}\right]}{\left[\varepsilon-\frac{N\Delta_0}{\pi}\ln\left|\frac{\varepsilon-\varepsilon_f}{-\varepsilon_f}\right|-\frac{N\Delta_0}{\pi}\frac{\pi^2}{6}(k_BT)^2\left[\frac{1}{\varepsilon_f^2}-\frac{1}{(\varepsilon-\varepsilon_f)^2}\right]\right]^2}\\
&-\frac{2\Delta_0^2}{(\varepsilon_f-\omega)}\frac{1}{N}\frac{N\Delta_0}{\pi}\int_{-D}^{0}d\varepsilon\frac{\left[\frac{1}{(\varepsilon_f-\omega)^2}-\frac{1}{(\varepsilon+\omega-\varepsilon_f)^2}\right]}{\varepsilon-\frac{N\Delta_0}{\pi}\ln\left|\frac{\varepsilon-\varepsilon_f}{-\varepsilon_f}\right|-\frac{N\Delta_0}{\pi}\frac{\pi^2}{6}(k_BT)^2\left[\frac{1}{\varepsilon_f^2}-\frac{1}{(\varepsilon-\varepsilon_f)^2}\right]}\\
&-2\left[\frac{\Delta_0}{\varepsilon_f}\right]^2\frac{1}{N}\frac{\mu}{1+\mu}\left[\left[\frac{\omega}{\varepsilon_f}\right]\frac{2+\mu}{1+\mu}+\left[\frac{\omega}{\varepsilon_f}\right]^2\left[\frac{13}{2}-\frac{5}{2}\frac{\mu}{1+\mu}\right]\right]\\
&+\frac{1}{2}\left[\frac{\Delta_0}{\varepsilon_f}\right]^2\frac{1}{N}\left[\frac{\mu}{1+\mu}\right]^2\left[\frac{\omega}{\varepsilon_f}\right]^2\left[\frac{4}{1+\mu}+\left[\frac{\mu}{1+\mu}\right]^2\right]+\frac{2\pi^2}{3}\left[\frac{\Delta_0}{\varepsilon_f}\right]^2\frac{1}{N}\left[\frac{k_BT}{\varepsilon_f}\right]^2\left[\frac{\mu}{1+\mu}\right]^2\left[1+\frac{1}{1+\mu}\right]\\
&\left.+\frac{\pi^2}{3}\left[\frac{\Delta_0}{\varepsilon_f}\right]^2\frac{1}{N}\left[\frac{k_BT}{\varepsilon_f}\right]^2\left[\frac{\mu}{1+\mu}\right]\left[3-\frac{\mu}{1+\mu}\right]+\frac{\pi^2}{2}\left[\frac{\Delta_0}{\varepsilon_f}\right]^2\frac{1}{N}\left[\frac{k_BT}{\varepsilon_f}\right]^2\left[\frac{\mu}{1+\mu}\right]^2\left[\frac{4}{1+\mu}+\left[\frac{\mu}{1+\mu}\right]^2\right]\right].
\end{aligned}
\tag{3.7}
$$

At $T=0$, the zero-frequency relaxation rate

$$R(0,0)=\frac{2}{\pi\rho}\left[\frac{\Delta_0}{\varepsilon_f}\right]^2[1-n_f(0)]^2\left[1-\frac{2}{N}\mu^2\int_0^{D/T_A}dx\frac{\left[\frac{x}{1+x}\right]^2}{[x+\mu\ln(1+x)]^2}+\frac{2}{N}\mu\int_0^{D/T_A}\frac{\left[1-\frac{1}{(1+x)^2}\right]}{x+\mu\ln(1+x)}\right], \tag{3.8}$$

where from (2.35)

$$\varepsilon_f=T_A\left[1+\frac{1}{N}\frac{\mu^2}{1+\mu}L_1\right], \tag{3.9}$$

therefore

$$R(0,0)=\frac{2\Delta_0^2[1-n_f(0)]^2}{\pi\rho T_A^2}\times\left[1+\frac{2\mu}{N}\left[L_2-\mu K-\frac{\mu}{1+\mu}L_1\right]\right]. \tag{3.10}$$

From (2.40), the f electron occupancy at $T=0$ is given by

$$n_f(0)=\frac{\mu}{1+\mu}\left[1+\frac{1}{N}\frac{\mu}{1+\mu}\left[L_2-\mu K-\frac{\mu}{1+\mu}L_1\right]\right], \tag{3.11}$$

hence we obtain

$$R(0,0)=\frac{2}{\pi\rho}\left[\frac{\pi n_f(0)}{N}\right]^2,$$

which is exactly the prediction of the Friedel sum rule which was shown to be valid for the Anderson model by Langreth.[16] In the next section we will derive the temperature-dependent corrections to the transport coefficients to order $(1/N)$.

IV. TRANSPORT COEFFICIENTS

As discussed in the Introduction the conductivity is given by

$$\sigma \propto \int \frac{-\partial f(\omega)}{\partial \omega}\tau(\omega,T)d\omega = \tau(0,T)+\frac{\pi^2}{6}(k_BT)^2\tau''(\omega,0)\,|_{\omega=0} \,, \tag{4.1}$$

which when expressed in terms of the relaxation rate calculated in the preceding section can be written as

$$\frac{\sigma}{\sigma_0}=1-(k_BT)^2\frac{\overline{R}}{R_0}+\frac{\pi^2}{6}(k_BT)^2\left[\frac{2(R')^2}{R_0^2}-\frac{R''}{R_0}\right]; \tag{4.2}$$

here σ_0 is the dc conductivity at zero temperature, R_0 is the relaxation rate at zero temperature and frequency, R' and R'' are the first and second frequency derivatives, respectively, of the zero-temperature relaxation rate evaluated at $\omega=0$, and $\overline{R}$ is the coefficient of $(k_BT)^2$ in the expansion of the zero-frequency relaxation rate. Expanding $R(\omega,0)$, (3.7), in powers of ω and using (3.9) to express ε_f in terms of T_A, we find

$$\frac{R'}{R_0}=\frac{2}{T_A}\left[1-\frac{\mu}{N}\left[\frac{\mu}{1+\mu}L_1-\mu(K_{11}-K_{12})+(L_2-2L_3)+\frac{2+\mu}{(1+\mu)^2}\right]\right] \tag{4.3}$$

and

$$\frac{R''}{R_0}=\frac{6}{T_A^2}\left[1+\frac{2\mu}{3N}\left\{\mu\left[2K_{11}-K_{12}-K_{13}-\frac{3}{1+\mu}L_1\right]-(2L_2-2L_3-3L_4)-\frac{1}{2(1+\mu)}\left[\left[13-\frac{\mu}{1+\mu}\right]\right]+\frac{5\mu}{(1+\mu)^2}\left\{\frac{\mu}{1+\mu}+\frac{1}{4}\left[\left[\frac{\mu}{1+\mu}\right]\right]^2\right\}\right\}\right], \tag{4.4}$$

hence

$$\frac{\tau''(\omega,0)}{\tau(0,0)}=\frac{2}{T_A^2}\left[1-\frac{2\mu}{N}\left\{\frac{\mu}{1+\mu}L_1+(2L_2-6L_3+3L_4)-\mu(2K_{11}-3K_{12}+K_{13})+\frac{3}{2(1+\mu)^2}+\frac{\mu}{(1+\mu)^2}\left[\frac{1}{1+\mu}+\frac{1}{4}\left[\frac{\mu}{1+\mu}\right]^2\right]\right\}\right]. \tag{4.5}$$

We now turn to the temperature dependence of the zero-frequency relaxation rate $\overline{R}$, (4.2). There are three distinct sources of temperature dependence as can be seen from (3.7). Explicit temperature dependence of the boson propagator which gives a contribution

$$\frac{\overline{R}_p}{R_0}=-\frac{\pi^2}{3}\frac{1}{T_A^2}\frac{\mu^2}{N}[2K_{13}-2\mu M_{112}+K_{22}]\,, \tag{4.6}$$

the temperature dependence of ε_f, (2.35), which contributes

$$\frac{\overline{R}_{\varepsilon_f}}{R_0}=-\frac{\pi^2}{3}\frac{\mu}{1+\mu}\frac{1}{T_A^2}\left\{1+\frac{1}{N}\left[\delta-\mu\left[\frac{\mu}{1+\mu}L_1-\mu K_{11}+L_2\right]\right]+\mu(2L_3-3\mu K_{12}+2\mu^2M_{111})\right\}, \tag{4.7}$$

and finally the temperature dependence of n_f which contributes

$$\frac{\overline{R}_{n_f}}{\overline{R}_0}=-\frac{\pi^3}{3}\frac{1}{T_A^2}\frac{1}{1+\mu}\left\{\frac{\mu(2+\mu)}{1+\mu}-\frac{1}{N}\left[\frac{\mu^3(2+\mu)}{(1+\mu)^2}\left[L_2-\mu K_{11}-\frac{\mu}{1+\mu}L_1\right]+\frac{\mu^3}{1+\mu}(2L_3-3\mu K_{12}+2\mu^2M_{111})+\mu^2(2\mu K_{13}+\mu K_{22}-2\mu^2M_{112})+\mu^2\left[\mu K_{13}-3L_4+\frac{\mu}{1+\mu}L_1\right]+\frac{2\mu^2}{1+\mu}\left[\delta-\frac{2\mu^2L_1}{1+\mu}\right]+\frac{2\mu^2}{1+\mu}\left[3-\frac{\mu}{1+\mu}\right]-\frac{2\mu^3}{(1+\mu)^2}\left[2-\frac{\mu}{1+\mu}\right]\right]\right\}. \tag{4.8}$$

Collecting together the results (4.3)—(4.8), the conductivity can now be written explicitly as

$$\frac{\sigma(T)}{\sigma_0}=1+\frac{\pi^2}{3}\left[\frac{T}{T_A}\right]^2\left[1+\frac{\mu}{1+\mu}+\frac{\mu(2+\mu)}{(1+\mu)^2}\right]$$
$$+\frac{\pi^2}{3}\left[\frac{T}{T_A}\right]^2\frac{1}{N}\left\{\frac{\mu}{1+\mu}\left[-6-\frac{4\mu}{1+\mu}+\frac{6\mu^2}{(1+\mu)^2}-\frac{10\mu}{(1+\mu)^2}-\frac{4\mu^3}{(1+\mu)^3}\right]\right.$$
$$-\left[4\mu+\frac{\mu^2}{1+\mu}+\frac{\mu^3(2+\mu)}{(1+\mu)^3}\right]\left[L_2-\mu K_{11}-\frac{\mu}{1+\mu}L_1\right]$$
$$+6\mu\left[2L_3-\mu K_{1,2}-\frac{\mu}{1+\mu}\left[1+\frac{\mu}{1+\mu}\right]L_1\right]+\frac{\mu^2}{(1+\mu)^2}\left[2L_3-3\mu K_{1,2}-\frac{2\mu}{1+\mu}L_1\right]$$
$$\left.+\frac{\mu}{1+\mu}(\mu K_{2,2}+4\mu K_{1,3}-6L_4-2\mu^2 M_{112})+\frac{\mu}{(1+\mu)^2}(\delta+2\mu^3 M_{111})\right\}. \tag{4.9}$$

In the Kondo limit, $n_f \to 1$, $\mu \to \infty$; the conductivity reduces to

$$\frac{\sigma(T)}{\sigma_0}=1+\pi^2\left[\frac{T}{T_A}\right]^2\left[1+\frac{2}{N}[2\mu L_3+\mu^2(K_{11}-K_{12})-\mu L_2-\mu L_1-\tfrac{4}{3}]\right]; \tag{4.10}$$

as the Kondo temperature, T_K, which can be deduced from the zero-temperature magnetic susceptibility $\chi_0=\frac{1}{3}(g\mu_B)^2 j(j+1)/T_K$, is given to $O(1/N)$ by[8,11]

$$T_K=T_A\left[1+\frac{\mu}{N}[\mu(K_{12}-K_{11})+L_1+L_2-2L_3]\right]; \tag{4.11}$$

we find the very simple result

$$\frac{\sigma(T)}{\sigma_0}=1+\pi^2\left[\frac{T}{T_K}\right]^2\left[1-\frac{8}{3N}\right]. \tag{4.12}$$

For Cerium compounds, $N=6$, the coefficient of the T^2 term in $\sigma(T)/\sigma_0$ is $5\pi^2/9$. This is to be compared with the predictions of numerical calculations based on the "noncrossing approximation" which gave a coefficient of 5.

The thermal power $S(T)$ given by (1.6a) and (1.5) is found to be

$$S(T)=\frac{2\pi^2}{3}\frac{k_B^2}{e}\left[\frac{T}{T_A}\right]\times\left[1+\frac{1}{N}\left[2\mu L_3-\mu L_2-\mu^2 K_{12}+\mu^2 K_{11}-\frac{\mu^2}{1+\mu}L_1-\frac{\mu(2+\mu)}{(1+\mu)^2}\right]\right], \tag{4.13}$$

which, on using the expressions for γ, (2.39), and n_f, (2.41), can be written as

$$S(T)=\frac{2T}{e}\frac{\gamma}{n_f} \tag{4.14}$$

in agreement in the large-N limit with the exact result derived in the Introduction (1.17). In the Kondo limit,

$$S(T)=\frac{2\pi^2}{3}\frac{k_B^2}{e}\frac{T}{T_K}\left[1-\frac{1}{N}\right]; \tag{4.15}$$

here the coefficient $(1-1/N)$ has its origin in the reduction of γ due to the suppression of charge fluctuations. Finally, the Lorentz ratio given by (1.6b) and (1.5) is given in the Kondo limit by

$$\frac{L(T)}{L_0}=1-\frac{4\pi^2}{15}\left[\frac{T}{T_K}\right]^2\times\left[1-\frac{8}{N}(1+\mu L_2-\mu^2 K_{11}+2\mu^2 K_{12}-4\mu L_3+3\mu L_4+\mu^2 K_{13})\right]. \tag{4.16}$$

Although all the integrals contained in (4.16) are divergent in the infrared $L(T)/L_0$ is finite and can be evaluated analytically. We find

$$\frac{L(T)}{L_0}=1-\frac{4\pi^2}{15}\left[\frac{T}{T_K}\right]^2\left[1-\frac{8}{N}(1-2\ln 2)\right]$$
$$=1-\frac{4\pi^2}{15}\left[\frac{T}{T_K}\right]^2\left[1+\frac{3.1}{N}\right]. \tag{4.17}$$

V. CONCLUSIONS

In conclusion we will summarize the technical and then the physical aspects of this work, adding some final interpretation.

In Sec. II, we reconsidered the low-temperature free energy of the Anderson impurity to order $1/N$, taking the large-N limit with Q fixed at 1. We uncovered, for the first time, fluctuation terms of order 1 (i.e., the same order as the mean-field terms), coming from the pole of the boson propagator (after zero modes had been properly eliminated) and from terms at zero frequency involving the constraint field λ. It was then shown to two-loop order that for physical quantities (free energy, f occupation) all these terms cancel to order $1/N$ leaving the same results found earlier. In the expression for $|\langle b\rangle|^2$, on the other hand, the new terms do not cancel. The appearance of the additional terms only in unobservable quantities such as $|\langle b\rangle|^2$, parallels the appearance of infrared divergences, discussed earlier;[11] the resolution of the infrared divergence problem[11] appears to be unaffected by the new terms as far as we have been able to check (the new terms in $|\langle b\rangle|^2$ like the old are multiplied at $T=0$ by a series of powers of logarithms, which is expected to exponentiate, so that $|\langle b\rangle|^2$ eventually vanishes). In fact in the correlation function $\langle\mathscr{T}\sigma(\tau)\sigma^*(0)\rangle$, $\sim V^2|\langle b\rangle|^2$ in lowest order, these terms together with their accompanying infrared divergencies are exactly cancelled to order $1/N$ by corresponding contributions to the one-loop correction to the correlation function. Assuming exponentiation of the remaining logarithmic corrections we find as in II the correlation function has power-law decay with exponent $\alpha=n_f^2/N$ again implying that $|\langle b\rangle|^2$ must vanish.

In the calculation of the F Green's function (Sec. III) again $\Delta=|\langle b\rangle|^2\Delta_0$ appears explicitly at lowest order and again we were able to show that such cancellations occur to one-loop order, keeping terms to order $1/N$ at low frequency and temperature. In summary, while we have not shown explicitly that our results are exact to order $(1/N)$, this seems overwhelmingly likely to be the case; we now discuss some additional facts which point in this direction.

Calculation of the free energy to more than two loops reveals still more terms of order 1 and $1/N$ however they necessarily involve zero frequency (λs) propagators. One can show that the order 1 class of free energy diagrams involving only (λs) and (ss) propagators, with arbitrary number of loops, cancels against terms arising from contractions of the Jacobian as in the calculations of Sec. II, such cancellations were mentioned also in the Appendix of I. It seems likely that all diagrams can be accounted for by extending our arguments. Furthermore, other methods exist,[7,8] which take the large-N limit in the same way we do. Though less flexible than our approach, they unambiguously show the existence of a large-N expansion for the free energy at low temperatures, of the kind discussed in the Introduction.

If instead the large-N limit is taken with $Q\propto N$, the pole terms still arise when the calculation is done in the "cartesian" gauge used here, and still have to be cancelled against the zero-frequency fluctuations. In this case, of course, the terms are of order N^{1-r} for r loops, as we explained in the Introduction. The cancellations are important, in that we now account quite generally for *all* types of fluctuations, including those of the constraint field λ. The structure of the calculation suggests that expansion in the number of boson loops, as used here, may be the more fundamental approach, and may be useful even when the $1/N$ expansion does not exist.

Turning to our results for transport quantities, we have presented the first calculations of the low-temperature conductivity, thermopower, and thermal conductivity in the $1/N$ expansion, to order $1/N$. Our results were given in Eqs. (4.12), (4.15), and (4.17). For the conductivity, our result for the T^2 coefficient in the Kondo limit when $N=6$ is $[1-(8/3N)]\pi^2\simeq 5\pi^2/9\simeq 5.6$ which is to be compared with the result 5 obtained by Bickers *et al.*[19] from the noncrossing approximation. In the case of the thermopower, our result agreed to order $1/N$ with the exact result derived in Sec. I B. In the Kondo region, this implies that the linear term in the thermopower is reduced by a factor of $(1-1/N)$ relative to the mean field result, and that this is an *exact* ratio, analogous to the well-known χ/γ ratio: we may write

$$\frac{S}{T\chi_0}=\frac{2\pi^2k_B^2}{g^2j(j+1)\mu_B^2e}\left[1-\frac{1}{N}\right]\frac{\pi}{N}\cot\left[\frac{\pi}{N}\right], \tag{5.1}$$

which is a universal result in the Kondo regime. We wish to emphasize that, like the χ/γ ratio, this arises in our calculation from a reduction of S or γ, not an enhancement of χ. If we think of S as a measurement of the ratio of heat carried by a quasiparticle to current carried, then we can give a simple interpretation of this result. Current is carried only by conduction electrons, not by the f quasiparticles which are localized at the impurity site (the f component has zero velocity even in the lattice case). The heat current, on the other hand, probes the total number of degrees of freedom available for the low-energy excitation. In the Kondo limit, the charge degree of freedom of the ion is fixed and the spin excitations of the ion dominate, giving $N-1$ degrees of freedom in the Fermi-liquid picture. Thus the quasiparticle carries f spin, but no f-charge degree of freedom[5,11] and we recover the value of (5.1).

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APPENDIX A

In this appendix we give expressions for the boson self-energies Γ introduced in Sec. II, Eqs. (2.11) and (2.12), see Fig. 1, and the boson propagators $D_{\sigma\sigma^*}$ and $D_{\sigma^*\sigma^*}$. All results are for large bandwidth $D\gg T_A$, and ω_ν:

$$\Gamma_0(i\omega_\nu)=\frac{N}{\beta}\sum_{\omega_n} g_f(i\omega_n)g_0(i(\omega_n-\omega_\nu))$$

$$=N\rho\left[\ln\left[\frac{\varepsilon_f}{D}\right]-\frac{\pi^2}{6}\left[\frac{k_BT}{\varepsilon_f}\right]^2+\ln\left[\frac{i\omega_\nu-\varepsilon_f+i\Delta\,\mathrm{sgn}\omega_\nu}{-\varepsilon_f+i\Delta\,\mathrm{sgn}\omega_\nu}\right]\right.$$

$$\left.+\frac{\pi^2}{6}(k_BT)^2\left[\frac{1}{(-\varepsilon_f+i\Delta\,\mathrm{sgn}\omega_\nu)^2}-\frac{1}{(i\omega_\nu-\varepsilon_f+i\Delta\,\mathrm{sgn}\omega_\nu)^2}\right]\right], \tag{A1}$$

$$\Gamma_1(i\omega_\nu)=\frac{N}{\beta}s_0^2\sum_{\omega_n} g_f(i\omega_n)g_0^2(i(\omega_n-\omega_\nu))g_f(i(\omega_n-\omega_\nu))$$

$$=\frac{N\rho\Delta^2}{|\omega_\nu|(|\omega_\nu|+2\Delta)}\left\{\ln\left[1+\frac{|\omega_\nu|(|\omega_\nu|+2\Delta)}{\varepsilon_f^2+\Delta^2}\right]\right.$$

$$+\frac{\pi^2}{3}(k_BT)^2\left[\left[\frac{1}{\varepsilon_f^2+(\Delta+|\omega_\nu|)^2}-\frac{1}{\varepsilon_f^2+\Delta^2}\right]\right.$$

$$\left.\left.-2\varepsilon_f^2\left[\frac{1}{[\varepsilon_f^2+(\Delta+|\omega_\nu|)^2]^2}-\frac{1}{(\varepsilon_f^2+\Delta^2)^2}\right]\right]\right\}, \tag{A2}$$

$$\Gamma_2(i\omega_\nu)=\frac{Ns_0^2}{\beta}\sum_{\omega_n} g_0(i\omega_n)g_f(i\omega_n)g_0(i(\omega_n+\omega_\nu))g_f(i(\omega_n+\omega_\nu))=\left[\frac{\Delta+|\omega_\nu|}{\Delta}\right]\Gamma_1(i\omega_\nu)\ , \tag{A3}$$

$$\Gamma_{\lambda s}=\frac{s_0}{V^2}+\frac{N}{\beta}\sum_n g_f^2(i\omega_n)g_0(i\omega_n)=\frac{s_0}{V^2}\left\{1+\frac{N\Delta_0}{\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}, \tag{A4}$$

$$\Gamma_{\bar\lambda\bar\lambda}=-\frac{N}{\beta}\sum_n g_f^2(i\omega_n)=\frac{N\Delta}{\pi\varepsilon_f}\frac{1}{\varepsilon_f}\left[1+\pi^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\right], \tag{A5}$$

$$\Gamma_{ss}=\frac{1}{V^2}(\varepsilon_f-E_0)+\Gamma_0(0)+\Gamma_1(0)+\Gamma_2(0)\cong\frac{1}{V^2}\left[\varepsilon_f-E_0+\frac{N\Delta_0}{\pi}\ln\left[\frac{\varepsilon_f}{D}\right]-\frac{N\Delta_0}{\pi}\frac{\pi^2}{6}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]=\eta^2/V^2\ . \tag{A6}$$

Given the self-energies we can compute the boson propagator $D_{\sigma\sigma^*}$ and the anomalous propagators $D_{\sigma^*\sigma^*}=D_{\sigma\sigma}$. To the accuracy required in this paper,

$$D_{\sigma\sigma^*}(\varepsilon+i\delta)=\frac{V^2}{\varepsilon-(\varepsilon_f-E_0)-V^2\Gamma_0(\varepsilon+i\delta,T)} \tag{A7}$$

and

$$D_{\sigma^*\sigma^*}(\varepsilon+i\delta)=\frac{V^4\Gamma_2(\varepsilon+i\delta)}{[-\varepsilon+(\varepsilon_f-E_0)+V^2\Gamma_0(\varepsilon+i\delta,T)][\varepsilon+(\varepsilon_f-E_0)+V^2\Gamma_0(-\varepsilon-i\delta,T)]}\ . \tag{A8}$$

Both propagators are explicitly temperature dependent via the self-energies Γ and implicitly temperature dependent via ε_f. In the text we will make frequent use of the real and imaginary parts of $D_{\sigma\sigma^*}$ which are given at the stationary point $\eta^2=0$ by

$$\mathrm{Re}D_{\sigma\sigma^*}(\varepsilon)=\frac{V^2}{\varepsilon-\frac{N\Delta_0}{\pi}\ln\left[1-\frac{\varepsilon}{\varepsilon_f}\right]-\frac{N\Delta_0}{\pi}\frac{\pi^2}{6}(k_BT)^2\left[\frac{1}{\varepsilon_f^2}-\frac{1}{(\varepsilon_f-\varepsilon)^2}\right]} \tag{A9}$$

and

$$\mathrm{Im}D_{\sigma\sigma^*}(\varepsilon)=\frac{V^2\frac{N\Delta_0}{\pi}\Delta\left[\left[\frac{1}{\varepsilon-\varepsilon_f}+\frac{1}{\varepsilon_f}\right]+\frac{\pi^2}{3}(k_BT)^2\left[\frac{1}{\varepsilon_f^3}+\frac{1}{(\varepsilon-\varepsilon_f)^3}\right]\right]}{\left[\varepsilon-\frac{N\Delta_0}{\pi}\ln\left[1-\frac{\varepsilon}{\varepsilon_f}\right]-\frac{N\Delta_0}{\pi}\frac{\pi^2}{6}(k_BT)^2\left[\frac{1}{\varepsilon_f^2}-\frac{1}{(\varepsilon_f-\varepsilon)^2}\right]\right]^2}\ . \tag{A10}$$

Both propagators have poles at the origin:

$$\mathrm{Re}D_{\sigma\sigma^*}(\varepsilon\cong 0)=\frac{1}{\varepsilon}\frac{V^2}{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]}\ , \tag{A11}$$

$$\mathrm{Im}D_{\sigma\sigma^*}(\varepsilon\cong 0)=-\frac{1}{\varepsilon}\frac{V^2\frac{N\Delta_0}{\pi\varepsilon_f}\frac{\Delta}{\varepsilon_f}\left[1+\pi^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}^2}\ , \tag{A12}$$

and

$$\mathrm{Im}D_{\sigma^*\sigma^*}(\varepsilon\cong 0)=\frac{1}{\varepsilon}\frac{V^2\frac{N\Delta_0}{\pi\varepsilon_f}\frac{\Delta}{\varepsilon_f}\left[1+\pi^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}^2}\ . \tag{A13}$$

APPENDIX B

In this Appendix we evaluate in detail the contribution of the finite-frequency sector to the action at one-loop order. To the order considered here this can be written as

$$\delta S^{(1)}=a\sum_{\omega_\nu(\neq 0)}\ln[\beta(-i\omega_\nu+\eta^2+V^2\widetilde{\Gamma}_0(i\omega_\nu))]\ , \tag{B1}$$

where using (2.23) and (A1)

$$\widetilde{\Gamma}_0(i\omega_\nu)=N\rho\left[\ln\left[\frac{i\omega_\nu-\varepsilon_f+i\Delta\,\mathrm{sgn}\omega_\nu}{-\varepsilon_f+i\Delta\,\mathrm{sgn}\omega_\nu}\right]+\frac{\pi^2}{6}(k_BT)^2\left[\frac{1}{(-\varepsilon+i\Delta\,\mathrm{sgn}\omega_\nu)^2}-\frac{1}{(i\omega_\nu-\varepsilon_f+i\Delta\,\mathrm{sgn}\omega_\nu)^2}\right]\right]\ . \tag{B2}$$

The sum over the Matsubara frequencies $\omega_\nu=2n\pi T$ can be written as a contour integral, the contour C is shown in Fig. 8:

$$\delta S^{(1)}=\frac{\beta}{2\pi i}\oint_C b(z)\ln[\beta(-z+\eta^2+V^2\widetilde{\Gamma}_0(z))]\ . \tag{B3}$$

The contour C can then be distorted around the cut on the real axis as shown in the figure. Now the Bose function $b(z)$ has a pole at the origin and the argument of the logarithm vanishes at

$$\tilde{\eta}^2=\frac{\eta^2}{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]}\left[1-\frac{\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\pi^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\frac{\eta^2}{\varepsilon_f}}{2\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}^2}\right]\ , \tag{B4}$$

hence

$$\delta S^{(1)}=-a\beta\frac{\Delta}{\Delta_0}\frac{1}{N}\left[\frac{N\Delta_0}{\pi}\right]^2\int_{-D}^{D}d\varepsilon\, b(\varepsilon)\frac{\left[\left[\frac{1}{\varepsilon_f}+\frac{1}{\varepsilon-\varepsilon_f}\right]+\frac{\pi^2}{3}(k_BT)^2\left[\frac{1}{\varepsilon_f^3}+\frac{1}{(\varepsilon-\varepsilon_f)^3}\right]\right]}{\left[\varepsilon-\eta^2-\frac{N\Delta_0}{\pi}\ln\left|1-\frac{\varepsilon}{\varepsilon_f}\right|-\frac{\pi^2}{6}(k_BT)^2\frac{N\Delta_0}{\pi}\left[\frac{1}{\varepsilon_f^2}-\frac{1}{(\varepsilon-\varepsilon_f)^2}\right]\right]}$$

$$-a\beta\int_{\tilde{\eta}^2}^{\infty}dx\frac{1}{e^{\beta x}-1}-a\ln(\beta\eta^2)\ . \tag{B5}$$

The first term in (B5) is the discontinuity of the logarithm across the branch cut Fig. 8. The second term due to the discontinuous change in phase of the logarithm at its branch point at $\tilde{\eta}^2$ and the last term from the pole of the Bose function.

Now

$$\beta \int_{\tilde{\eta}^2}^{\infty} dx \frac{1}{e^{\beta x}-1} = -\ln(\beta\tilde{\eta}^2) - \ln(1-\tfrac{1}{2}\beta\tilde{\eta}^2) \cong -\ln(\beta\tilde{\eta}^2) + \tfrac{1}{2}\beta\tilde{\eta}^2 , \tag{B6}$$

hence on combining the pole terms and using (B4) we find

$$\delta S^{(1)} = -a \left[\beta \frac{\Delta}{\Delta}\frac{1}{N}\left[\frac{N\Delta_0}{\pi}\right]^2 \int_{-D}^{D} d\varepsilon\, b(\varepsilon) \frac{\left[\left[\dfrac{1}{\varepsilon_f}+\dfrac{1}{\varepsilon-\varepsilon_f}\right]+\dfrac{\pi^2}{3}(k_B T)^2\left[\dfrac{1}{\varepsilon_f^3}+\dfrac{1}{(\varepsilon-\varepsilon_f)^3}\right]\right]}{\left[\varepsilon-\eta^2-\dfrac{N\Delta_0}{\pi}\ln\left|1-\dfrac{\varepsilon}{\varepsilon_f}\right|-\dfrac{\pi^2}{6}(k_B T)^2\dfrac{N\Delta_0}{\pi}\left[\dfrac{1}{\varepsilon_f^2}-\dfrac{1}{(\varepsilon-\varepsilon_f)^2}\right]\right]} \right.$$

$$+\ln\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_B T}{\varepsilon_f}\right]^2\right]\right\}$$

$$\left. + \frac{\dfrac{N\Delta_0}{\pi\varepsilon_f}\left[1+\pi^2\left[\dfrac{k_B T}{\varepsilon_f}\right]^2\right]\eta^2}{2\varepsilon_f\left\{1+\dfrac{N\Delta_0}{\pi\varepsilon_f}\left[1+\dfrac{\pi^2}{3}\left[\dfrac{k_B T}{\varepsilon_f}\right]^2\right]^2\right\}} + \frac{\tfrac{1}{2}\beta\eta^2}{1+\dfrac{N\Delta_0}{\pi\varepsilon_f}\left[1+\dfrac{\pi^2}{3}\left[\dfrac{k_B T}{\varepsilon_f}\right]^2\right]} \right] . \tag{B7}$$

This result is quoted in (2.25).

APPENDIX C

In this Appendix we evaluate in detail the contribution of the diagram of Fig. 6(a) to the relaxation rate and list the contributions of all diagrams in the finite-frequency sector Fig. 6 and zero-frequency sector Fig. 7:

$$R^{1}_{6(\mathrm{a})}(\omega_\nu \neq 0) = \frac{2}{\beta} \sum_{\omega_\nu (\neq 0)} D_{\sigma\sigma^*}(i\omega_\nu) g_f(i(\omega_n+\omega_\nu)) . \tag{C1}$$

The sum over the Matsubara frequencies $\omega_\nu = 2n\pi T$ can be written as a contour integral, the contour C is shown in Fig. 9:

$$R^{1}_{6(\mathrm{a})}(\omega_\nu \neq 0) = \frac{1}{\pi i} \oint dz\, b(z) D_{\sigma\sigma^*}(z) g_f(z+i\omega_\nu) . \tag{C2}$$

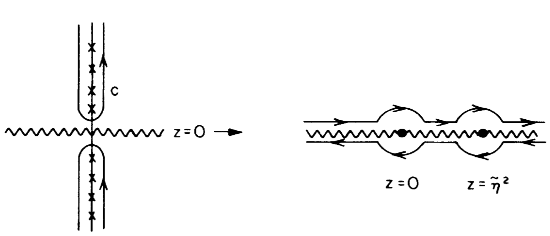

FIG. 8. Contour C, Eq. (B3).

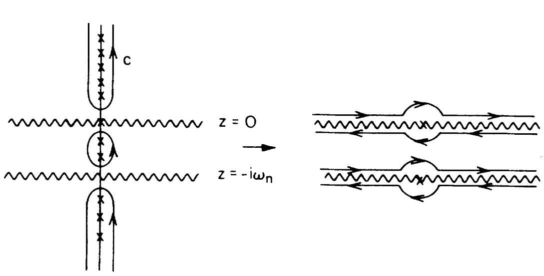

FIG. 9. Contour C, Eq. (C2).

The contour can then be distorted around the branch cuts of the Bose propagator and "f" electron propagator as shown in Fig. 9. Shifting variable on the lower contour, we find

$$R^{1}_{6(a)}(\omega_\nu \neq 0)=2\left[\frac{1}{\pi}\int_{-D}^{D} d\varepsilon\, b(\varepsilon)\mathrm{Im}[D_{\sigma\sigma^*}(\varepsilon+i\delta)]g_f(\varepsilon+i\omega_\nu) - \frac{1}{\pi}\int_{-D}^{D} d\varepsilon\, f(\varepsilon)\mathrm{Im}[g_f(\varepsilon+i\delta)]D(\varepsilon-i\omega_n)+\text{"pole terms"}\right]. \tag{C3}$$

The terms written explicitly in (C3) are simply the discontinuities of the integrand of (C2) across the cuts shown in Fig. 9. As both the boson propagator $D_{\sigma\sigma^*}(z)$ and Bose function $b(z)$ have poles at the origin there are additional pole terms. To obtain the contribution to the relaxation rate we analytically continue $i\omega_n \to \omega+i\delta$ in (C3) and then take the imaginary part and find

$$R^{1}_{6(a)}(\omega_\nu \neq 0)=2\left\{\frac{1}{\pi}\int_{-D}^{D}[b(\varepsilon)+f(\varepsilon+\omega)]\mathrm{Im}[D(\varepsilon+i\delta)]\mathrm{Im}[g_f(\varepsilon+\omega+i\delta)] - i\int_{-D}^{D} f(\varepsilon)\delta(\varepsilon-\omega)\mathrm{Im}g_f(\varepsilon+i\delta)\frac{V^2}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}} - \mathrm{Im}\frac{d}{dz}[z^2 b(z)D_{\sigma\sigma^*}(z)g_f(z+\omega+i\delta)]\,|_{z=0}\right\}, \tag{C4}$$

where in (C4) the pole terms have now been written explicitly. The integrand over $\mathrm{Im}D(\varepsilon+i\delta)$, given in (A10), is infrared divergent. Extracting this divergence the remaining analytic part of the integral can be evaluated using the Sommerfeld-Watson expansion. As we wish to determine the conductivity $O(T^2)$, we must retain terms to order ω^2 and T^2 in $R(\omega,T)$. Using the results listed in Appendix A we find

$$R^{1}_{6(a)}(\omega_\nu \neq 0)=\frac{2a}{\pi\rho}\left[\frac{\Delta}{\Delta_0}\right]^2\left\{\frac{\Delta_0^2}{(\varepsilon_f-\omega)^2}\frac{1}{N}\frac{\left[\frac{N\Delta_0}{\pi\varepsilon_f}\right]^2\left[1+\pi^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}^2}\int_{-D}^{D} d\varepsilon\frac{[b(\varepsilon)+f(\varepsilon+\omega)]}{\varepsilon}\right. \tag{C5a}$$

$$+\frac{\pi^2}{2}\left[\frac{k_BT}{\varepsilon_f}\right]^2\frac{\Delta_0^2}{\varepsilon_f^2}\left[\frac{\mu}{1+\mu}\right]^2\frac{1}{N}\left[4+\frac{2}{1+\mu}-\frac{5\mu}{1+\mu}+\frac{3}{4}\left[\frac{\mu}{1+\mu}\right]^2\right] \tag{C5b}$$

$$\left.+\frac{\Delta_0^2}{(\varepsilon_f-\omega)^2}\left[\frac{\mu}{1+\mu}\right]^2\frac{1}{N}\left\{-\frac{1}{1+\mu}\frac{\omega}{\varepsilon_f}-\frac{2\omega}{\varepsilon_f-\omega}+\left[\frac{\omega}{\varepsilon_f}\right]^2\left[2+\frac{1}{1+\mu}-\frac{5}{6}\frac{\mu}{1+\mu}+\frac{3}{8}\left[\frac{\mu}{1+\mu}\right]^2\right]\right\}\right\} \tag{C5c}$$

$$-\frac{2a}{\pi\rho}\frac{\Delta}{\Delta_0}\left\{\frac{2\Delta_0^2}{(\omega-\varepsilon_f)^3}\frac{1}{\beta}\frac{1}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}}+\frac{\Delta_0^2}{(\omega-\varepsilon_f)^2}\frac{1}{2\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}}\right.$$

$$+\frac{\Delta_0^2}{(\omega-\varepsilon_f)^2}\frac{1}{2\beta\varepsilon_f}\frac{\left[\frac{N\Delta_0}{\pi\varepsilon_f}\right]\left[1+\pi^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}^2}$$

$$-f(\omega)\frac{\Delta_0^2}{(\omega-\varepsilon_f)^2}\frac{1}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\pi_f}\right]^2\right]\right\}}\Bigg] . \tag{C5d}$$

The last four terms in (C5) are pole terms. We now list the contributions of the remaining diagrams of Fig. 6:

$$R^1_{6(b)}(\omega_\nu\neq 0)=\frac{2a}{\pi\rho}\left[\frac{\Delta}{\Delta_0}\right]^2\left[\frac{2\Delta_0^2}{(\varepsilon_f-\omega)^2}\frac{1}{N}\frac{\left[\frac{N\Delta_0}{\pi\varepsilon_f}\right]^2\left[1+\pi^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}^2}\int_{-D}^{D}d\varepsilon\frac{b(\varepsilon)}{\varepsilon}\right. \tag{C6a}$$

$$+\frac{2\Delta_0^2}{(\varepsilon_f-\omega)^3}\frac{1}{N}\left[\frac{N\Delta_0}{\pi}\right]\frac{1}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}}\int_{-D}^{D}d\varepsilon\frac{f(\varepsilon+\omega)}{\varepsilon} \tag{C6b}$$

$$-\frac{2\Delta_0^2}{(\varepsilon_f-\omega)}\frac{1}{N}\left[\frac{N\Delta_0}{\pi}\right]^2\int_{-D}^{0}d\varepsilon\left\{\frac{\left[\left[\frac{1}{\varepsilon_f}+\frac{1}{\varepsilon-\varepsilon_f}\right]+\frac{\pi^2}{3}(k_BT)^2\left[\frac{1}{\varepsilon_f^3}+\frac{1}{(\varepsilon-\varepsilon_f)^3}\right]\right]\frac{1}{\varepsilon+\omega-\varepsilon_f}}{\left[\varepsilon-\frac{N\Delta_0}{\pi}\ln\left|\frac{\varepsilon-\varepsilon_f}{-\varepsilon_f}\right|-\frac{N\Delta_0}{\pi}\frac{\pi^2}{6}(k_BT)^2\left[\frac{1}{\varepsilon_f^2}-\frac{1}{(\varepsilon-\varepsilon_f)^2}\right]\right]^2}\right.$$

$$\left.+\frac{1}{\varepsilon}\frac{\left[1+\pi^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\frac{1}{\varepsilon_f-\omega}}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}^2}\right\} \tag{C6c}$$

$$+\frac{2\Delta_0^2}{(\varepsilon_f-\omega)}\frac{1}{N}\left[\frac{N\Delta_0}{\pi}\right]\int_{-D}^{0}d\varepsilon\left\{\frac{\frac{1}{(\varepsilon+\omega-\varepsilon_f)^2}}{\varepsilon-\frac{N\Delta_0}{\pi}\ln\left|\frac{\varepsilon-\varepsilon_f}{-\varepsilon_f}\right|-\frac{N\Delta_0}{\pi}\frac{\pi^2}{6}(k_BT)^2\left[\frac{1}{\varepsilon_f^2}-\frac{1}{(\varepsilon-\varepsilon_f)^2}\right]}\right.$$

$$\left.-\frac{1}{\varepsilon}\frac{\frac{1}{(\varepsilon_f-\omega)^2}}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}}\right\} \tag{C6d}$$

$$+\frac{2\pi^2}{3}\left[\frac{\Delta_0}{\varepsilon_f}\right]^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\frac{1}{N}\left[\frac{\mu}{1+\mu}\right]^2\left[2+\frac{1}{1+\mu}-\frac{5}{3}\frac{\mu}{1+\mu}+\frac{3}{4}\left[\frac{\mu}{1+\mu}\right]^2\right]$$
$$+\frac{\pi^2}{3}\left[\frac{\Delta_0}{\varepsilon_f}\right]^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\frac{1}{N}\frac{\mu}{1+\mu}\left[3-\frac{4}{3}\frac{\mu}{1+\mu}+\frac{1}{4}\left[\frac{\mu}{1+\mu}\right]^2\right] \tag{C6e}$$

$$+\frac{2\Delta_0^2}{(\varepsilon_f-\omega)^3}\frac{\varepsilon_f}{N}\left[\frac{\mu}{1+\mu}\right]\left\{\frac{1}{2}\left[\frac{\omega}{\varepsilon_f}\right]\frac{\mu}{1+\mu}-\frac{2\omega}{\varepsilon_f-\omega}+\left[\frac{\omega}{\varepsilon_f}\right]^2\left[-\frac{2}{3}\frac{\mu}{1+\mu}+\frac{3}{2}+\frac{1}{8}\left[\frac{\mu}{1+\mu}\right]^2\right]\right\}\Bigg] \tag{C6f}$$

$$-\frac{2a}{\pi\rho}\frac{\Delta}{\Delta_0}\left[\frac{\Delta_0^2}{(\varepsilon_f-\omega)^2}\frac{1}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}}+\frac{\Delta_0^2}{(\varepsilon_f-\omega)^2}\frac{1}{\beta\varepsilon_f}\frac{\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\pi^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}}\right.$$

$$\left.-\frac{2\Delta_0^2}{(\varepsilon_f-\omega)^3}\frac{1}{\beta}\frac{1}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}}\right], \tag{C6g}$$

$$R^{1}_{6(c)}(\omega_\nu\neq 0)=\frac{2a}{\pi\rho}\left[\frac{\Delta}{\Delta_0}\right]^2\left[\frac{\Delta_0^2}{(\varepsilon_f-\omega)^2}\frac{1}{N}\frac{\left[\frac{N\Delta_0}{\pi\varepsilon_f}\right]^2\left[1+\pi^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}^2}\int_{-D}^{D}d\varepsilon\,\frac{[b(\varepsilon)+f(\varepsilon-\omega)]}{\varepsilon}\right. \tag{C7a}$$

$$-\frac{2\Delta_0^2}{(\varepsilon_f-\omega)^3}\frac{1}{N}\frac{N\Delta_0}{\pi}\frac{1}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}}\int_{-D}^{D}d\varepsilon\,\frac{f(\varepsilon-\omega)}{\varepsilon} \tag{C7b}$$

$$-\frac{2\Delta_0^2}{(\varepsilon_f-\omega)^3}\frac{1}{N}\frac{N\Delta_0}{\pi}\int_{-D}^{0}d\varepsilon\left[\frac{1}{\left[\varepsilon-\frac{N\Delta_0}{\pi}\ln\left|\frac{\varepsilon-\varepsilon_f}{-\varepsilon_f}\right|-\frac{N\Delta_0}{\pi}\frac{\pi^2}{6}(k_BT)^2\left[\frac{1}{\varepsilon_f^2}-\frac{1}{(\varepsilon-\varepsilon_f)^2}\right]\right]}-\frac{1}{\varepsilon}\frac{1}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}}\right] \tag{C7c}$$

$$+\frac{\pi^2}{2}\left[\frac{\Delta_0}{\varepsilon_f}\right]^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\frac{1}{N}\left[\frac{\mu}{1+\mu}\right]^2\left[1-\frac{5}{3}\frac{\mu}{1+\mu}+\frac{3}{4}\left[\frac{\mu}{1+\mu}\right]^2\right]$$

$$-\frac{\pi^2}{3}\left[\frac{\Delta_0}{\varepsilon_f}\right]^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\frac{1}{N}\left[\frac{\mu}{1+\mu}\right]\left[-\frac{1}{3}\left[\frac{\mu}{1+\mu}\right]+\frac{1}{4}\left[\frac{\mu}{1+\mu}\right]^2\right] \tag{C7d}$$

$$+\frac{\Delta_0^2}{(\varepsilon_f-\omega)^2}\frac{1}{N}\left[\frac{\mu}{1+\mu}\right]^2\left[\frac{1}{1+\mu}\frac{\omega}{\varepsilon_f}+\frac{1}{2}\left[\frac{\omega}{\varepsilon_f}\right]^2-\frac{5}{6}\left[\frac{\mu}{1+\mu}\right]\left[\frac{\omega}{\varepsilon_f}\right]^2+\frac{3}{8}\left[\frac{\mu}{1+\mu}\right]^2\left[\frac{\omega}{\varepsilon_f}\right]^2\right]$$

$$-\frac{2\Delta_0^2}{(\varepsilon_f-\omega)^3}\frac{\varepsilon_f}{N}\left[\frac{\mu}{1+\mu}\right]\left[-\frac{1}{2}\left[\frac{\mu}{1+\mu}\right]\left[\frac{\omega}{\varepsilon_f}\right]-\frac{1}{6}\left[\frac{\mu}{1+\mu}\right]\left[\frac{\omega}{\varepsilon_f}\right]^2 + \frac{1}{8}\left[\frac{\mu}{1+\mu}\right]^2\left[\frac{\omega}{\varepsilon_f}\right]^2\right] \quad \text{(C7e)}$$

$$-\frac{2a}{\pi\rho}\frac{\Delta}{\Delta_0}\left\{\frac{1}{2}\frac{\Delta_0^2}{(\varepsilon_f-\omega)^2}\frac{1}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}}+\frac{1}{2}\frac{\Delta_0^2}{(\varepsilon_f-\omega)^2}\frac{1}{\beta\varepsilon_f}\frac{\left[\frac{N\Delta_0}{\pi\varepsilon_f}\right]\left[1+\pi^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}^2}\right.$$

$$\left.-f(-\omega)\frac{\Delta_0^2}{(\varepsilon_f-\omega)^2}\frac{1}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}}\right\}, \quad \text{(C7f)}$$

and finally the contribution of the anomalous diagrams Fig. 6(a):

$$R^{1}_{6(a)}(\omega_\nu\neq0)=-\frac{2a}{\pi\rho}\left[\frac{\Delta}{\Delta_0}\right]^2\left\{\frac{2\Delta_0^2}{(\varepsilon_f-\omega)^2}\frac{1}{N}\frac{\left[\frac{N\Delta_0}{\pi\varepsilon_f}\right]^2\left[1+\pi^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}^2}\int_{-D}^{D}d\varepsilon\frac{[b(\varepsilon)+f(\varepsilon+\omega)]}{\varepsilon}\right. \quad \text{(C8a)}$$

$$+\pi^2\left[\frac{\Delta_0}{\varepsilon_f}\right]^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\frac{1}{N}\left[\frac{\mu}{1+\mu}\right]^2\left[\frac{3}{2}-\frac{2}{3}\frac{\mu}{1+\mu}+\frac{1}{4}\left[\frac{\mu}{1+\mu}\right]^2\right] \quad \text{(C8b)}$$

$$\left.+\frac{2\Delta_0^2}{(\varepsilon_f-\omega)^2}\frac{1}{N}\left[\frac{\mu}{1+\mu}\right]^2\left\{-\frac{\omega}{\varepsilon_f-\omega}+\left[\frac{\omega}{\varepsilon_f}\right]^2\left[\frac{3}{4}-\frac{1}{3}\frac{\mu}{1+\mu}+\frac{1}{8}\left[\frac{\mu}{1+\mu}\right]^2\right]\right\}\right\}. \quad \text{(C8c)}$$

It is now clear that the infrared divergent terms (C5a), (C7a), and (C8a) sum to zero as do (C6b) and (C7b); (C6c) and (C7c) combine to give a finite integral. The only remaining divergent contribution to $R^{(1)}(\omega_\nu\neq0)$, (C6a), cancels the infrared divergence of $R^0(\omega_\nu\neq0)$, (3.4) of the text, which had its origin in the equation of state. The pole terms, (C5d), (C6g) and (C7f) sum to

$$R^{(1)}_{\text{pole}}(\omega_\nu\neq0)=-\frac{2a}{\pi\rho}\frac{\Delta}{\Delta_0}\left\{\frac{\Delta_0^2}{(\varepsilon_f-\omega)^2}\frac{1}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}}\right.$$

$$+\frac{2\Delta_0^2}{(\varepsilon_f-\omega)^2}\frac{1}{\beta\varepsilon_f}\frac{\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\pi^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}^2}$$

$$\left.-\frac{4\Delta_0^2}{(\varepsilon_f-\omega)^3}\frac{1}{\beta}\frac{1}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}}\right\}. \quad \text{(C9)}$$

Combining (C5)–(C9) gives (3.5) of the text.

Finally, we list the contribution of the zero-frequency sector Fig. 7,

$$R^1_{7(a)}(0)=\frac{2a}{\pi\rho}\frac{\Delta_0}{\Delta}\frac{1}{4}\frac{\Delta_0^2}{(\varepsilon_f-\omega)^2}\frac{1}{\beta\varepsilon_f}\frac{\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\pi^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}^2}, \tag{C10}$$

$$R^1_{7(b)}(0)=\frac{2a}{\pi\rho}\frac{\Delta}{\Delta_0}\frac{1}{2}\frac{\Delta_0^2}{(\varepsilon_f-\omega)^2}\frac{1}{\beta\varepsilon_f}\frac{\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\pi^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}^2}, \tag{C11}$$

$$R^1_{7(c)}(0)=\frac{2a}{\pi\rho}\frac{\Delta}{\Delta_0}\frac{1}{4}\frac{\Delta_0^2}{(\varepsilon_f-\omega)^2}\frac{1}{\beta\varepsilon_f}\frac{\left[\frac{N\Delta_0}{\pi\varepsilon_f}\right]\left[1+\pi^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}^2}, \tag{C12}$$

$$R^1_{7(d)}(0)=\frac{2a}{\pi\rho}\frac{\Delta}{\Delta_0}\frac{1}{2}\frac{\Delta_0^2}{(\varepsilon_f-\omega)^2}\frac{1}{\beta\varepsilon_f}\frac{\left[\frac{N\Delta_0}{\pi\varepsilon_f}\right]\left[1+\pi^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}^2}, \tag{C13}$$

$$R^1_{7(e)}(0)=-\frac{2a}{\pi\rho}2\left[\frac{\Delta}{\Delta_0}\right]\frac{\Delta_0^2}{(\varepsilon_f-\omega)^3}\frac{1}{\beta}\frac{1}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}}, \tag{C14}$$

$$R^1_{7(f)}(0)=-\frac{2a}{\pi\rho}\left[\frac{\Delta}{\Delta_0}\right]\frac{\Delta_0^2}{(\varepsilon_f-\omega)^3}\frac{1}{\beta}\frac{1}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}}, \tag{C15}$$

$$R^1_{7(g)}(0)=-\frac{2a}{\pi\rho}\left[\frac{\Delta}{\Delta_0}\right]\frac{\Delta_0^2}{(\varepsilon_f-\omega)^3}\frac{1}{\beta}\frac{1}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}}. \tag{C16}$$

These terms sum to

$$R^1(0)=\frac{2a}{\pi\rho}\frac{\Delta}{\Delta_0}\left[\frac{3}{2}\frac{\Delta_0^2}{(\varepsilon_f-\omega)^2}\frac{1}{\beta\varepsilon_f}\frac{\left[\frac{N\Delta_0}{\pi\varepsilon_f}\right]\left[1+\pi^2\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]}{\left\{1+\left[\frac{N\Delta_0}{\pi\varepsilon_f}\right]\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}^2}\right.$$
$$\left.-\frac{4\Delta_0^2}{(\varepsilon_f-\omega)^3}\frac{1}{\beta}\frac{1}{\left\{1+\frac{N\Delta_0}{\pi\varepsilon_f}\left[1+\frac{\pi^2}{3}\left[\frac{k_BT}{\varepsilon_f}\right]^2\right]\right\}}\right]. \tag{C17}$$

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[1]For a recent review, see P. A. Lee, T. M. Rice, J. W. Serene, L. J. Sham, and J. W. Wilkins, Comments Condensed Matter Phys. **12**, 99 (1986).

[2]Proceedings of the 4th International Conference on Valence Fluctuations, edited by E. Müller Hartmann, B. Roden, and D. Wohlleben [J. Magn. Magn. Mater. **47-48** (1985)].

[3]P. W. Anderson and T. V. Ramakrishnan, in *Valence Fluctuations in Solids,* edited L. M. Falicov, W. Hanke, and M. B. Maple (North-Holland, Amsterdam, 1981); T. V. Ramakrishnan, *ibid.* T. V. Ramakrishnan and K. Sur, Phys. Rev. B **26**,

1798 (1982).
[4]O. Gunnarsson and K. Schönhammer, Phys. Rev. B **28**, 4315 (1983).
[5]N. Read and D. M. Newns, J. Phys. C **16**, 3273 (1983).
[6]N. Read and D. M. Newns, J. Phys. C **16**, L1055 (1983).
[7]F. C. Zhang and T. K. Lee, Phys. Rev. B **28**, 33 (1983).
[8]J. W. Rasul and A. C. Hewson, J. Phys. C **16**, L933 (1983); **17**, 2555 (1984); **17**, 3337 (1984).
[9]B. Coqblin and J. R. Schrieffer, Phys. Rev. **185**, 847 (1969).
[10](a) P. Coleman, Phys. Rev. B **29**, 3035 (1984); (b) see also S. Barnes, J. Phys. F **6**, 1375 (1975); **7**, 2637 (1977).
[11]N. Read, J. Phys. C **18**, 2651 (1985).
[12]N. Read, D. M. Newns, and S. Doniach, Phys. Rev. B **30**, 384 (1984); D. M. Newns, N. Read, and A. C. Hewson, in *Moment Formation in Solids,* edited by W. J. L. Buyers (Plenum, New York, 1984); N. Read and D. M. Newns, Solid State Commun. **52**, 993 (1984).
[13]D. M. Newns and A. C. Hewson, J. Phys. F **10**, 2429 (1980).
[14]P. Nozières and A. Blandin, J. Phys. (Paris) **41**, 193 (1980).
[15]G. D. Mahan, *Many Particle Physics* (Plenum, New York, 1981), pp. 592—597.
[16]D. C. Langreth, Phys. Rev. **150**, 712 (1966).
[17]A. Yoshimori, Prog. Theor. Phys. **55**, 67 (1976).
[18]P. Nozières, J. Low Temp. Phys. **17**, 31 (1974).
[19]N. E. Bickers, D. L. Cox, and J. W. Wilkins, Phys. Rev. Lett. **54**, 230 (1985).
[20]N. Grewe, Z. Phys. B **52**, 193 (1982); Y. Kuramoto, *ibid.* **53**, 37 (1983).
[21](a) H. Kojina, Y. Kuramoto, and M. Tachiki, Z. Phys. B **54**, 95 (1984); Y. Kuramoto and H. Kojima, *ibid.* **57** (1984); (b) E. Müller-Hartmann, *ibid.* **57**, 281 (1984).
[22]N. Read (unpublished).
[23]F. D. M. Haldane, J. Phys. C **11**, 5015 (1978).
[24]N. Read (unpublished).
[25]P. Coleman, J. Magn. Magn. Mater. **47-48**, 2 (1985).
[26]P. Coleman and N. Andrei, J. Phys. C **19**, 3211 (1986).
[27]N. Read and D. M. Newns (unpublished); D. L. Cox (unpublished and private communication).
[28]N. Read, Ph.D. thesis, University of London, 1986 (unpublished), Chap. 4, Sec. 6.