Noise exponents of the random resistor network

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(Received 15 January 1986)

We consider the critical properties of the two-point resistance and its fluctuations due to microscopic noise in a randomly diluted resistor network near the percolation threshold p_c. We introduce a $n\times m$ replicated Hamiltonian in order to treat separately the configuration average over the randomly occupied bonds denoted $[\ ]_{\rm av}$ and the average over probability distribution function of the fluctuating microscopic bond conductance, denoted $\{\ \}_f$. We evaluate a family of exponents $\{\psi_l\}$ ($l=2,3,\ldots$) whose values are $1+O(\epsilon)$ with $\epsilon=6-d$ where d is the spatial dimensionality. Each ψ_l governs the critical behavior of the lth cumulant of the resistance between the sites $\mathbf{x},\mathbf{x}'$ conditionally averaged subject to the sites being in the same cluster such that $\overline{C_R}^{(l)}(\mathbf{x},\mathbf{x}')\sim|\mathbf{x}-\mathbf{x}'|^{\psi_l/\nu_p}$ for p near p_c, where ν_p is the correlation-length exponent for percolation. Furthermore, $\psi_2=1+\epsilon/105$ determines the dependence of the variance of the resistance in a finite network on size L as L^{ψ_2/ν_p}.

I. INTRODUCTION

The randomly diluted resistor network is one of the simplest models of random media. Recent analysis has shown that its critical properties near the percolation threshold[1] are far more complicated than a simple analogy with traditional second order thermodynamic phase transitions would suggest. Harris *et al.*[2,3] calculated corrections to first order in $\epsilon=6-d$ where d is the spatial dimension for an infinite set of crossover exponents $\{\phi_n\}$ that reduce to unity at the upper critical dimension $d_c=6$. The first member is the previously identified[4,5] exponent ϕ_1 determining the conductivity exponent[6,7]

$$t=(d-2)\nu_p+\phi_1\ , \tag{1.1}$$

where ν_p is the percolation correlation length exponent. Higher ϕ_n's describe correction to scaling in the distribution function for the resistance between two points. Rammal *et al.*[8] proposed the existence of another infinite set of exponents $\{x_n\nu_p\}$ describing resistance fluctuations arising from the microscopic noise in the resistors comprising the random network. They raised the question of a possible connection between the sets $\{x_n\}$ and $\{\phi_n\}$. A set of exponents $\{\zeta_{2n}\}$ equivalent to $\{-x_n\nu_p\}$ was also proposed by de Arcangelis *et al.*[9] In addition, de Arcangelis *et al.*[10] have proposed the following relation between $\phi_1(r)$, the crossover exponent for the nonlinear ($V\sim I^r$) network, and the $\{x_n\}$:

$$\phi_1(r)=-x_{(1+r)/2}\nu_p\ . \tag{1.2}$$

A different relation of this type is obtained in the following paper.[11] Thus the status of the x_n has many ramifications.

In this paper, we will develop a field theory for a random noisy network in which each resistor has voltage fluctuation proportional to I^2 for constant current I. We will introduce an infinite set of crossover exponent $\{\psi_n\}$ describing the effects of noise on fluctuations in the resistance between two points. We then calculate the correction to first order in ϵ for $\{\psi_n\}$. The set $\{\psi_n\}$ is identical to the set $\{-x_n\nu_p\}$ introduced by Rammal *et al.*,[8] but for $n>1$ is distinct from the set $\{\phi_n\}$ introduced by Harris *et al.*[2,3] A trivial homogeneity requirement shows that ψ_1 and ϕ_1 are identical. The present formulation also shows the existence of a further extended family of exponents whose values are not calculated here. Thus the noisy diluted resistor has several families of crossover exponents reducing to unity at $d_c=6$.

This paper is organized as follows. In Sec. II we define various quantities which characterize the resistance noise and higher-order resistance fluctuations on the randomly diluted resistor network near the percolation threshold p_c. We introduce exponents $\{\psi_n\}$ to describe the large separation behavior of cumulants of the resistance at the threshold and, using finite-size scaling, establish their identity with $\{-x_n\nu_p\}$. In Sec. III we develop a field theory for the noisy network using the formulation introduced by Stephen[12] and show how it can be used to calculate the cumulants of resistance introduced in Sec. II. In Sec. IV, we calculate the order-ϵ correction to $\{\phi_n\}$ and in Sec. V, we discuss our results and compare with existing theories.

II. SUMMARY OF THE RESULTS: NOISE IN THE RANDOM RESISTOR NETWORK

A. Noise in a resistor network

We first consider an arbitrary network of sites $\mathbf{x}$ connected by bonds b. Each bond b has a conductance σ_b which is an independent random variable with mean σ_0 and variance Δ ($\Delta\ll\sigma_0$). For convenience, we will take the distribution function for each σ_b to be the Gaussian:

$$f(\sigma)=(2\pi\Delta^2)^{-1/2}\exp[-(\sigma-\sigma_0)^2/2\Delta^2]\ . \tag{2.1}$$

In general quantities such as resistance between two sites will depend on the set of values $\{\sigma_b\}$ of all conductances of the network. Averages over f of a variable A depending on conductances $\{\sigma_b\}$ will be denoted by $\{A\}_f$:

$$\{A\}_f \equiv \int \prod_b d\sigma_b f(\sigma_b) A(\{\sigma_b\}) . \tag{2.2}$$

For a given realization of conductances $\{\sigma_b\}$, the resistance between sites $\mathbf{x}$ and $\mathbf{x}'$ is denoted by $R(\mathbf{x},\mathbf{x}';\{\sigma_b\})$. Then the statistical properties of the resistance are characterized in part by the average resistance, $\{R(\mathbf{x},\mathbf{x}')\}_f$, and its variance

$$\{\delta R(\mathbf{x},\mathbf{x}')\delta R(\mathbf{x},\mathbf{x}')\}_f = \{[R(\mathbf{x},\mathbf{x}') - \{R(\mathbf{x},\mathbf{x}')\}_f]^2\}_f \equiv C_R{}^{(2)} , \tag{2.3}$$

where $C_R{}^{(2)}$ is the second cumulant of the resistance. We can also consider higher cumulants $C_R{}^{(m)}(\mathbf{x},\mathbf{x}')$ of the resistance. The first cumulant of resistance $C_R{}^{(1)}(\mathbf{x},\mathbf{x}')$ is just the average resistance $\{R(\mathbf{x},\mathbf{x}')\}_f$. Following Rammal *et al.*[8] we introduce the noise correlation function $S_R(\mathbf{x},\mathbf{x}')$ as

$$S_R(\mathbf{x},\mathbf{x}') = \frac{C_R{}^{(2)}(\mathbf{x},\mathbf{x}')}{\{R(\mathbf{x},\mathbf{x}')\}_f^2} . \tag{2.4}$$

B. Random resistor network

We now consider a randomly diluted network on a d-dimensional lattice such that each nearest-neighbor bond is occupied with probability p and unoccupied with probability $1-p$. The f-averaged resistance and its cumulants introduced above will depend on the configuration c of the randomly occupied bonds. We therefore introduce an additional average over these configurations. This average will be denoted $[\ ]_{\rm av}$. In performing this average, it is important to recognize that the resistance between disconnected sites is infinite. Accordingly we consider only sites in the same cluster using the pair connectedness indicator function $\nu(\mathbf{x},\mathbf{x}';c)$ which, for the configuration c, is unity if sites $\mathbf{x},\mathbf{x}'$ are in the same cluster and is zero otherwise. Then, for the random resistor network, the moments of the resistance and the noise correlation function between sites $\mathbf{x},\mathbf{x}'$ conditionally averaged subject to the sites being in the same cluster are given respectively by

$$\overline{R}^m(\mathbf{x},\mathbf{x}') = \frac{[\nu(\mathbf{x},\mathbf{x}';c)\{R(\mathbf{x},\mathbf{x}';c)\}_f^m]_{\rm av}}{\chi_p(\mathbf{x},\mathbf{x}')} , \tag{2.5}$$

$$\overline{S}(\mathbf{x},\mathbf{x}') = \frac{[\nu(\mathbf{x},\mathbf{x}';c)S_R(\mathbf{x},\mathbf{x}';c)]_{\rm av}}{\chi_p(\mathbf{x},\mathbf{x}')} , \tag{2.6}$$

where $\chi_p(\mathbf{x},\mathbf{x}') \equiv [\nu(\mathbf{x},\mathbf{x}';c)]_{\rm av}$ is the susceptibility function for percolation. We may also consider conditional averages of higher-order cumulants of the resistance

$$\overline{C_R}^{(m)}(\mathbf{x},\mathbf{x}') \equiv \frac{[\nu(\mathbf{x},\mathbf{x}';c)C_R^{(m)}(\mathbf{x},\mathbf{x}';c)]_{\rm av}}{\chi_p(\mathbf{x},\mathbf{x}')} . \tag{2.7}$$

Note that the cumulants have been defined with respect to an average over f whereas the moments are defined with respect to average over the configurations c.

In the following section, we will develop a field theory, by which we can calculate the resistance and its fluctuations near the percolation threshold p_c. It will be shown in Sec. III that the moments and cumulants of the resistance have the following scaling form near p_c:

$$\overline{R}^m(\mathbf{x},\mathbf{x}') \sim \sigma_0^{-m} \,|\mathbf{x}-\mathbf{x}'|^{m\phi_1/\nu_p} X_m \left[\frac{|\mathbf{x}-\mathbf{x}'|}{\xi_p}\right] , \tag{2.8}$$

$$\overline{C_R}^{(m)}(\mathbf{x},\mathbf{x}') \sim \sigma_0^{-m} \,|\mathbf{x}-\mathbf{x}'|^{\psi_m/\nu_p} Y_m \left[\frac{|\mathbf{x}-\mathbf{x}'|}{\xi_p}\right] , \tag{2.9}$$

where $\xi_p \sim |p-p_c|^{-\nu_p}$ is the percolation correlation length and ψ_n is a member of a new family of exponent whose values are $1+O(\epsilon)$. Here X_m and Y_m are undetermined scaling functions.

We may also consider the average resistance $\overline{R}(p,L)$ and normalized noise $\overline{S}_R(p,L)$ for a finite network of linear size L at bond concentration p by averaging over the choice of the sites $\mathbf{x}$ and $\mathbf{x}'$:

$$\overline{R}(p,L) = \frac{\sum_{\mathbf{x}} \overline{R}(\mathbf{x},\mathbf{x}')\chi_p(\mathbf{x},\mathbf{x}')}{\sum_{\mathbf{x},\mathbf{x}'} \chi_p(\mathbf{x},\mathbf{x}')} , \tag{2.10}$$

$$\overline{S}_R(p,L) = \frac{\sum_{\mathbf{x},\mathbf{x}'} [\overline{S}(\mathbf{x},\mathbf{x}')\chi_p(\mathbf{x},\mathbf{x}')]}{\sum_{\mathbf{x}} \chi_p(\mathbf{x},\mathbf{x}')} . \tag{2.11}$$

Note that $\overline{R}(p,L)$ is the average of the two-point resistance $R(\mathbf{x},\mathbf{x}')$ over the choice of the probe sites $\mathbf{x}$ and $\mathbf{x}'$ rather than the configuration average of the resistance of a sample which is placed between two (equipotential) bus bars. Since near p_c, $\chi_p(\mathbf{x},\mathbf{x}';c)$ obeys the scaling relation,

$$\chi_p(\mathbf{x},\mathbf{x}';c) = |\mathbf{x}-\mathbf{x}'|^{-(d-2+\eta_p)} \times X_0 \left[\frac{|\mathbf{x}-\mathbf{x}'|}{\xi_p}, \frac{L}{\xi_p}\right] , \tag{2.12}$$

we obtain in the limit $\Delta/\sigma_0 \ll 1$

$$\overline{R}(p,L) = \sigma_0^{-1} L^{\phi_1/\nu_p} g_1 \left[\frac{L}{\xi_p}\right] , \tag{2.13}$$

$$\overline{C_R}^{(n)}(p,L) \sim \sigma_0^{-n} L^{\psi_n/\nu_p} g_n \left[\frac{L}{\xi_p}\right] , \tag{2.14}$$

where the g_n's are undetermined scaling functions. A somewhat more elaborate analysis to be presented in Sec. III predicts

$$\overline{S}_R(p,L) = L^{(\psi_2-2\phi_1)/\nu_p} \frac{\Delta^2}{\sigma_0^2} h \left[\frac{L}{\xi_p}\right] , \tag{2.15}$$

where h is a scaling function.

Now we infer the behavior of these scaling functions $g_n(x)$ and $h(x)$ in the limits of $x \ll 1$ and $x \gg 1$. For small x, i.e., if we are sufficiently close to p_c such that $\xi_p \gg L$ holds, then

$$\overline{R}(p,L) \sim L^{\phi_1/\nu_p} \sigma_0^{-1} , \tag{2.16}$$

$$\overline{S}_R(p,L) \sim L^{(\psi_2-2\phi_1)/\nu_p} \frac{\Delta^2}{\sigma_0^2} . \tag{2.17}$$

The scaling forms Eqs. (2.13) and (2.14) are the same as suggested by Rammal *et al.*[8] based on the self-similar structure of the percolating random resistor network near p_c. Their exponents $\{x_n\}$ which are defined by the characteristic size dependence of the $2n$th moment of the current in a microscopic bond are equivalent to our $\{\psi_n\}$ with $x_n = -\psi_n/\nu_p$. In this paper, we obtain $\psi_2 = 1+\epsilon/105+O(\epsilon^2)$.

Using $\phi_1 = 1+\epsilon/42$ and $\nu_p = \frac{1}{2}+\frac{5}{84}\epsilon$, we have

$$x_1 \equiv -\frac{\phi_1}{\nu_p} = -2+\frac{4}{21}\epsilon , \tag{2.18}$$

$$b \equiv \frac{2\phi_1-\psi_2}{\nu_p} = 2-\frac{17}{105}\epsilon , \tag{2.19}$$

for the exponents introduced by Rammal *et al.*,[8] $\beta = x_1$ and b, which describe the scaling behavior of the resistance and noise, respectively. Note that our results for ψ_2 satisfy both the inequality[8,13,14] $\psi_2 \le \phi_1$ and also the bound[8,13,14] $\phi_1 \ge 1$. Furthermore, our results for ψ_n obey the convexity relation $\partial^2\psi_n/\partial n^2 \ge 0$ given by Blumenfeld *et al.*[15]

Now we consider the scaling behavior of $g_1(x)$ and $h(x)$ for large x. We note that in the homogeneous network ($p=1$, $L \gg \xi_p$), $\overline{R}$ and $\overline{S}_R$ scale as[8,13]

$$\overline{R}(p=1,L) \sim L^{-(d-2)} , \tag{2.20}$$

$$\overline{S}_R(p=1,L) \sim L^{-d} . \tag{2.21}$$

By comparing these equations with Eqs. (2.13) and (2.14), we get the asymptotic behavior of the scaling function g_1 and h for $x \gg 1$:

$$g_1(x) \sim x^{-(d-2)-\phi_1/\nu_p} , \tag{2.22}$$

$$h(x) \sim x^{-d-(\psi_2-2\phi_1)/\nu_p} . \tag{2.23}$$

Then the scaling behaviors of $\overline{R}$ and $\overline{S}_R$ for $L \gg \xi_p$ are respectively

$$\overline{R}(p,L) \sim \frac{1}{L^{d-2}} |p-p_c|^{-t} , \tag{2.24}$$

$$\overline{S}_R(p,L) \sim \frac{1}{L^d} |p-p_c|^{-\kappa} , \tag{2.25}$$

where

$$t = (d-2)\nu_p + \phi_1 = 3 - \frac{5}{21}\epsilon \tag{2.26}$$

and

$$\kappa = d\nu_p - (2\phi_1 - \psi_2) = 2 - \frac{19}{105}\epsilon . \tag{2.27}$$

Hence if the network is sufficiently large ($L \gg \xi_p$), the average noise $\overline{S}_R$ retains the characteristic scaling behavior of the homogeneous network with the critical dependence of $|p-p_c|^{-\kappa}$.

III. FIELD THEORY FOR THE RANDOMLY DILUTED RESISTOR NETWORK

A. Replica formalism

In this subsection, we will develop a field theory for a model when we randomly dilute the noisy network. The probability distribution for the resistance between sites **x** and **x**′ is determined by the generating function

$$G_{\overleftrightarrow{k}}(\mathbf{x},\mathbf{x}') = \left[\prod_{\beta=1}^{m} \left\{ \prod_{\alpha=1}^{n} \exp[-\tfrac{1}{2}k_{\alpha\beta}^2 R(\mathbf{x},\mathbf{x}';c)] \right\}_f \right]_{\rm av} , \tag{3.1a}$$

$$\equiv \left[\nu_p(\mathbf{x},\mathbf{x}';c) \prod_{\beta} \left\{ \prod_{\alpha} \exp[-\tfrac{1}{2}k_{\alpha\beta}^2 R(\mathbf{x},\mathbf{x}';c)] \right\}_f \right]_{\rm av} , \tag{3.1b}$$

where $\overleftrightarrow{k}$ is a tensor with components $k_{\alpha\beta}$. Equation (3.1b) follows from the fact $R(\mathbf{x},\mathbf{x}';c)$ is infinite if **x** and **x**′ are in different clusters. The conditionally averaged cumulants of the resistance introduced in Sec. II can be obtained using the identity

$$G_{\overleftrightarrow{k}}(\mathbf{x},\mathbf{x}') = \left[\nu_p(\mathbf{x},\mathbf{x}';c) \exp\left(\sum_{l=1}^{\infty} K_l\, C_R^{(l)}(\mathbf{x},\mathbf{x}') \right) \right]_{\rm av} , \tag{3.2}$$

where K_l is defined by

$$K_l \equiv \frac{(-1)^l}{2^l(l!)} \left[\sum_{\beta} \left(\sum_{\alpha} k_{\alpha\beta}^2 \right)^l \right] . \tag{3.3}$$

Since each K_l is algebraically independent, we can obtain the conditionally averaged resistance and its cumulants defined in Eqs. (2.5) and (2.7) simply by:

$$\overline{R}(\mathbf{x},\mathbf{x}')X_p(\mathbf{x},\mathbf{x}') \equiv [\nu_p(\mathbf{x},\mathbf{x}')\{R(\mathbf{x},\mathbf{x}';c)\}_f]_{\rm av} = \lim_{n\to 0} \left. \frac{\partial G_{\overleftrightarrow{k}}(\mathbf{x},\mathbf{x}')}{\partial K_1} \right|_{\overleftrightarrow{k}=0} , \tag{3.4a}$$

$$\overline{C_R}^{(l)}(\mathbf{x},\mathbf{x}')X_p(\mathbf{x},\mathbf{x}') \equiv [\nu_p(\mathbf{x},\mathbf{x}')C_R^{(l)}(\mathbf{x},\mathbf{x}')]_{\rm av} = \lim_{n\to 0} \left. \frac{\partial G_{\overleftrightarrow{k}}(\mathbf{x},\mathbf{x}')}{\partial K_l} \right|_{\overleftrightarrow{k}=0} . \tag{3.4b}$$

Likewise the conditionally averaged noise correlation function defined in Eq. (2.6) is obtained by

$$\bar{S}(\mathbf{x},\mathbf{x}')\chi_p(\mathbf{x},\mathbf{x}') = \int_{-\infty}^{0} dK_1 K_1 \left. \frac{\partial G_{\vec{k}}(\mathbf{x},\mathbf{x}')}{\partial K_2} \right|_{K_2=K_3=\cdots=0} . \quad (3.5)$$

We note that $G_{\vec{k}}(\mathbf{x},\mathbf{x}')$ can be obtained from the replicated Gaussian field theory for the resistor network using

$$\exp[-\tfrac{1}{2}k_{\alpha\beta}^2 R(\mathbf{x},\mathbf{x}';c)] = \frac{1}{Z}\int dV_{\alpha\beta}\exp[-H_{\alpha\beta}(\sigma_b)]\exp\{ik_{\alpha\beta}[V_{\alpha\beta}(\mathbf{x})-V_{\alpha\beta}(\mathbf{x}')]\} , \quad (3.6)$$

where $H_{\alpha\beta}$ is defined by

$$H_{\alpha\beta}(\sigma_b) = \tfrac{1}{2}\sum_{\langle \mathbf{x},\mathbf{x}'\rangle} \sigma_b[V_{\alpha\beta}(\mathbf{x})-V_{\alpha\beta}(\mathbf{x}')]^2 \quad (3.7)$$

and

$$Z \equiv \int dV_{\alpha\beta}\exp[-H_{\alpha\beta}(\sigma_b)] . \quad (3.8)$$

Thus $G_{\vec{k}}(\mathbf{x},\mathbf{x}')$ may be written as

$$G_{\vec{k}}(\mathbf{x},\mathbf{x}') = \left[\int DV \prod_{\beta} \left\{ Z^{-n}\prod_{\alpha}\exp\{ik_{\alpha\beta}[V_{\alpha\beta}(\mathbf{x})-V_{\alpha\beta}(\mathbf{x}')]\}\exp(-H_{\alpha\beta}) \right\}_f \right]_{\text{av}} , \quad (3.9)$$

where $DV \equiv \prod_{\alpha\beta} dV_{\alpha\beta}$. This equation and the relations $\lim_{n\to 0} Z^n = 1$ and $\lim_{nm\to 0}[\{Z^n\}_f^m]_{\text{av}} = 1$ imply

$$G_{\vec{k}}(\mathbf{x},\mathbf{x}') = \lim_{n\to 0}\langle \Psi_{\vec{k}}(\mathbf{x})\Psi_{-\vec{k}}(\mathbf{x}')\rangle , \quad (3.10)$$

where

$$\Psi_{\vec{k}}(\mathbf{x}) \equiv \exp\left[\sum_{\beta}\sum_{\alpha} ik_{\alpha\beta}V_{\alpha\beta}(\mathbf{x})\right] \quad (3.11)$$

and where $\langle\ \rangle$ signifies an average with respect to the H_{eff} defined by

$$\exp(-H_{\text{eff}}) \equiv \left[\prod_{\beta} \left\{ \exp\left[-\sum_{\alpha} H_{\alpha\beta}\right] \right\}_f \right]_{\text{av}} , \quad (3.12)$$

provided that

$$\lim_{n\to 0} Z^n = \lim_{n\to 0}\int \prod_{\alpha=1}^{n} dV_{\alpha\beta}\exp\left[\sum_{\alpha=1}^{n} H_{\alpha\beta}\right] = 1 . \quad (3.13)$$

B. Field theoretic formulation

In order to discuss the scaling form of $G_{\vec{k}}(\mathbf{x},\mathbf{x}')$ near the percolation threshold p_c, we turn to the calculation of H_{eff}. According to Eq. (3.12), we have

$$\begin{aligned} H_{\text{eff}} &= -\sum_{\langle \mathbf{x},\mathbf{x}'\rangle} \ln\left[1-p+p\prod_{\beta=1}^{m}\left\{\exp\left[-\tfrac{1}{2}\sigma_b\sum_{\alpha=1}^{n}[V_{\alpha\beta}(\mathbf{x})-V_{\alpha\beta}(\mathbf{x}')]^2\right]\right\}_f\right] \\ &\equiv -\sum_{\langle \mathbf{x},\mathbf{x}'\rangle} \ln\left[1-p+p\prod_{\beta=1}^{m}\int f(\sigma_b)\exp\left[-\tfrac{1}{2}\sigma_b\sum_{\alpha=1}^{n}[V_{\alpha\beta}(\mathbf{x})-V_{\alpha\beta}(\mathbf{x}')]^2\right]d\sigma_b\right] \\ &= -\sum_{\langle \mathbf{x},\mathbf{x}'\rangle} \ln\left\{1-p+p\exp\{-\tfrac{1}{2}\sigma_0[\overleftrightarrow{V}(\mathbf{x})-\overleftrightarrow{V}(\mathbf{x}')]^2\}\prod_{\beta=1}^{m}\exp\left[\frac{\Delta^2}{8}\left[\sum_{\alpha=1}^{n}[V_{\alpha\beta}(\mathbf{x})-V_{\alpha\beta}(\mathbf{x}')]^2\right]^2\right]\right\} . \end{aligned} \quad (3.14)$$

Notice that since we choose a Gaussian distribution, Eq. (2.1), only the second cumulant of bond conductance Δ^2 appears.

We follow here the discrete version[2] of Stephen's treatment of the resistor network[12] in which $V(\mathbf{x})$ takes on a discrete set of values separated by interval ΔV in order to ensure

$$\lim_{n\to 0} Z^n \equiv \lim_{n\to 0}\sum_{V_{1\beta},V_{2\beta},\ldots,V_{n\beta}}^{n} (\Delta V)^n \exp\left[\sum_{\alpha=1}^{n} H_{\alpha\beta}\right] = 1 . \quad (3.13')$$

We will regain the continuous model by taking $\Delta V \to 0$ at the appropriate point of calculation. In terms of the order parameter $\Psi_{\overleftrightarrow{k}}(\mathbf{x})$, $H_{\rm eff}$ can be written as

$$H_{\rm eff}=-\sum_{\langle \mathbf{x},\mathbf{x}'\rangle}\sum_{\overleftrightarrow{k}} B_{\overleftrightarrow{k}}\Psi_{\overleftrightarrow{k}}(\mathbf{x})\Psi_{-\overleftrightarrow{k}}(\mathbf{x}') , \qquad (3.15)$$

where

$$B_{\overleftrightarrow{k}}=\sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}v^l\prod_{\beta=1}^{m}F_l(\{\mathbf{k}_\beta\}) , \qquad (3.16)$$

where $v=p/1-p$ and where $\mathbf{k}_\beta$ is the vector $(k_{1\beta},k_{2\beta},\ldots,k_{n\beta})$ and

$$F_l(\{\mathbf{k}_\beta\})=\sum_{U_{1\beta}}\sum_{U_{2\beta}}\cdots\sum_{U_{n\beta}}(\Delta V)^n \exp\left[-\sum_\alpha ik_{\alpha\beta}U_{\alpha\beta}\right]\exp\left[-l\frac{\sigma_0}{2}\sum_\alpha U_{\alpha\beta}^2\right]\exp\left[l\left[\frac{\Delta^2}{8}\sum_\alpha U_{\alpha\beta}^2\right]^2\right] . \qquad (3.17)$$

Since $H_{\rm eff}$ is quadratic in $\Psi_{\overleftrightarrow{k}}(\mathbf{x})$, we introduce the Hubbard-Stratanovich transformation to obtain a continuum field theory of the form

$$Z = {\rm Tr}e^{-H_{\rm eff}}\equiv\int D\Phi e^{-L} , \qquad (3.18)$$

where $\Phi_{\overleftrightarrow{k}}(\mathbf{x})$ is a variable conjugate to $\Psi_{\overleftrightarrow{k}}(\mathbf{x})$, and $D\Phi$ denotes integration over all $\Phi_{\overleftrightarrow{k}}(\mathbf{x})$. The action L expanded to the third order in $\Phi_{\overleftrightarrow{k}}$ is

$$L=\tfrac{1}{2}\int d^dx\sum_{\overleftrightarrow{k}\neq 0}[r_{\overleftrightarrow{k}}\Phi_{\overleftrightarrow{k}}(\mathbf{x})\Phi_{-\overleftrightarrow{k}}(\mathbf{x})+\nabla\Phi_{\overleftrightarrow{k}}(\mathbf{x})\nabla\Phi_{-\overleftrightarrow{k}}(\mathbf{x})]\Delta k+\frac{1}{3!}u\int d^dx{\sum}'\Phi_{\overleftrightarrow{k}_1}(\mathbf{x})\Phi_{\overleftrightarrow{k}_2}(\mathbf{x})\Phi_{-\overleftrightarrow{k}_1-\overleftrightarrow{k}_2}(\mathbf{x})\Delta k_1\Delta k_2 , \qquad (3.19)$$

where $\sum'$ indicates a summation with the terms $\overleftrightarrow{k}_1=0$, $\overleftrightarrow{k}_2=0$, and $\overleftrightarrow{k}_1+\overleftrightarrow{k}_2=0$ omitted, and $r_{\overleftrightarrow{k}}=(zB_{\overleftrightarrow{k}})^{-1}-1$, where z is the number of nearest neighbors. Terms in Eq. (3.11) of order Φ^4 or higher are irrelevant near $d=6$ and are therefore omitted.

When $\sigma_0^{-1}=0$ and $\Delta=0$, then $\prod_{\beta=1}^m F_l(\{\mathbf{k}_\beta\})=1$, so that all $r_{\overleftrightarrow{k}}$ are equal to $r_0\sim(p_c-p)$, but for $\sigma_0^{-1}>0$ and $\Delta>0$, $\prod_\beta F_l(\{\mathbf{k}_\beta\})$ depends on $\{K_s\}$ defined in Eq. (3.3). Hence near p_c, $r_{\overleftrightarrow{k}}$ can be expressed as

$$r_{\overleftrightarrow{k}}=r_0+\sum_{m=1}^{\infty}w_m(\overleftrightarrow{k}^2)^m+\sum_{s=2}^{\infty}v_sK_s , \qquad (3.20)$$

where $w_m\sim\sigma_0^{-m}$, $v_{2s}\sim\Delta^{2s}/\sigma_0^{4s}$, and $v_{2s-1}\sim\Delta^{2s}/\sigma_0^{4s-1}$. The v_1 potential is incorporated into w_1. It is important to note that $v_{2s}/w_1^{2s}\sim v_{2s-1}/w_1^{2s-1}\sim(\Delta/\sigma_0)^{2s}$. In situations of physical interest, $\Delta<\sigma_0$ and $v_s<w_1^s$. This means that in calculating crossover exponents associated with v_s, we must always take the limit $v_s\to 0$ before the limit $w_1\to 0$. A similar situation was encountered in the study of networks with a singular distribution of resistances.[16] In Eq. (3.20), we omitted higher order terms proportional to $\prod_i K_{s_i}^{p_i}$, with at least one $s_i>1$ and $\sum_i p_i>1$. There is an independent crossover exponent of order $1+O(\epsilon)$ with the coefficient of each of these terms. We will not calculate these exponents in this paper.

We find that the averaged order parameter correlation function $G_{\overleftrightarrow{k}}(\mathbf{x},\mathbf{x}')$ has the following scaling form near p_c :

$$G_{\overleftrightarrow{k}}(\mathbf{x},\mathbf{x}')=|\mathbf{x}-\mathbf{x}'|^{-(d-2+\eta_p)}S\left[\frac{|\mathbf{x}-\mathbf{x}'|}{\xi_p},\{w_n\overleftrightarrow{k}^{2n}|p-p_c|^{-\phi_n}\},\{v_sK_s(\overleftrightarrow{k})|p-p_c|^{-\psi_s}\}\right] , \qquad (3.21)$$

where $\{\phi_n\}$ is the same family of exponents as Harris *et al.* identified[2,3] and $\{\psi_s\}$ is a new family of crossover exponents associated with $\{v_s\}$.

We can now calculate the conditionally averaged resistance, resistance fluctuation, and noise between sites $\mathbf{x}$ and $\mathbf{x}'$ using Eqs. (3.4) and (3.5) and the scaling relation Eq. (3.21):

$$\overline{R}(\mathbf{x},\mathbf{x}')\sim\sigma_0^{-1}|\mathbf{x}-\mathbf{x}'|^{\phi_1/\nu_p}\tilde{g}\left[\frac{|\mathbf{x}-\mathbf{x}'|}{\xi_p}\right], \qquad (3.22)$$

$$\overline{C_R}^{(m)}(\mathbf{x},\mathbf{x}')\sim v_m|\mathbf{x}-\mathbf{x}'|^{\psi_m/\nu_p}\tilde{g}_2\left[\frac{|\mathbf{x}-\mathbf{x}'|}{\xi_p}\right] , \qquad (3.23)$$

$$S_R(\mathbf{x},\mathbf{x}')\sim\frac{\Delta^2}{\sigma_0^2}|\mathbf{x}-\mathbf{x}'|^{(\psi_2-2\phi_1)/\nu_p}\tilde{h}\left[\frac{|\mathbf{x}-\mathbf{x}'|}{\xi_p}\right] . \qquad (3.24)$$

IV. ϵ EXPANSION

In this section we will develop the momentum shell recursion relation[17,18] for the continuum Hamiltonian L in $6-\epsilon$ dimensions. We integrate out degrees of freedom with wave numbers in the annulus $\Lambda/b^{-1}<q<\Lambda$, where Λ is a cutoff determined by the lattice constant a by $\Lambda\sim a^{-1}$, and rescale the fields by $\Phi_{\overleftrightarrow{k}}(\mathbf{q}/b)\to b^{(1/2)(d-2+\eta_p)}\Phi_{\overleftrightarrow{k}}(\mathbf{q})$. When all the $\{w_n\}$ and

$\{v_s\}$ are zero, i.e., when $\sigma_0=\Delta=0$, we will regain the usual percolation result[18,19] with the third-order vertex function u reaching a fixed point where $g\equiv K_d(u^*)^2/2=\epsilon/7$ and $\eta_p=-\epsilon/21$.

When $\Delta>0$ and $\sigma_0^{-1}>0$, we get the following momentum shell recursion relation with $b=e^{\delta l}$:

$$\frac{dr_{\overleftrightarrow{k}}}{dl}=(2-\eta_p)r_{\overleftrightarrow{k}}-g\Sigma_{\overleftrightarrow{k}}. \quad (4.1)$$

Here $\Sigma_{\overleftrightarrow{k}}$ is the contribution from the one-loop diagram shown in Fig. 1. Its analytic form is

$$\Sigma_{\overleftrightarrow{k}}\equiv\sum_{\overleftrightarrow{p},\overleftrightarrow{p}+\overleftrightarrow{k}\neq 0}G(\overleftrightarrow{k}+\overleftrightarrow{p})G(-\overleftrightarrow{p}) \quad (4.2a)$$

$$=-2G(\overleftrightarrow{k})G(0)+\sum_{\overleftrightarrow{p}}G(\overleftrightarrow{k}+\overleftrightarrow{p})G(-\overleftrightarrow{p}) \quad (4.2b)$$

$$\equiv-2G(\overleftrightarrow{k})G(0)+\tilde{\Sigma}_{\overleftrightarrow{k}}, \quad (4.2c)$$

where $G(\overleftrightarrow{k})$ is a mean-field propagator evaluated at $|\mathbf{q}|=1$, which is given by

$$G(\overleftrightarrow{k})^{-1}=1+r_0+\sum_{n=1}^{\infty}w_n(\overleftrightarrow{k}^2)^n+\sum_{l=2}^{\infty}v_lK_l(\overleftrightarrow{k}). \quad (4.3)$$

Here $r_0\sim p-p_c$ and $\overleftrightarrow{k}^2\equiv\sum_{\alpha\beta}k_{\alpha\beta}^2$. In the Appendix, we show there are contributions to $\tilde{\Sigma}_{\overleftrightarrow{k}}$ linear in w_1 and all $v_s(s>2)$ which can be written in the form

$$\tilde{\Sigma}_{\overleftrightarrow{k}}=1-2r_0+c_1w_1(\overleftrightarrow{k}^2)+\sum_{s>2}d_sv_sK_s\ , \quad (4.4)$$

where c_1 is the coefficient given previously[2,3] which determines the crossover exponent ϕ_1 for w_1 such that $\phi_1=1-(\epsilon/14)c_1$ and

$$d_s=-\frac{2}{(s+1)(2s+1)}\ . \quad (4.5)$$

There are additional nonlinear terms in $\tilde{\Sigma}_{\overleftrightarrow{k}}$ which do not affect the evaluation of exponents. Using Eqs. (4.1), (4.2c), and (4.4), we find the recursion relations

$$\frac{dv_s}{dl}=[2-\eta_p-g(2+d_s)]v_s \quad (4.6)$$

from which we obtain the crossover exponent

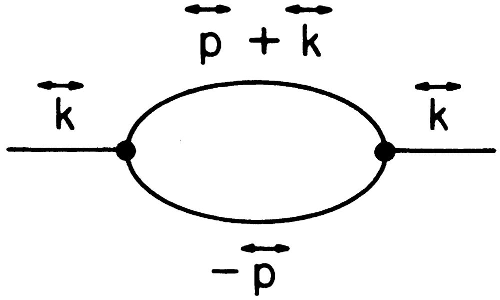

FIG 1. One-loop contribution to $\Sigma(\overleftrightarrow{k})$ in Eq. (4.1).

$$\psi_s=\nu_p[2-\eta_p-g(2+d_s)]=1-\frac{\epsilon}{14}d_s \quad (4.7a)$$

$$=1+\frac{\epsilon}{7(s+1)(2s+1)}\ . \quad (4.7b)$$

This equation yields, for example, $\psi_0=1+\epsilon/7$, $\psi_1=1+\epsilon/42$, and $\psi_2=1+\epsilon/105$.

V. DISCUSSIONS AND CONCLUSIONS

In this subsection, we will discuss additional interpretation of the exponents $\{\psi_n\}$. In a given configuration c, we may expand the resistance in a power series in the deviations $\delta r_b\equiv r_b-r_b^0$ of the resistance of each bond b from its average value r_b^0:

$$R(\mathbf{x},\mathbf{x}';c)=R^0(\mathbf{x},\mathbf{x}';c)+\sum_b\frac{\partial R(\mathbf{x},\mathbf{x}';c)}{\partial r_b}\delta r_b+\tfrac{1}{2}\sum_{bb'}\frac{\partial^2R(\mathbf{x},\mathbf{x}';c)}{\partial r_b\partial r_{b'}}\delta r_b\delta r_{b'}+\cdots\ , \quad (5.1)$$

where $R^0(\mathbf{x},\mathbf{x}';c)$ is the resistance when $\delta r_b=0$ for every bond b. Here δr_b is a random variable with a probability distribution $s^{-1}f(\delta r_b/s)$, where s is a variable with units of resistance which sets the scale of the distribution f. As $s\to 0$, $s^{-1}f(\delta r_b/s)$ approaches a δ function. With this form for f, the lth cumulant $\Delta^{(l)}$ of δr_b tends to zero as s^l. Equation (5.1) allows us to express $\overline{C_R}^{(l)}(\mathbf{x},\mathbf{x}')$ in terms of the sets of cumulants $\{\Delta^{(l)}\}$. In general, $\overline{C_R}^{(l)}(\mathbf{x},\mathbf{x}';c)$ depends on all the cumulants $\{\Delta^{(l)}\}$. In the limit $s\to 0$, however, the leading term in $\overline{C_R}^{(l)}$ is proportional to $\Delta^{(l)}$ so that

$$\left.\frac{\partial\overline{C_R}^{(l)}(\mathbf{x},\mathbf{x}')}{\partial\Delta^{(l)}}\right|_{s=0}=\chi^{-1}(\mathbf{x},\mathbf{x}')\times\left[\sum_b\nu(\mathbf{x},\mathbf{x}';c)\left[\frac{\partial R(\mathbf{x},\mathbf{x}';c)}{\partial r_b}\right]^l\right]_{\text{av}}. \quad (5.2)$$

Since $\delta R(\mathbf{x},\mathbf{x}';c)/\delta r_b$ is[20] the square of current $i_b(\mathbf{x},\mathbf{x}';c)$ that flows in the bond b if a unit current is injected at $\mathbf{x}$ and removed in $\mathbf{x}'$,

$$\left.\frac{\partial\overline{C_R}^{(l)}(\mathbf{x},\mathbf{x}')}{\partial\Delta^{(l)}}\right|_{s=0}=\chi^{-1}(\mathbf{x},\mathbf{x}')\left[\sum_b\nu(\mathbf{x},\mathbf{x}';c)i_b^{2l}\right]_{\text{av}}. \quad (5.3)$$

The right-hand side of Eq. (5.3) is equivalent to the noise function introduced by Rammal *et al.*[8] and de Arcangelis *et al.*[9] which scales as $|\mathbf{x}-\mathbf{x}'|^{-x_l}$. The left-hand side, on the other hand, scales as $|\mathbf{x}-\mathbf{x}'|^{\psi_l/\nu_p}$ since it has the same behavior as $\overline{C_R}^{(l)}$. This establishes the equivalence of the family $\{\psi_l\}$ of exponents introduced here and those studied in Refs. 8 and 9.

In the limit of $l\to 0$, i_b^{2l} becomes an indicator variable that is unity if current flows in the bond b and zero otherwise. Thus,

$$N_2(\mathbf{x},\mathbf{x}')\equiv\lim_{l\to 0}\frac{\partial\overline{C_R}^{(l)}}{\partial\Delta^{(l)}} \quad (5.4)$$

is the average number of bonds that carry current when unit currents are inserted and removed at sites $\mathbf{x}$ and $\mathbf{x}'$ in the same cluster. When $|\mathbf{x}-\mathbf{x}'|\to\infty$ and $p>p_c$, this is, by definition, the number of bonds in the backbone of the infinite cluster. ($\mathbf{x}$ and $\mathbf{x}'$ could be on "dangling bonds," but the fraction of current carrying bonds on the dangling ends will tend to zero for large separation of sites $\mathbf{x}$ and $\mathbf{x}'$.) Since the probability that a bond is on the backbone is proportional to $(p-p_c)^{\beta_2}$, where β_2 is the critical exponent for the backbone, N_2 becomes

$$\lim_{|\mathbf{x}-\mathbf{x}'|\to\infty} N_2 \sim L^d (p-p_c)^{\beta_2} \quad (p>p_c) \tag{5.5}$$

where L^d is the total number of bonds in a d-dimensional lattice of size L. Thus, by scaling at p_c,

$$N_2(\mathbf{x},\mathbf{x}') \sim |\mathbf{x}-\mathbf{x}'|^{d-\beta_2/\nu_p} . \tag{5.6}$$

We note that the exponent $d-\beta_2/\nu_p$ is the fractal dimension of the backbone. On the other hand, from its definition $N_2(\mathbf{x},\mathbf{x}') \sim |\mathbf{x}-\mathbf{x}'|^{\psi_0/\nu_p}$, so that we can make the identification

$$\psi_0 = d\nu_p - \beta_2 . \tag{5.7}$$

The exponent β_2 (Refs. 21 and 22) is $2\beta_p$ plus correction of order ϵ^2 so that $d\nu_p-\beta_2 \approx d\nu_p - 2\beta_p = \gamma_p = 1+\epsilon/7$, in agreement with Eq.(4.7b) for ψ_0.

ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation under Grant No. DMR85-25380, and one of us (Y.P.) would like to acknowledge partial support from the Laboratory for the Research on the Structure of Matter (LRSM), University of Pennsylvania, under Grant No. DMR 85-19059.

APPENDIX

In this appendix, we calculate d_l of Eq. (4.5). Here we take $G_{\overleftrightarrow{k}}$ as the following form:

$$G_{\overleftrightarrow{k}}^{-1} = r_0 + w_1 \overleftrightarrow{k}^2 + \sum_{s>2} v_s K_s \tag{A1}$$

since we may set $w_n=0$ for $n\geq 2$ in the calculation of crossover exponent for v_l. Then $\tilde{\Sigma}_{\overleftrightarrow{k}}$ becomes

$$\tilde{\Sigma}_{\overleftrightarrow{k}} = \lim_{n\to 0} \sum_{\overleftrightarrow{p}} (\Delta p)^{nm} G(\overleftrightarrow{p}) G(\overleftrightarrow{k}+\overleftrightarrow{p})$$

$$= \lim_{n\to 0} \int Dp_{\alpha\beta} \int_0^\infty du \int_0^\infty dt \exp[-uG^{-1}(\overleftrightarrow{p})] \exp[-tG^{-1}(\overleftrightarrow{p}+\overleftrightarrow{k})] \tag{A2a}$$

$$= \lim_{n\to 0} \int Dp_{\alpha\beta} \int_0^\infty du \int_0^\infty dt \exp[-(u+t)(1+r_0)] \exp\left[-w_1 \left[u \sum_{\alpha\beta} p_{\alpha\beta}^2 + t \sum_{\alpha\beta} (k_{\alpha\beta}+p_{\alpha\beta})^2\right]\right]$$

$$\times \exp\left\{-\sum_{s=2}^{\infty} v_s \left[u \sum_{\beta} \left[\sum_{\alpha} p_{\alpha\beta}^2\right]^s + t \sum_{\beta} \left[\sum_{\alpha} (k_{\alpha\beta}+p_{\alpha\beta})^2\right]^s\right]\right\} . \tag{A2b}$$

We expand to the linear order in v_l and set $p_{\alpha\beta} = \xi_{\alpha\beta} - tk_{\alpha\beta}/(u+t)$, so that we have

$$\tilde{\Sigma}_{\overleftrightarrow{k}} = \lim_{n\to 0} \int D\xi_{\alpha\beta} \int_0^\infty du \int_0^\infty dt \exp[-(u+t)(1+r_0)] \exp[-w_1(u+t)\overleftrightarrow{\xi}^2]$$

$$\times \left[1 - w_1 \left[\frac{ut}{u+t}\left[\sum_{\alpha\beta} k_{\alpha\beta}^2\right]\right] - \sum_{s=2}^{\infty} v_s \frac{(-1)^s}{2^s(s!)} \left[\sum_{\beta} u \left\{\sum_{\alpha}\left[\xi_{\alpha\beta} - \left[\frac{tk_{\alpha\beta}}{u+t}\right]\right]^2\right\}^s + \sum_{\beta} t \left\{\sum_{\alpha}\left[\xi_{\alpha\beta} + \left[\frac{uk_{\alpha\beta}}{u+t}\right]\right]^2\right\}^s\right]\right] . \tag{A3}$$

The integrals in (A3) are convergent provided the potentials v_s are taken to be zero before w_1. As we argued in Sec. III, this is the physically relevant order of limits. To linear order in r_0, Eq. (A3) can be written as

$$\tilde{\Sigma}_{\overleftrightarrow{k}} = 1 - 2r_0 + d_1 w_1 \overleftrightarrow{k}^2 + \sum_{s>2} d_s v_s K_s + \sum_{s'>s} m_{ss'} K_s v_{s'} w_1^{-(s'-s)} + \cdots \tag{A4}$$

where isotropic terms of order $\overleftrightarrow{k}^4$ and higher which contribute to w_{2n} have been omitted, and where

$$d_s \equiv -\lim_{n\to 0} \int_0^\infty dx \int_{-x}^{x} dy \left\{\frac{e^{-x}}{2^{2s+2}}\left[(x-y)\left[\frac{x+y}{x}\right]^{2s} + (x+y)\left[\frac{x-y}{x}\right]^{2s}\right]\right\} , \tag{A5}$$

which gives Eq. (4.5). Note that $d_1 = c_1$ so that crossover exponent ϕ_1 and ψ_1 have the same value as expected. Since $m_{ss'}$ is a triangular matrix and we require v_s to go to zero before w_1, the terms proportional to $v_{s'} w_1^{-(s'-s)}$ affect the scaling functions but not the crossover exponents. There are, of course, terms in $\tilde{\Sigma}_{\vec{k}}$ of higher than linear order in v_s. These again do not affect the exponents.

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