# Field-theoretic formulation of the randomly diluted nonlinear resistor network

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A field-theoretic formulation is used to describe the resistive properties of a randomly diluted network consisting of nonlinear conductances for which $V \sim I^r$. The nonlinear resistance $R(\mathbf{x},\mathbf{x}')$ between sites $\mathbf{x}$ and $\mathbf{x}'$ is expressed in terms of an analytic continuation in an associated crossover field. The renormalization-group recursion relations are analyzed within this analytic continuation to order $\epsilon = 6-d$, where $d$ is the spatial dimension. For $r$ near unity a perturbative calculation to first order in $(r-1)$ agrees with both the result obtained here for general $r$ and with the approximate relation proposed by de Arcangelis *et al.* between the nonlinear conductivity and the noise characteristics of a linear network. For arbitrary $r$ and $d$ a generalization of this perturbative treatment gives $(r+1)d\phi(r)/dr = \partial\psi(q,r)/\partial q \mid_{q=1}$, where $\phi(r)$ is the resistance crossover exponent and $\psi(q,r)$ a generalized noise crossover exponent associated with $\mid \partial R/\partial\sigma_b \mid^q$, both quantities referred to the nonlinear system, where $\sigma_b$ is the conductance of an individual bond. For $r$ not near unity our results to first order in $\epsilon$ for $\phi(r)$ and $\psi(q,r)$ satisfy the above relation but not that of de Arcangelis *et al.* For $q=0$, $\psi(q,r)/\nu_p$ is the fractal dimension of the backbone, where $\nu_p$ is the correlation length exponent for percolation. As is known, $\phi(0)/\nu_p$ is an exponent associated with the chemical length, for which our result agrees with that given by Cardy and Grassberger and by Janssen.

## I. INTRODUCTION

Although much progress has been made in recent years in understanding the critical properties of random resistor networks, relatively little is known about nonlinear networks. Here we consider the model of a nonlinear network as proposed by Kenkel and Straley[1] for which each circuit element obeys a nonlinear generalization of Kirchoff's Law, which can be written in either of two equivalent ways as:

$$[V(\mathbf{x})-V(\mathbf{x}')] = I_{\mathbf{x}\to\mathbf{x}'} \mid I_{\mathbf{x}\to\mathbf{x}'} \mid^{(r-1)} r_b \;, \tag{1.1a}$$

$$\sigma_b \,[V(\mathbf{x})-V(\mathbf{x}')] \mid V(\mathbf{x})-V(\mathbf{x}') \mid^{s-1} = I_{\mathbf{x}\to\mathbf{x}'} \;, \tag{1.1b}$$

where $\sigma_b$ ($r_b$) is the nonlinear conductance (resistance) of the bond $b$ connecting sites $\mathbf{x}$ and $\mathbf{x}'$, $r$ and $s$ are the exponents describing the nonlinearity with $s=r^{-1}$, and $I_{\mathbf{x}\to\mathbf{x}'}$ is the current in the bond flowing from site $\mathbf{x}$ to site $\mathbf{x}'$. We will consider a version of this model in which each bond randomly is present (with conductance $\sigma_0$) with probability $p$ and is absent (with zero conductance) with probability $1-p$. Such a model has also been studied recently by Blumenfeld and Aharony[2] on fractal lattices. From an analysis of a fractal nonlinear network, de Arcangelis *et al.*[3] have proposed an approximate relation between exponents of the *linear* network which describe the critical behavior of noise characteristics[4] (or of the current distribution[5]) and the exponent for the conductivity, $\Sigma$, of the *nonlinear* network.

The nonlinear conductivity,[1] $\Sigma$, may be defined in terms of the current $I$ which results from imposing a potential difference $\Delta V$ between opposite faces of a hypercube of side $L$ in $d$ spatial dimensions as

$$\Sigma \mid \Delta V/L \mid^{s} = I/L^{d-1} \;. \tag{1.2}$$

For $p$ near the percolation[6] threshold at $p_c$ the critical exponent associated with $\Sigma$ is defined via

$$\Sigma(p) \sim \mid p-p_c \mid^{t(r)} \tag{1.3}$$

for a given nonlinear exponent $r$. It can be shown,[1] using the node-link picture,[7,8] that

$$t(r) = (d-1-r^{-1})\nu_p + r^{-1}\phi(r), \tag{1.4}$$

where $\nu_p$ is the correlation-length exponent for percolation and $\phi(r)$ is a crossover exponent associated with the scaling behavior of the two-point resistance, defined in Eq. (1.8), below. For linear networks on percolating clusters, Eq. (1.4) reduces to the familiar result,[7–12] $t=(d-2)\nu_p+\phi$.

In any configuration of occupied and unoccupied bonds we define the two-point resistance $R(\mathbf{x},\mathbf{x}')$ as follows. We solve the circuit equations

$$\sum_{\mathbf{y}'} \sigma_{\mathbf{y},\mathbf{y}'} \,[V(\mathbf{y})-V(\mathbf{y}')] \mid V(\mathbf{y})-V(\mathbf{y}') \mid^{s-1} = I_{\mathrm{ext}}(\mathbf{y}) \;, \tag{1.5}$$

when an external current $I_{\mathrm{ext}}$ is put into the network at site $\mathbf{x}$ and taken out at site $\mathbf{x}'$:

$$I_{\mathrm{ext}}(\mathbf{y}) = I_0(\delta_{\mathbf{y},\mathbf{x}} - \delta_{\mathbf{y},\mathbf{x}'}). \tag{1.6}$$

Then $R(\mathbf{x},\mathbf{x}')$ is defined to be

$$R(\mathbf{x},\mathbf{x}') = [V(\mathbf{x})-V(\mathbf{x}')]/I_0^r. \tag{1.7}$$

We may consider the average resistance between sites known to be in the same cluster. Denoting this quantity[13] by $\overline{R}$ we have

 

$$\bar{R}(\mathbf{x},\mathbf{x}') \sim (\sigma_0)^{-r} \, |(\mathbf{x}-\mathbf{x}')/a|^{\phi(r)/\nu_p} \times f_\phi[(\mathbf{x}-\mathbf{x}')/\xi_p] , \quad (1.8)$$

where $\xi_p$ is the percolation correlation length, $f_\phi$ is some scaling function, and $a$ is the lattice constant, which henceforth we take to be unity. The principle objective of this paper is to establish a field-theoretic formulation from which we can obtain a controlled calculation of $\phi(r)$.

As we have mentioned, de Arcangelis *et al.*[3] have proposed that resistive properties of the nonlinear network are qualitatively related to suitable noise characteristics of linear ($r=1$) networks. The noise characteristics are expressed in terms of the distribution of moments of the currents in individual bonds,[4] $i_b$. Using Cohn's theorem,[14] one can express these currents for a linear network as

$$i_b^2 = -\left[\frac{\partial \bar{R}(\mathbf{y},\mathbf{y}')}{\partial \sigma_b}\right] \sigma_0^2 I_0^2 , \quad (1.9)$$

and the results of the preceding paper[13] for a linear ($r=1$) network may be written as

$$\left|\frac{\sigma_0^2 \partial \bar{R}(\mathbf{y},\mathbf{y}')}{\partial \sigma_{\mathbf{x},\mathbf{x}'}}\right|^q \sim \xi_p^{\psi(q,1)/\nu_p - d} F_\psi^{(q,1)}\left[\frac{\mathbf{r}}{\xi_p}\right] \times X_p\left[\frac{|\mathbf{y}-\mathbf{y}'|}{\xi_p}\right] , \quad (1.10)$$

where $F_\psi^{(q,r)}$ is a scaling function of position coordinates and $\psi(q,r)$ a crossover exponent, both quantities for the network with nonlinear exponent $r$ in Eq. (1.1a). For symmetric fractals de Arcangelis *et al.*[3] found the relation

$$\phi(r) = \psi\left[\frac{r+1}{2},1\right] . \quad (1.11)$$

They noted that this relation would not hold for asymmetric structures, such as we would expect to find within percolation clusters. They speculated that such a breakdown in Eq. (1.11) would be most pronounced for $d=2$. Since we now have (in Ref. 13) a controlled calculation of $\psi(q,1)$ within the $\epsilon=6-d$ expansion, it is of increasing interest to develop a similar calculation for $\phi(r)$ to test the above conjecture of de Arcangelis *et al.*[3] From such a calculation we could tell whether it is permissible to regard the blobs as symmetric for $d=6-\epsilon$. To this end, we give here the first $\epsilon$-expansion calculation for a nonlinear network and the result given in Eq. (3.15), below indicates that Eq. (1.11) is not exact, i.e., the asymmetry of the blobs can not be neglected. Note that in the limit $r\rightarrow 0$ the resistance between two points $\mathbf{x}$ and $\mathbf{x}'$ becomes essentially equal to the length of the shortest path between the two points.[2] The exponent $\nu_{\text{ch}}$ for this so-called "chemical distance" denoted $r_{\text{ch}}$ is defined by $r_{\text{ch}} \sim |\mathbf{x}-\mathbf{x}'|^{\nu_{\text{ch}}/\nu_p}$ for $p \sim p_c$ and our result to order $\epsilon$ is $\nu_{\text{ch}}=1+(\epsilon/28)$, in agreement with $\epsilon$ expansions obtained for $\nu_{\text{ch}}$ by Cardy and Grassberger[15] and also by Janssen.[16] For $r\rightarrow\infty$, Blumenfeld and Aharony[2] and de Arcangelis *et al.*[3] have shown that $\phi(r)\rightarrow 1$.

Since the technique that we use may not be entirely convincing, we also give perturbative analysis for small nonlinearity (i.e., for $r-1 \ll 1$). In the limit $r\rightarrow 1$ this analysis is consistent with both our $\epsilon$-expansion calculation for general $r$ and the relation Eq. (1.11) proposed by de Arcangelis *et al.*[3] However, since our calculation disagrees with Eq. (1.11), we conclude that there is probably no simple relationship between linear and nonlinear networks for arbitrary $r$. This conclusion is supported by a generalization of the perturbative analysis, using which we derive the following relation between the resistivity crossover exponent and the noise exponent for systems having the *same* nonlinearity exponent, $r$:

$$(r+1)d\phi(r)/dr = \partial\psi(q,r)/\partial q\,|_{q=1} . \quad (1.12)$$

This relation holds for arbitrary spatial dimension. Note that our Eq. (1.12) and Eq. (1.11) become equivalent as $r\rightarrow 1$. In order to check that our results do indeed satisfy Eq. (1.12) we calculated the noise exponents, $\psi(q,r)$ for the nonlinear network and these results are given in Eq. (A9) of the Appendix. Hopefully the results presented here can be tested via series calculations or other numerical approaches.

Briefly, this paper is organized as follows. In Sec. II we describe the analytic continuation needed to study the general nonlinear network. The notation and general formalism are the same as that in the preceding paper[13] and are not described in full detail, except where differences arise. In Sec. III a detailed analysis of the analytic continuation of the renormalization group recursion relations to order $\epsilon$ is presented. In Sec. IV the perturbative calculation for small deviations from linearity is given. This calculation is rather straightforward and substantiates the more elusive calculation of Sec. III. This treatment is generalized in Sec. V to yield Eq. (1.12) for arbitrary nonlinearity. In the Appendix the noise crossover exponents are calculated to order $\epsilon$ for arbitrary nonlinearity and are shown to satisfy Eq. (1.12). Finally, in Sec. VI, we summarize our results.

## II. FIELD THEORY FOR NONLINEAR NETWORKS

We wish to develop a field theory from which can be obtained the properties of the solutions to the nonlinear network equations of Eq. (1.5). In analogy with Stephen's procedure for the linear network[10,12,17] we consider the correlation function $G(\mathbf{x},\mathbf{x}';\lambda)$:

$$G(\mathbf{x},\mathbf{x}';\lambda) \equiv \int DVe^{-H(\{V\})}e^{i\lambda[V(\mathbf{x})-V(\mathbf{x}')]} , \quad (2.1)$$

where $DV$ indicates an integration over all variables $\{V(\mathbf{x})\}$ and the "Hamiltonian" is

$$H = \sum_b \frac{1}{s+1}\sigma_b\,|V(\mathbf{y})-V(\mathbf{y}')|^{s+1} , \quad (2.2)$$

where the $\sigma_b$'s are arbitrary. In the limit when $\lambda$ is imaginary and $\lambda^{s+1}$ is much larger in magnitude than any of the $\sigma$'s, the integrand in Eq. (2.1) becomes sharply peaked so that the integral is dominated by contributions from the maximum in the integrand. From the fact that $\delta H/\delta V(\mathbf{y})$ generates the left-hand side of Eq. (1.5), we see that the location of the maximum in the integrand is determined by the solution to the nonlinear circuit equations (1.5). Thus in the asymptotic limit $\lambda=\pm i\lambda_0$ with $\lambda_0\to+\infty$, we have

$$G(\mathbf{x},\mathbf{x}';\lambda)=a\,|H_{VV}|^{-1/2}\exp[\lambda_0^{r+1}R(\mathbf{x},\mathbf{x}')/(r+1)]\ , \tag{2.3}$$

where $R(\mathbf{x},\mathbf{x}')$ is the nonlinear resistance between sites $\mathbf{x}$ and $\mathbf{x}'$, $a$ is an unimportant constant, $H_{VV}$ denotes the determinant of second derivatives at the maximum, and we used $rs=1$. Fluctuation corrections to Eq. (2.3) are of order $\lambda_0^{-(1+r)}$, relative to the dominant term. It should be noted that for nonintegral $s$ it is essential to continue $G$ to the regime where $\lambda$ is large and $arg\lambda\approx\pm\pi/2$. Otherwise the integral in Eq. (2.1) would be dominated by contributions from branch cuts whose relation to the solution to the network equations is totally unclear.

To treat the random network we introduce replicas in the usual way. We introduce an $n$-replicated effective Hamiltonian $H_{\text{eff}}$ given by

$$\exp(-H_{\text{eff}})=\left[\prod_{\alpha=1}^{n}\exp(-H(\{V_\alpha\}))\right]_{\text{av}}\ , \tag{2.4}$$

where $[\ \ ]_{\text{av}}$ indicates an average over all random configurations of occupied and unoccupied bonds. Now we consider the generalized correlation function

$$G(\mathbf{x},\mathbf{x}';\boldsymbol{\lambda})=\frac{\int D\mathbf{V}\,\exp(-H_{\text{eff}})\exp\{i\boldsymbol{\lambda}\cdot[\mathbf{V}(\mathbf{x})-\mathbf{V}(\mathbf{x}')]\}}{\int D\mathbf{V}\,\exp(-H_{\text{eff}})}\ , \tag{2.5}$$

where the vectors $\boldsymbol{\lambda}$ and $\mathbf{V}(\mathbf{x})$ have $n$ components, one for each replica and $D\mathbf{V}\equiv\prod_\alpha DV_\alpha$. In Eq. (2.5) the denominator becomes unity in the limit $n\to0$, which we utilize in the replica scheme. In the limit $n\to0$ we write the analog of Eq. (2.3) for $\lambda_\alpha=i\lambda_{\alpha,0}$ as

$$G(\mathbf{x},\mathbf{x}';\boldsymbol{\lambda})=\left[\exp\left[\sum_{\alpha=1}^{n}\lambda_{\alpha,0}^{r+1}R(\mathbf{x},\mathbf{x}')/(r+1)\right]\right]_{\text{av}}\ ,\quad \lambda_{\alpha,0}\to\infty\ . \tag{2.6}$$

It is not generally possible to consider this $G$ as a small perturbation away from a critical correlation function. This can be traced back to the fact that in order to make the maximum dominate the integral in Eq. (2.1) we had to continue $\lambda$ to large imaginary values. However, utilizing a point of view developed in treating the problem of biconnectedness,[18] we will overcome this difficulty with Eq. (2.6) by evaluating it near the limit when all the components of $\boldsymbol{\lambda}$ are equal. Accordingly, consider the case when $\lambda_\alpha=i\lambda_0$ for all $\alpha$. Then for $\lambda_0\to\infty$ Eq. (2.6) becomes

$$G(\mathbf{x},\mathbf{x}';\boldsymbol{\lambda})=[\exp[n\lambda_0^{r+1}R(\mathbf{x},\mathbf{x}')/(r+1)]]_{\text{av}}\ , \tag{2.7}$$

where $R(\mathbf{x},\mathbf{x}')$ is proportional to $\sigma_0^{-r}$. More generally, we set

$$\lambda_\alpha=i\lambda_0+\xi_\alpha \tag{2.8a}$$

subject to the constraint

$$\sum_{\alpha=1}^{n}\xi_\alpha=0\ . \tag{2.8b}$$

Then in the regime where $\lambda_0$ is near the positive real axis and

$$|\lambda_0^{r+1}|/\sigma_0^r\gg1\ , \tag{2.9a}$$

$$n\,|\lambda_0|^{r+1}/\sigma_0^r\ll1\ , \tag{2.9b}$$

and

$$|\xi^2\lambda_0^{r-1}|/\sigma_0^r\ll1\ , \tag{2.9c}$$

where $\xi^2=\sum_\alpha\xi_\alpha^2$, we have

$$G(\mathbf{x},\mathbf{x}';\boldsymbol{\lambda})=[\exp\{[n\lambda_0^{r+1}/(r+1)-\xi^2r\lambda_0^{r-1}]R(\mathbf{x},\mathbf{x}')\}]_{\text{av}}\ . \tag{2.10a}$$

We interpret this as

$$G(\mathbf{x},\mathbf{x}';\boldsymbol{\lambda})=[\nu(\mathbf{x},\mathbf{x}')\{1+[n\lambda_0^{r+1}/(r+1)-\xi^2r\lambda_0^{r-1}]R(\mathbf{x},\mathbf{x}')\}]_{\text{av}}\ , \tag{2.10b}$$

$$=\left[\nu(\mathbf{x},\mathbf{x}')\left[1+R(\mathbf{x},\mathbf{x}')\sum_\alpha(-i\lambda_\alpha)^{r+1}/(r+1)\right]\right]_{\text{av}}\ , \tag{2.10c}$$

where $\nu(\mathbf{x},\mathbf{x}')$ is the pair connectedness indicator function of percolation.[13] The inequality (2.9a) is essential for the saddle point evaluation as in Eqs. (2.6) and (2.7) to be valid. Inequalities (2.9b) and (2.9c) are invoked to justify the expansion of the exponential as in Eq. (2.10b). This limit then becomes analogous to the limit $\sigma_0\to\infty$, which has been used[10,12] to connect the randomly diluted resistor network with the percolation problem.

To carry out this program we construct $H_{\text{eff}}$ in this limit in terms of the order parameter $\Psi_{\boldsymbol{\lambda}}(\mathbf{x})$ defined as

$$\Psi_{\boldsymbol{\lambda}}(\mathbf{x})=\exp[i\boldsymbol{\lambda}\cdot\mathbf{V}(\mathbf{x})]\ . \tag{2.11}$$

From the definition of $H_{\text{eff}}$ we find that

$$H_{\text{eff}}=-\sum_{\langle\mathbf{x},\mathbf{x}'\rangle}\sum_{\boldsymbol{\lambda}}B_{\boldsymbol{\lambda}}\Psi_{\boldsymbol{\lambda}}(\mathbf{x})\Psi_{-\boldsymbol{\lambda}}(\mathbf{x}')\ , \tag{2.12}$$

where the sum over $\langle\mathbf{x},\mathbf{x}'\rangle$ is carried over pairs of nearest neighboring sites. Here

$$B_\lambda = -\sum_{l=1}^{\infty} \frac{(-1)^l}{l} \left[\frac{p}{1-p}\right]^l \prod_{\alpha=1}^{n} F_l(\lambda_\alpha) , \tag{2.13}$$

where $F$ is given as

$$F_l(\lambda) = \int_{-\infty}^{\infty} \exp\left[-\frac{l\sigma_0}{s+1}|V|^{s+1} - i\lambda V\right] dV . \tag{2.14}$$

We now obtain an asymptotic expansion of $F_l(i\lambda_0)$ for $l$ finite and $\lambda_0$ large and near the positive real axis such that Eq. (2.9a) holds, in which case

$$F_l(i\lambda_0) = \int_{-\infty}^{\infty} \exp\left[-\frac{l\sigma_0}{s+1}|V|^{s+1} + \lambda_0 V\right] dV \tag{2.15a}$$

$$\sim \int \exp\left[-\frac{l\sigma_0}{s+1}V^{s+1} + \lambda_0 V\right] dV , \tag{2.15b}$$

where the integral in Eq. (2.15b) is dominated by the contribution from the region near the maximum of the integrand at $V=(\lambda_0/l\sigma_0)^r \equiv V_0$. We therefore set $V=V_0+x\lambda_0^{(r-1)/2}/(l\sigma_0)^{r/2}$ so that

$$F_l(i\lambda_0) \sim \frac{\lambda_0^{(r-1)/2}}{(l\sigma_0)^{r/2}} \int dx \exp\left[\frac{\lambda_0^{r+1}}{(r+1)(l\sigma_0)^r} - \frac{s}{2}x^2 - \frac{s(s-1)}{3!}\mu x^3 - \frac{s(s-1)(s-2)}{4!}\mu^2 x^4 \cdots\right] , \tag{2.16}$$

where the expansion parameter $\mu$ is

$$\mu^2 = (l\sigma_0)^r/\lambda_0^{r+1} \ll 1 . \tag{2.17}$$

The condition $\mu \ll 1$ is equivalent to Eq. (2.9a). Thus we have

$$F_l(i\lambda_0) \sim c\frac{\lambda_0^{(r-1)/2}}{\sigma_0^{r/2}} \exp\left[\frac{\lambda_0^{r+1}}{(r+1)(l\sigma_0)^r}\right] . \tag{2.18}$$

We now consider the evaluation of $B_\lambda$ for $\lambda$ as in Eq. (2.8). Then we find that $B_\lambda$ is of the form

$$B_\lambda \sim B_0 - \sum_{k=1}^{\infty} a_k \left[\sum_\alpha (-i\lambda_\alpha)^{r+1}\right]^k , \tag{2.19}$$

with coefficients $a_k$, which we do not evaluate.

The percolation critical point occurs for a critical value of $B_0$, denoted $B_{0,c}$, of order[10] $1/z$, where $z$ is the coordination number of the lattice. In order to obtain the resistive behavior from the properties of this critical point, we should arrange for $B_\lambda$ to differ only perturbatively from $B_0$. To satisfy this condition we restrict $\lambda$ to be in the regime of Eq. (2.9). In the field-theoretic formulation[10,12,13,17] the inverse propagator $r(\lambda)$ has a bare value $r_0(\lambda) \sim (zB_\lambda)^{-1} - 1$. From Eq. (2.19) we see that near criticality we have

$$r_0(\lambda) \approx r_0(0) - \sum_{k=1}^{\infty} w_k \left[\sum_\alpha (-i\lambda_\alpha)^{r+1}\right]^k , \tag{2.20}$$

where the $w_k$'s are constants. By comparison with Eq. (2.10c) we see that the crossover exponent associated with $w_1 \sim \sigma_0^{-r}$ is identified as $\phi(r)$ of Eq. (1.8).

## III. $\epsilon$ EXPANSION FOR THE NONLINEAR NETWORK

In this section we will study the analytic continuation needed to generate the two-point resistance. To obtain the correlation function required in Eq. (2.5) it suffices to obtain the propagator in the replicated field theory for the randomly diluted network. The recursion relation[19,20] for the inverse propagator $r(\lambda)$ has been used in several calculations[12,13,17] and for convenience we start our analysis from Eqs. (4.1) and (4.2) of the preceding paper,[13] which we write as

$$\frac{dr(\lambda)}{dl} = (2-\eta_p)r(\lambda) - g\Sigma(\lambda) , \tag{3.1}$$

where $\eta_p$ is the exponent describing critical correlations at the percolation threshold, $g$ is a coupling constant whose fixed point value is $\epsilon/7$, and $\Sigma(\lambda)$ is given as

$$\Sigma(\lambda) = -2G(\lambda)G(0) + \sum_\tau G(\lambda-\tau)G(\tau), \tag{3.2}$$

$$\equiv -2G(\lambda)G(0) + \tilde{\Sigma}(\lambda) , \tag{3.3}$$

where $G(\lambda)$ is the propagator which we take in the form

$$G(\lambda)^{-1} = 1 + r(0) + \delta r(\lambda) . \tag{3.4}$$

For notational convenience we will set $r_0 = r(0)$. In view of the discussion in Sec. II, we will consider the analytic continuation of this recursion relation for $\lambda$ of the form of Eq. (2.8) in the regime described by Eq. (2.9). As mentioned after Eq. (2.20), to determine $\phi(r)$ we may set

$$\delta r(\lambda) = -w_1 \sum_\alpha (-i\lambda_\alpha)^{r+1} . \tag{3.5}$$

The evaluation of $\tilde{\Sigma}$ in Eq. (3.2b) now is carried out along the lines of the Appendix in the preceding paper.[13] We start by expressing Eq. (3.2) as

$$\tilde{\Sigma}(\lambda) = \int_0^\infty du \int_0^\infty dv \int_{-\infty}^{\infty} D\tau \exp[-uG^{-1}(\tfrac{1}{2}\lambda+\tau) - vG^{-1}(\tfrac{1}{2}\lambda-\tau)] \tag{3.6a}$$

$$= \int_0^\infty du \int_0^\infty dv \exp[-(u+v)(1+r_0)] \int_{-\infty}^{\infty} D\tau \exp[-u\delta r(\tfrac{1}{2}\lambda+\tau) - v\delta r(\tfrac{1}{2}\lambda-\tau)] , \tag{3.6b}$$

where $D\tau \equiv \prod_\alpha d\tau_\alpha$. One should recall that we are interested in $\boldsymbol{\lambda}$ of the form of Eqs. (2.8) and (2.9). Then we may use Eq. (3.5) with appropriate arguments to evaluate $\delta r(\frac{1}{2}\boldsymbol{\lambda} \pm \boldsymbol{\tau})$, so that

$$\tilde{\Sigma}(\boldsymbol{\lambda}) = \int_0^\infty du \int_0^\infty dv \exp[-(u+v)(1+r_0)] \prod_{\alpha=1}^n I(\lambda_\alpha) , \tag{3.7}$$

where

$$I(\lambda) = \int d\tau \exp[w_1 u(-\tfrac{1}{2}i\lambda + i\tau)^{r+1} + w_1 v(-\tfrac{1}{2}i\lambda - i\tau)^{r+1}] \tag{3.8a}$$

$$\equiv \int d\tau \exp[H(\tau)] . \tag{3.8b}$$

Since $w_1 \lambda_0^{1+r}$ is large, we evaluate $I(\lambda)$ by steepest descents. The saddle point in the integral is at $\tau = \tau_0$ determined by $H'(\tau) = 0$ or

$$u(-\tfrac{1}{2}i\lambda + i\tau_0)^r = v(-\tfrac{1}{2}i\lambda - i\tau_0)^r \tag{3.9a}$$

so that

$$\tau_0 = \tfrac{1}{2}\lambda[u^s - v^s]/[u^s + v^s] . \tag{3.9b}$$

For $w_1 | \lambda_0^{r+1} | \sim | \lambda_0^{r+1} | / \sigma_0^r \gg 1$, we may neglect higher-than quadratic fluctuations about the saddle-point, so that

$$I(\lambda) = \{r(r+1)w_1(uv)^{1-s}(-i\lambda)^{r-1}/[2\pi(u^s+v^s)^{r-2}]\}^{-1/2} \times \exp[w_1(-i\lambda)^{r+1}uv/(u^s+v^s)^r] . \tag{3.10}$$

When this result is substituted into Eq. (3.7), the dominant terms come from the exponential and are of order $I(\lambda)^n \sim 1 + n w_1 \lambda_0^{r+1}$. The contributions from the prefactor to the exponential in Eq. (3.10) lead to terms in Eq. (3.7) of order $n \ln \lambda_0$ which we neglect. In addition, we neglect terms of order $r_0$. Thus we have

$$\tilde{\Sigma} = \int_0^\infty du \int_0^\infty dv \exp[-(u+v)] \exp\left[ w_1 \sum_{\alpha=1}^n (-i\lambda_\alpha)^{r+1} uv/[u^s+v^s]^r \right] . \tag{3.11}$$

Therefore we obtain

$$\tilde{\Sigma} \sim 1 + \left[ w_1 \sum_{\alpha=1}^n (-i\lambda_\alpha)^{r+1} \right] c(r) , \tag{3.12}$$

where

$$c(r) = \tfrac{1}{2} \int_{-1}^1 d\xi \frac{(1-\xi^2)}{[(1+\xi)^{1/r} + (1-\xi)^{1/r}]^r} . \tag{3.13}$$

From Eq. (3.12) we write the recursion relation for $w_1$ as

$$\frac{\partial w_1}{\partial l} = [1 + \tfrac{1}{2} g c(r)] \frac{w_1}{\nu_p} \equiv \phi(r) \frac{w_1}{\nu_p} . \tag{3.14}$$

Inserting the fixed point value $g = \epsilon/7$ we have

$$\phi(r) = 1 + \frac{\epsilon}{14} c(r) . \tag{3.15}$$

From $\phi(r)$ one obtains the scaling of the two-point resistance as in Eq. (1.8) and the conductivity exponent as in Eq. (1.4).

Several special cases of this result may be noted. In order to check this result we perform, in the next section, a perturbative calculation of $\phi(r)$ for $r$ near 1, i.e. for the nearly linear network. For $r = 1 + \delta$ with $\delta \ll 1$, Eq. (3.15) gives

$$\phi(r) = 1 + \frac{\epsilon}{42} \left[ 1 - \frac{7\delta}{12} \right] . \tag{3.16}$$

A more general test of this type is discussed in Eq. (5.12), below. Also, we note that for $r \to 0$, the resistance between two points, $\mathbf{x}$ and $\mathbf{x}'$, becomes equal to the "chemical distance" or length of the shortest path, $r_{ch}$, between the two points.[2] Thus we write

$$r_{ch} \sim | \mathbf{x} - \mathbf{x}' |^{\phi(0)/\nu_p} . \tag{3.17}$$

and Eq. (3.15) gives $\phi(0) = 1 + (\epsilon/28)$. This result agrees with the previous work of Cardy and Grassberger[15] and of Janssen.[16] Finally, Eq. (3.15) gives the limit $\phi(\infty) = 1$, as expected.[2,3]

## IV. PERTURBATIVE TREATMENT OF THE ALMOST LINEAR NETWORK

We now study the circuit equations in the form

$$\sum_{\mathbf{x}'} \sigma_{\mathbf{x},\mathbf{x}'} [V(\mathbf{x}) - V(\mathbf{x}')] \, | V(\mathbf{x}) - V(\mathbf{x}') |^{-\delta} = I_{ext}(\mathbf{x}), \tag{4.1}$$

where $s = (1-\delta)$. To calculate $R(\mathbf{y},\mathbf{y}')$ we take the source terms to be $I_{ext}(\mathbf{x}) = I(\delta_{\mathbf{x},\mathbf{y}} - \delta_{\mathbf{x},\mathbf{y}'})$. We consider the case $\delta \ll 1$ so that $r \approx 1 + \delta$. Note, however, that throughout this section the spatial dimension, $d$, is arbitrary. We have, correct to first order in $\delta$:

$$\sum_{\mathbf{x}'} \sigma_{\mathbf{x},\mathbf{x}'} [V(\mathbf{x}) - V(\mathbf{x}')][1 - \delta \ln | V(\mathbf{x}) - V(\mathbf{x}') | ] = I_{ext}(\mathbf{x}), \tag{4.2}$$

which we write as

$$\sum_{\mathbf{x}'} \sigma_{\mathbf{x},\mathbf{x}'} [V(\mathbf{x}) - V(\mathbf{x}')] = J(\mathbf{x}), \tag{4.3}$$

where

$$J(\mathbf{x}) = I_{ext}(\mathbf{x}) + \delta \sum_{\mathbf{x}'} \sigma_{\mathbf{x},\mathbf{x}'} [V(\mathbf{x}) - V(\mathbf{x}')] \times \ln | V(\mathbf{x}) - V(\mathbf{x}') | . \tag{4.4}$$

For $\delta = 0$ we have the usual solution denoted $V^{(0)}$:

$$V^{(0)}(\mathbf{x})=[G(\mathbf{x},\mathbf{y})-G(\mathbf{x},\mathbf{y}')]I, \tag{4.5}$$

where $G(\mathbf{x},\mathbf{x}')$ is the Green's function for the circuit equations. A solution to Eq. (4.3) correct to first order in $\delta$ is obtained by iteration:

$$V(\mathbf{x})=\sum_{\mathbf{x}'} G(\mathbf{x},\mathbf{x}')J^{(0)}(\mathbf{x}'), \tag{4.6}$$

where $J^{(0)}(\mathbf{x})$ is the value of $J(\mathbf{x})$ for $V=V^{(0)}$.

In this way we obtain

$$V(\mathbf{y})-V(\mathbf{y}')=R^{(0)}(\mathbf{y},\mathbf{y}')I+\tfrac{1}{2}\delta\sum_{\mathbf{x},\mathbf{x}'}\sigma_{\mathbf{x},\mathbf{x}'}I[G(\mathbf{y},\mathbf{x})-G(\mathbf{y}',\mathbf{x})-G(\mathbf{y},\mathbf{x}')+G(\mathbf{y}',\mathbf{x}')]^2 \times \ln[\,|G(\mathbf{y},\mathbf{x})-G(\mathbf{y}',\mathbf{x})-G(\mathbf{y},\mathbf{x}')+G(\mathbf{y}',\mathbf{x}')|\,I\,]\,, \tag{4.7}$$

where $R^{(0)}$, the two-point resistance in the linear network, is

$$R^{(0)}(\mathbf{y},\mathbf{y}')=[G(\mathbf{y},\mathbf{y})-G(\mathbf{y}',\mathbf{y})-G(\mathbf{y},\mathbf{y}')+G(\mathbf{y}',\mathbf{y}')]\,. \tag{4.8}$$

Cohn's theorem[14] in this notation is

$$\frac{\partial R^{(0)}(\mathbf{y},\mathbf{y}')}{\partial\sigma_{\mathbf{x},\mathbf{x}'}} = -[G(\mathbf{y},\mathbf{x})-G(\mathbf{y}',\mathbf{x})-G(\mathbf{y},\mathbf{x}')+G(\mathbf{y}',\mathbf{x}')]^2, \tag{4.9}$$

so that we have

$$V(\mathbf{y})-V(\mathbf{y}')=R^{(0)}(\mathbf{y},\mathbf{y}')I-\frac{\delta}{4}\sum_{\mathbf{x},\mathbf{x}'}\sigma_{\mathbf{x},\mathbf{x}'}\frac{\partial R^{(0)}(\mathbf{y},\mathbf{y}')}{\partial\sigma_{\mathbf{x},\mathbf{x}'}}I\ln\left[-\frac{\partial R^{(0)}(\mathbf{y},\mathbf{y}')}{\partial\sigma_{\mathbf{x},\mathbf{x}'}}I^2\right]. \tag{4.10}$$

If the conductances $\sigma_b$ of every bond $b$ are incremented by the same fraction, the effect on the two-point resistance is obtained by linearity. Thus

$$R^{(0)}(\mathbf{y},\mathbf{y}')=-\sum_b\frac{\partial R^{(0)}(\mathbf{y},\mathbf{y}')}{\partial\sigma_b}\sigma_b\,. \tag{4.11}$$

In view of this relation we write Eq. (4.10) as

$$V(\mathbf{y})-V(\mathbf{y}')=-\sum_b\sigma_b\left[1+\tfrac{1}{2}\delta\ln\left[-\frac{\partial R^{(0)}(\mathbf{y},\mathbf{y}')}{\partial\sigma_b}I^2\right]\right]\frac{\partial R^{(0)}(\mathbf{y},\mathbf{y}')}{\partial\sigma_b}I \tag{4.12a}$$

$$\approx\sum_b\sigma_b\left[-\frac{\partial R^{(0)}(\mathbf{y},\mathbf{y}')}{\partial\sigma_b}\right]^{1+(\delta/2)}I^{1+\delta}\,, \tag{4.12b}$$

where we used $\sum_{\mathbf{x},\mathbf{x}'}=2\sum_b$. In accord with Eq. (1.7) we have the nonlinear resistance as

$$R(\mathbf{y},\mathbf{y}')\sim\sum_b\sigma_b\left[-\frac{\partial R^{(0)}(\mathbf{y},\mathbf{y}')}{\partial\sigma_b}\right]^{1+(\delta/2)}. \tag{4.13}$$

From the accompanying paper[13] we have the scaling behavior

$$\left[\nu(\mathbf{y},\mathbf{y}')\left|\frac{\partial R^{(0)}(\mathbf{y},\mathbf{y}')}{\partial\sigma_{\mathbf{x},\mathbf{x}'}}\right|^q\right]_{av}[\chi_p(\mathbf{y},\mathbf{y}')]^{-1} \sim \xi_p^{[\psi(q)/\nu_p-d]}F_\psi^{(q)}\left[\frac{\mathbf{r}}{\xi_p}\right], \tag{4.14}$$

where $\chi_p(\mathbf{y},\mathbf{y}')\equiv[\nu(\mathbf{y},\mathbf{y}')]_{av}$ is the susceptibility function for percolation, $F$ a scaling function of the spatial coordinates, and $\psi(q)$ a crossover exponent for the linear network calculated[13] to first order in $\epsilon=6-d$ as $\psi(q)=1+\epsilon/[7(2q+1)(q+1)]$. Combining Eqs. (4.13) and (4.14) we have

$$[\nu(\mathbf{y},\mathbf{y}')R(\mathbf{y},\mathbf{y}')]_{av}/\chi_p(\mathbf{y},\mathbf{y}') \sim \xi_p^{\psi[1+(\delta/2)]/\nu_p}. \tag{4.15}$$

Since we also identify the left-hand side of this equation as $\xi_p^{\phi(r=1+\delta)/\nu_p}$, we have the nonlinear resistive cross-over exponent to first order in the nonlinearity $\delta$ and $\epsilon$ as

$$\phi(r=1+\delta)\sim\psi(1+\tfrac{1}{2}\delta) \tag{4.16a}$$

$$=1+\frac{\epsilon}{42}\left[1-\frac{7}{12}\delta\right]. \tag{4.16b}$$

This result agrees with that, Eq. (3.16) found via the $\epsilon$ expansion.

The importance of the above analysis is that it shows that our procedure of analytic continuation of the recursion relations can be substantiated to first order in $r-1$.

## V. GENERAL RELATION BETWEEN NONLINEARITY AND RESISTANCE EXPONENTS

In this section we obtain a general relation between the resistance crossover exponent $\phi(r)$ and the generalized noise exponent, $\psi(q,r)$ for a system with nonlinear exponent $r$. In the Appendix we show that our $\epsilon$-expansion results satisfy this relation.

We calculate the nonlinear resistance between sites $\mathbf{x}$ and $\mathbf{x}'$ for the nonlinearity exponent $s$ of Eq. (1.5) assuming the value $s+\Delta s$. To do this we write the circuit equations as

$$\sum_{\mathbf{y}'} \sigma_{\mathbf{y},\mathbf{y}'}[V(\mathbf{y})-V(\mathbf{y}')]\,|V(\mathbf{y})-V(\mathbf{y}')|^{s-1+\Delta s}=I_{\rm ext}(\mathbf{y})\ , \tag{5.1}$$

where $I_{\rm ext}(\mathbf{y})$ is given by Eq. (1.6). For small $\Delta s$ we have

$$\sum_{\mathbf{y}'} \sigma_{\mathbf{y},\mathbf{y}'}[V(\mathbf{y})-V(\mathbf{y}')]\,|V(\mathbf{y})-V(\mathbf{y}')|^{s-1}[1+\Delta s \ln|V(\mathbf{y})-V(\mathbf{y}')|\,]=I_{\rm ext}(\mathbf{y})\ . \tag{5.2}$$

Using Eq. (1.7) for $\Delta s=0$, we obtain

$$V(\mathbf{x})-V(\mathbf{x}')=I_0^r\left[R(s;\mathbf{x},\mathbf{x}')+\tfrac{1}{2}\sum_{\mathbf{y},\mathbf{y}'}\frac{\partial R(s,\mathbf{x},\mathbf{x}')}{\partial \sigma_{\mathbf{y},\mathbf{y}'}}\sigma_{\mathbf{y},\mathbf{y}'}\Delta s\ \ln|V(\mathbf{y})-V(\mathbf{y}')|\right], \tag{5.3}$$

where the first argument in $R$ is the value of the nonlinear exponent, $s$ and the factor $\frac{1}{2}$ is included to count each bond $\mathbf{y},\mathbf{y}'$ once. For a given configuration, the Lagrangian corresponding to Eq. (2.1) is

$$L=\frac{1}{2(s+1)}\sum_{\mathbf{y},\mathbf{y}'}\sigma_{\mathbf{y},\mathbf{y}'}\ |V(\mathbf{y})-V(\mathbf{y}')|^{s+1}-\sum_{\mathbf{y}}I_{\rm ext}(\mathbf{y})V(\mathbf{y})\ . \tag{5.4}$$

When this Lagrangian is evaluated at the maximum of the integrand in Eq. (2.1), i.e. when the circuit equations are solved, then we have

$$L=-\frac{s}{s+1}R(s,\mathbf{x},\mathbf{x}')I_0^{r+1}\ . \tag{5.5}$$

We now differentiate $L$ with respect to $\sigma_{\mathbf{y},\mathbf{y}'}$. In so doing, we should, in principle, take account of the fact that the voltages are implicit functions of $\sigma_{\mathbf{y},\mathbf{y}'}$. However, the resulting terms vanish due to the fact that $\partial L/\partial V(\mathbf{y})=0$ is equivalent to the circuit equations. From Eqs. (5.4) and (5.5) we find that

$$-|V(\mathbf{y})-V(\mathbf{y}')|^{s+1}=sI_0^{r+1}\ \frac{\partial R(s,\mathbf{x},\mathbf{x}')}{\partial \sigma_{\mathbf{y},\mathbf{y}'}}\ . \tag{5.6}$$

We substitute this position dependence of $V(\mathbf{y})$ into Eq. (5.3) to get

$$V(\mathbf{x})-V(\mathbf{x}')=I_0^r\left[R(s;\mathbf{x},\mathbf{x}')+\tfrac{1}{2}\sum_{\mathbf{y},\mathbf{y}'}\frac{\partial R(s,\mathbf{x},\mathbf{x}')}{\partial \sigma_{\mathbf{y},\mathbf{y}'}}\frac{\sigma_{\mathbf{y},\mathbf{y}'}\Delta s}{s+1}\ln\left[-\frac{\partial R(s,\mathbf{x},\mathbf{x}')}{\partial \sigma_{\mathbf{y},\mathbf{y}'}}sI_0^{r+1}\right]\right]. \tag{5.7}$$

Homogeneity gives

$$\sum_b \frac{\partial R(s,\mathbf{x},\mathbf{x}')}{\partial \sigma_b}\sigma_b=-\frac{1}{s}R(s,\mathbf{x},\mathbf{x}')\ , \tag{5.8}$$

so that

$$V(\mathbf{x})-V(\mathbf{x}')=-I_0^r\sum_b\left\{\left[\frac{\partial R(s,\mathbf{x},\mathbf{x}')}{\partial \sigma_b}\right]\sigma_b\left[s-\frac{\Delta s}{s}\ln I_0-\frac{\Delta s}{s+1}\ln s-\frac{\Delta s}{s+1}\ln\left[-\frac{\partial R(s,\mathbf{x},\mathbf{x}')}{\partial \sigma_b}\right]\right]\right\}. \tag{5.9}$$

To order $\Delta s$ we write this as

$$R(s+\Delta s,\mathbf{x},\mathbf{x}')\equiv[V(\mathbf{x})-V(\mathbf{x}')]I_0^{-(r+\Delta r)}=\sum_b\sigma_b\left[-s\,\frac{\partial R(s;\mathbf{x},\mathbf{x}')}{\partial \sigma_b}\right]^{1-\Delta s/s(s+1)}, \tag{5.10}$$

where $\Delta r=-\Delta s/s^2$.

From Eq. (5.10) we can determine the way $R(s+\Delta s)$ scales with distance and thereby identify $\phi(s+\Delta s)$. The dependence of these quantities on distance [in the sense of Eq. (4.14)] in the nonlinear system is

$$[\nu(\mathbf{x},\mathbf{x}')R(s,\mathbf{x},\mathbf{x}')]_{\rm av}\{[\nu(\mathbf{x},\mathbf{x}')]_{\rm av}\}^{-1}\sim\xi_p^{\phi(s)/\nu_p} \tag{5.11a}$$

$$\left[\nu(\mathbf{x},\mathbf{x}')\left[-\frac{\partial R(s,\mathbf{x},\mathbf{x}')}{\partial \sigma_b}\right]^q\right]_{\rm av}\{[\nu(\mathbf{x},\mathbf{x}')]_{\rm av}\}^{-1}\sim\xi_p^{[\psi(q,s)/\nu_p-d]}\ , \tag{5.11b}$$

so that Eq. (5.10) gives $\phi(s+\Delta s)=\psi(1-\Delta s/s(s+1),s)$. Since $\phi(s)=\psi(1,s)$, we have

$$s(s+1)d\phi(s)/ds=-\partial\psi(q,s)/\partial q\,|_{q=1}\,. \tag{5.12a}$$

In terms of the variable $r$ this relation takes the form

$$(r+1)d\phi(r)/dr=\partial\psi(q,r)/\partial q\,|_{q=1}\,. \tag{5.12b}$$

This relation seems to be more plausible than that of de Arcangelis *et al.*[3] After all, there seems to be no reason why noise properties of one system should be related to resistance properties of another with a very different value of $r$. In the Appendix we calculate $\psi(q,r)$ and show that our results satisfy Eq. (5.12b).

As discussed in Ref. 13 for the linear ($r=1$) case, $\psi(0,r)$ measures whether or not $R(\mathbf{x},\mathbf{x}')$ depends on $\sigma_b$. Thus for $q=0$ the left-hand side of Eq. (5.11b) is unity if $\sigma_b$ affects $R(\mathbf{x},\mathbf{x}')$ and is zero otherwise, i.e. when the bond $b$ is in the backbone. Thus $\psi(0,r)/\nu_p$ may be identified as the fractal dimension of the backbone. This argument is independent of the value of $r$, so we conclude that $\psi(0,r)$ should be independent of $r$. Our result in Eq. (A9) of the Appendix for $\psi(q,r)$ satisfies this requirement.

## VI. CONCLUSIONS

We may summarize our work as follows:

(1) We have calculated the crossover exponent $\phi(r)$ for the critical behavior of the resistance in a randomly diluted nonlinear ($V\sim I^r$) network. The result to first order in $\epsilon$ is given in Eq. (3.15).

(2) For $r\to 0$, $\phi(r)$ is the crossover exponent for the critical behavior [see Eq. (3.17)] of the chemical length[2] (shortest path via occupied bonds) between two sites. To first order in $\epsilon=6-d$ we find $\phi(0)=1+(\epsilon/28)$, in agreement with previous results.[15,16] For $r\to\infty$, we obtain $\phi(r)\to 1$, as expected.[2,3]

(3) We have calculated the above-mentioned crossover exponent $\psi(q,r)$ associated with $|\partial R(\mathbf{y},\mathbf{y}')/\partial\sigma_{\mathbf{x}\mathbf{x}'}|^q$ which partially characterizes the noise characteristics of the system, and the result is given in Eq. (A9) in the Appendix.

(4) We obtain a relation valid for arbitrary spatial dimension and $r$ between $\phi(r)$ and the crossover exponent $\psi(q,r)$:

$$(r+1)\frac{d\phi(r)}{dr}=\left.\frac{\partial\psi(q,r)}{\partial q}\right|_{q=1}. \tag{6.1}$$

Our $\epsilon$-expansion results satisfy this relation. At least for $d$ near 6 this relation indicates that $\phi(r)$ is a decreasing function of $r$.

(5) For $q=0$, $\psi(q,r)$ is independent of $r$. Also $\psi(0,r)/\nu_p$ is the fractal dimension of the backbone whose value is in agreement with known results.[18]

(6) Although our calculations satisfy several nontrivial self-consistency checks and do reproduce known results for $r\to 0$ and for $r\to\infty$, they do involve an analytic continuation whose status is not beyond question. Accordingly, calculations either to second order in $\epsilon$ or of the exponent associated with $w_2$ in Eq. (2.20) would be useful to further test the method of analytic continuation used here.

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## APPENDIX: $\epsilon$ EXPANSION CALCULATION OF THE NOISE EXPONENTS FOR ARBITRARY NONLINEARITY

To see if our calculations are consistent with Eq. (5.12) we need to generalize to the nonlinear case the calculations of the preceding paper for the noise sensitivity crossover exponent $\psi(q)$, utilizing the two replica averaging scheme. We treat the Hamiltonian of Eq. (2.2) of the present paper when each $\sigma_b$ is independently averaged over $f(\sigma_b)$. However, to study the crossover associated with $(\partial R/\partial\sigma_b)^q$, it suffices, in view of Eq. (5.2) of Ref. 13, to include only the $q$th cumulant of the bond conductance denoted $\Delta^q$. Accordingly, we write the effective Hamiltonian [cf. Eq. (3.12) of Ref. 13] as

$$H_{\rm eff}=-\sum_{\langle\mathbf{x},\mathbf{x}'\rangle}\ln\left\{1-p+p\exp\left[-\frac{\sigma_0}{s+1}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{m}|V_{\alpha\beta}(\mathbf{x})-V_{\alpha\beta}(\mathbf{x}')|^{s+1}\right]\right.$$
$$\left.\times\prod_{\beta=1}^{m}\exp\left[\frac{(-1)^q\Delta^q}{q!(s+1)^q}\left[\sum_{\alpha=1}^{n}|V_{\alpha\beta}(\mathbf{x})-V_{\alpha\beta}(\mathbf{x}')|^{s+1}\right]^q\right]\right\}, \tag{A1}$$

where $\sigma_0$ is the average of $\sigma_b$ over $f(\sigma_b)$. The resulting generalization of Eqs. (2.11)–(2.14) of the present paper consists mainly in replacing $\lambda$ by $\overleftrightarrow{\lambda}$. Using Eq. (A1) we see that in the presence of noise characterized by $\Delta^q$, Eq. (2.14) of the present paper becomes

$$F_l(\lambda_\beta)=\sum_{U_{1\beta}}\sum_{U_{2\beta}}\cdots\sum_{U_{n\beta}}(\Delta V)^n\exp\left[-i\sum_{\alpha=1}^{n}\lambda_{\alpha\beta}U_{\alpha\beta}\right]\exp\left[-\frac{l\sigma_0}{s+1}\sum_{\alpha=1}^{n}|U_{\alpha\beta}^{s+1}|+\frac{l(-1)^q\Delta^q}{q!(s+1)^q}\left[\sum_{\alpha=1}^{n}|U_{\alpha\beta}^{s+1}|\right]^q\right], \tag{A2}$$

where $\lambda_\beta$ denotes the vector $(\lambda_{1\beta},\lambda_{2\beta},\cdots,\lambda_{n\beta})$ and $\Delta V$ is the mesh size used in the discrete version of the field theory.[12,13,17] In evaluating $F_l$ we can treat $\Delta^q$ perturbatively, i.e., we can ignore it in determining the maximum in the integrand for large imaginary $\lambda_{\alpha\beta}$. This maximum occurs at $U_{\alpha\beta}=[-i\lambda_{\alpha\beta}/(l\sigma_0)]^r$. We now consider $B_{\overleftrightarrow{\lambda}}$ for $\lambda_{\alpha\beta}=i\lambda_{0\beta}+\xi_{\alpha\beta}$, with $\lambda_{0\beta}$ large and near the positive real axis and with $|\xi_{\alpha\beta}|$ much smaller than $|\lambda_{0\beta}|$. In this regime we have

$$B_{\overleftrightarrow{\lambda}} \sim B_{\overleftrightarrow{\lambda}}(\Delta^q=0)+a''\sum_{\beta}\left[\lambda_{0\beta}^{r-1}\sum_{\alpha=1}^{n}\xi_{\alpha\beta}^2\right]^q , \tag{A3}$$

$a''\sim\Delta^q$ is an unimportant constant. Correspondingly, $\delta r(\overleftrightarrow{\lambda})$ in Eq. (3.5) now becomes

$$\delta r(\overleftrightarrow{\lambda})=-w\sum_{\alpha\beta}(-i\lambda_{\alpha\beta})^{r+1}+w'\sum_{\beta}\left[\sum_{\alpha}[-i\lambda_{\alpha\beta}]^{r+1}\right]^q , \tag{A4}$$

where $w'\sim\Delta^q$. Then the analog of Eq. (3.7) is

$$\widetilde{\Sigma}=\int_0^\infty du\int_0^\infty dv\,\exp[-(u+v)(1+r_0)]\prod_{\beta=1}^{m}I(\lambda_\beta) , \tag{A5}$$

where $I(\boldsymbol{\lambda})$ is given by

$$I(\boldsymbol{\lambda})=\int D\tau\exp\left[w\sum_{\alpha}[u(-\tfrac{1}{2}i\lambda_\alpha+i\tau_\alpha)^{r+1}+v(-\tfrac{1}{2}i\lambda_\alpha-i\tau_\alpha)^{r+1}]\right]$$
$$\times\exp\left\{w'u\left[\sum_{\alpha=1}^{n}(-\tfrac{1}{2}i\lambda_\alpha+i\tau_\alpha)^{r+1}\right]^q+w'v\left[\left[\sum_{\alpha=1}^{n}(-\tfrac{1}{2}i\lambda_\alpha-i\tau_\alpha)^{r+1}\right]^q\right]\right\} . \tag{A6}$$

We now evaluate $I(\boldsymbol{\lambda})$ for large $\lambda_0$ by steepest descents treating $w'$ perturbatively. Thus we may determine the location of the saddle at $\tau_\alpha=\tau_{\alpha 0}$ setting $w'=0$. As in Eq. (3.9) we find

$$\tau_{\alpha 0}=\tfrac{1}{2}\lambda_\alpha(u^s-v^s)/(u^s+v^s) . \tag{A7}$$

Recalling the developments after Eq. (3.10), we see that only the value of the integrand at the saddle was relevant. The prefactor in Eq. (3.10) did not contribute to leading order in $\lambda_0$, and this will also be the case here. Thus we finally get

$$\widetilde{\Sigma}=\int_0^\infty du\int_0^\infty dv\,\exp[-(u+v)(1+r_0)]\exp\left[\sum_{\alpha\beta}w(-i\lambda_{\alpha\beta})^{r+1}uv/(v^s+u^s)\right]$$
$$\times\left[1-2w'u\sum_{\beta}\left[\sum_{\alpha}[-i\lambda_{\alpha\beta}]^{r+1}\right]^q\left[\frac{v^s}{u^s+v^s}\right]^{q(r+1)}\right] . \tag{A8}$$

This result implies that the recursion relation for $w'$ is $\partial w'/\partial l=\psi(q,r)w'/v_p$ with

$$\psi(q,r)=1+\epsilon I(q,s=1/r,r)/14 , \tag{A9}$$

where $I(q,s,r)$ is given by

$$I(q,s,r)=\int_{-1}^{1}(1+\xi)\left[\frac{(1-\xi)^s}{[(1-\xi)^s+(1+\xi)^s]}\right]^{q(r+1)}d\xi. \tag{A10}$$

For the nonlinear system the value of $\psi(q,r)$ determines the scaling behavior of $(\partial R/\partial\sigma_b)^q$ as indicated in Eq. (5.11b) and also various noise scaling exponents as discussed in the preceding paper.[13]

This result satisfies several checks. Firstly, from Eq. (5.8) we see that $\psi(1,r)$ should equal $\phi(r)$ and our result in Eq. (A9) satisfies this relation. Also, for $r=1$ we recover the result for the linear network of the accompanying paper,[13] viz. $\psi(q,1)=1+\epsilon/[7(q+1)(2q+1)]$. Finally, we check that our results do satisfy Eq. (5.12). Since $\phi(r)=\psi(1,r)$, we wish to verify that

$$(r+1)\partial\psi(1,r)/\partial r=\partial\psi(q,r)/\partial q\,|_{q=1} \tag{A11a}$$

or

$$(r+1)\left[\frac{\partial I(q,s,r)}{\partial r}-\frac{\partial I(q,s,r)}{r^2\partial s}\right]=\frac{\partial I(q,s,r)}{\partial q} ,$$
$$\text{for } q=rs=1 . \tag{A11b}$$

To facilitate the derivation of Eq. (A11) we have artificially expressed $I(q,s,r)$ as a function of both $r$ and $s$. In view of the dependence of $I(q,s,r)$ on the variable

$q(r+1)$ one sees that

$$(r+1)\frac{\partial I(1,s,r)}{\partial r} = \left.\frac{\partial I(q,s,r)}{\partial q}\right|_{q=1} . \tag{A12}$$

One can also show that the derivative with respect to $s$ of the integrand in Eq. (A10) is an odd function of $\xi$ for $q=1$, so that

$$\partial I(1,s,r)/\partial s = 0 . \tag{A13}$$

Combining Eqs. (A12) and (A13) yields Eq. (A11b). Thus Eq. (A11a) is satisfied by our result, Eq. (A9), for the noise sensitivity exponent $\psi(q,r)$ of the nonlinear network.


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