Branched polymers on fractal lattices

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The asymptotic properties of large branched polymers placed on fractal lattices are studied using exact recursion relations. Simple examples are given on quasilinear fractals to illustrate the general method and explain how the critical exponents θ and ν are obtained. For more complex systems we find that a collapse transition may occur in the presence of attractive interactions between the individual monomers (i.e., for a polymer in a bad solvent). Above the critical temperature the polymer has the geometry of a random lattice animal, while below T_c it shrinks into a compact globule state. The geometrical and thermal exponents at the transition are also obtained exactly and compared to recent work on regular Euclidean lattices. For one of the lattices studied we find an essential singularity in the polymer generating function, rather than a power-law singularity. This suggests that a similar behavior may also occur in other systems lacking translational invariance, such as polymers on randomly dilute lattices.

I. INTRODUCTION

There are two essential motivations for studying models of branched polymers on fractal lattices. Firstly, many results have been obtained in recent years for polymers on periodic, Euclidean lattices,[1,2] but the situation for disordered lattices presents subtle difficulties and is not yet clarified.[3–5] Fractals may be considered as intermediate between the two cases, as they offer a class of systems where the consequences of the loss of translational invariance may be studied in detail. This approach has been very fruitful for random walks and diffusion[6] and it is natural to extend it to models of polymers in dilute solution, i.e., self-avoiding walks (SAW) for linear polymers, and random lattice animals for branched polymers. We present here results for the latter case, on various fractal lattices, showing how the asymptotic properties of large polymers are modified with respect to Euclidean lattices. We calculate the critical exponents θ and ν exactly, by studying the fixed points of a closed system of recursion equations. The variables in these equations are partial generating functions corresponding to different polymer configurations for a given size of the fractal lattice. The mathematical structure of the recursion system in the neighborhood of the fixed points may be rather intricate, and in one case we even find that the global polymer generating function has an essential singularity, rather than the standard power-law singularity found for periodic lattices.

The second motivation comes from the existence, on some of the lattices studied, of a collapse transition for a single isolated polymer, when effective attractive interactions are introduced to model the effect of a poor solvent. This transition is of the same nature as the collapse phenomenon which is observed on periodic lattices[7,8] and in some experiments.[9] The polymer shrinks from an extended, random-animal state to a compact, globule state when the temperature is lowered. In the limit where the number N of monomer units goes to infinity, the transition may be described as a critical phenomenon, analogous to a tricritical point for magnetic systems,[10] and the specific heat has a singularity. For large but finite polymers the average gyration radius R_N behaves as $R_N \sim N^{\nu_t}$, where the exponent ν_t is intermediate between the value ν for random animals and the value $\nu_c = 1/D$ for compact globules, D being the fractal dimension of the lattice.

On the finitely ramified fractal lattices studied here, it is possible to write down a closed set of recursion equations describing the transition and to obtain the critical exponents exactly. This is of great interest because fractals offer a whole domain where one may hope to obtain new and deep insights into critical phenomena. This expectation has met with a serious obstacle: Either the models are solvable but they possess no transition at finite temperature (e.g., the Ising or Potts models on finitely ramified fractals[11]), or a nontrivial transition exists but the model can be solved only approximately (e.g., Ising spins on infinitely ramified fractals[12]). Together with a few other problems (collapse of linear polymers,[13] Lee-Yang singularity[14]), the study of the collapse transition of branched polymers is very instructive on fractals because one can analyze the system in detail. It turns out moreover that some of the results are in good quantitative agreement with those for the Euclidean case. For instance, the exponent ν_t for the Sierpinski gasket is extremely close to the value ν_c for the compact phase: A similar situation was recently found numerically for the square lattice.[7]

The paper is organized as follows. In Sec. II the general formalism is presented and illustrated by two simple examples, the "T fractal," a treelike lattice we introduce, and a modified Koch curve with loops on all scales. Then we study in detail the properties of branched animals on the two- and three-dimensional Sierpinski gaskets, obtaining the phase diagram as a function of the effective interaction strength between monomers. In particular, we analyze the collapse transition occurring on these lattices. In Sec. IV we consider a modified gasket, for which the

critical behavior turns out to be very different and the generating function has an essential singularity, in contrast with the power-law singularity found in the previous cases.

II. GENERAL FORMALISM AND SIMPLE EXAMPLES

A. Lattice models of branched polymers

In the following we use the standard statistical approach to branched polymers and model them by connected graphs of N links embedded in a lattice (L), each link representing a monomer unit. The existence of loops in the graphs will be explicitly shown to be irrelevant as far as large-scale properties are concerned. If the only important interaction is the excluded volume effect between monomers, the problem reduces to the "animal" problem, where a weight factor x is associated with each link of a graph and all possible configurations of N links have the same weight. The generating function is then

$$G(x)=\sum_{N=0}^{\infty}\overline{\Omega}_N x^N , \tag{1}$$

where $\overline{\Omega}_N$ is the number of different configurations per site of an animal having N bonds, averaged over all lattice sites. If one *assumes* that for large N this number has the same type of behavior as on periodic lattices

$$\overline{\Omega}_N \sim \mu^N N^{-\theta} , \tag{2}$$

then one finds that $G(x)$ has a power-law singularity[15]

$$G(x)\sim(1-\mu x)^{\theta-1} \quad (x\to 1/\mu) . \tag{3}$$

Here $\mu=1/x_c$ is the lattice-dependent connectivity constant and θ is the "magnetic" critical exponent for the lattice-animal problem. The geometrical critical exponent ν is defined by the relation

$$\langle R_N^2\rangle \sim N^{2\nu} \tag{4}$$

for large N, where R_N is the gyration radius and the average is taken over all animals. The corresponding generating function is

$$R^2(x)=\lim_{\mathbb{N}\to\infty}\mathbb{N}^{-1}\sum_{(A)} R^2(A)x^{N(A)} ,$$

where $R(A)$ is the gyration radius of the random-animal A, having $N(A)$ bonds, and the summation extends over all animals that can be drawn on the finite lattice (L) of $\mathbb{N}$ sites. For large $\mathbb{N}$, $R(x)$ behaves as[16]

$$R(x)\sim(1-\mu x)^{\theta-1-2\nu} .$$

A more convenient way to obtain the exponent ν is to use standard renormalization techniques. For the animal problem on finitely ramified lattices,[15] one may obtain a closed set of recursion relations between a finite number of coupling constants. By linearizing these recursion relations around the appropriate fixed point we determine the "thermal" eigenvalues of the transformation matrix. The largest eigenvalue gives the correlation length exponent, and we have checked explicitly on several problems of polymers on fractals that this exponent is the same as ν defined in Eq. (4).

B. Polymers on the T fractal lattice

We derive now exact results for two quasilinear lattices which provide good pedagogical examples. The simplest fractal lattice on which a nontrivial branched polymer problem may be defined is the treelike T fractal (Fig. 1), whose fractal dimension is $D=\ln 3/\ln 2$. A closed set of recursion equations is obtained with only two restricted generating functions A_r and B_r described in Fig. 2. The renormalization transformation consists in summing over the internal configurations of the polymer on the rth-order T fractal that are consistent with the nature of the generating functions at iteration order $(r+1)$. One easily finds

$$\begin{aligned} A_{r+1}&=A_r^2(A_r+B_r) ,\\ B_{r+1}&=B_r(1+A_rB_r+A_r^2) , \end{aligned} \tag{5}$$

with the initial values

$$A_1=x^2+x^3,\ B_1=1+x+x^2 \tag{6}$$

(equivalently $A_0=x$, $B_0=1$, with the meaning that at order zero either the single link is occupied—case A, weight x—or it is empty—case B, weight x^0).

The global generating function $\Xi_r(x)$ for the whole rth-order T fractal can also be constructed recursively:

$$\Xi_r(x)=3\Xi_{r-1}(x)+(A_{r-1}+B_{r-1})^2(3+A_{r-1}+B_{r-1}) . \tag{7}$$

Iterating this relation one obtains for the generating function per site

$$G(x)=x+x^2+x^3/3+\sum_{r=1}^{\infty}3^{-r}(A_r+B_r)^2(3+A_r+B_r) . \tag{8}$$

If the value of the fugacity x is less than $x_c=\mu^{-1}=0.46922311\ldots$, then under successive iterations $A_r(x)$ decreases to zero while $B_r(x)$ and $G(x)$ tend to certain constants. If x is larger than μ^{-1}, $A_r(x)$ and $B_r(x)$ tend to infinity for large r and $G(x)$ is infinite (Fig. 3). At the random-animal fixed point $x=x_c$ the asymptotic behavior of the partial generating functions is

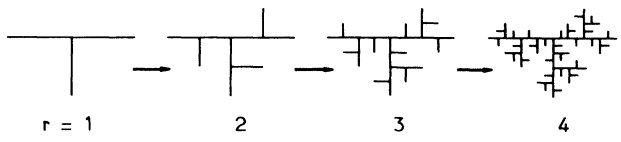

FIG. 1. First four stages in the iterative construction of the T fractal. The final object has a fractal dimension $D=\ln 3/\ln 2$ and is nonoverlapping in the plane. Note that it is very different from a Cayley tree, which has an *infinite* effective dimension.

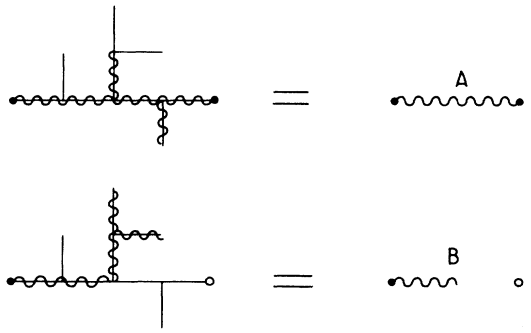

FIG. 2. Two examples of configurations for branched polymers on the T-fractal lattice (the wiggly lines represent monomer units). The configuration at the top contributes to A_r: This generating function corresponds to configurations where a part of the polymer joins the two opposite horizontal vertices of an rth order lattice, and its diagrammatic representation is a "superlink." The configuration at the bottom contributes to B_r, which enumerates the configurations where a part of the polymer is attached to only one of these vertices.

$$A_r \sim \bar{A} k^{-r},\ B_r \sim \bar{B} k^r \quad (r \to \infty) \tag{9}$$

where $\bar{A}$, $\bar{B}$, and $k > 1$ are finite constants. It is more convenient to work with new, nondiverging variables and the form of Eq. (9) suggests the choice

$$y = A,\ z = AB\ . \tag{10}$$

One has now, asymptotically,

$$y_{r+1} = y_r z_r,\ z_{r+1} = z_r^2 (1 + z_r)\ . \tag{11}$$

The nontrivial fixed point of Eq. (11) has coordinates

$$(y^*, z^*) = [0, (\sqrt{5} - 1)/2]$$

so the constant in Eq. (9) is given by $k = (z^*)^{-1} = (1 + \sqrt{5})/2$. Linearizing the recursion relations around this fixed point, we obtain only one eigenvalue greater than one: $\lambda = (7 - \sqrt{5})/2$. The exponent ν controlling the average linear size of the polymer [Eq. (4)] is then given by

$$\nu = \ln(b)/\ln(\lambda) = 0.79862\ldots\ ,$$

where $b = 2$ is the linear scale factor between two successive iterations. The exponent θ can be obtained most directly by considering $G_L(x_c)$, the global generating function of animals on a lattice of length L at the critical fugacity. This function is expected to diverge as $L^{(1-\theta)/\nu}$, by a standard finite-size-scaling argument. On the other hand, a direct evaluation of the dominant terms in $G(x)$ gives

$$G_L(x_c) \sim \sum_{r=1}^{R} 3^{-r} (1 + z^*)^{3r}\ ,$$

where $L = 2^R$. The dominant term in the sum is obtained when $r = R$, and we finally get

$$(1 - \theta)/\nu = 3 \ln(1 + z^*)/\ln 2 - D = \delta - D\ , \tag{12}$$

where $D = \ln 3/\ln 2$ is the fractal dimension of the T fractal. The numerical values are

$$\theta = 0.6025,\ \delta = \ln(2 + \sqrt{5})/\ln 2 = 2.0827\ldots\ .$$

These values agree with the results obtained from a complete calculation of $G(x)$ following the lines of Dhar's work on linear polymers.[16]

One has to note that the value obtained for δ is only slightly larger than the value $\delta = 2$ valid for Euclidean lattices in any dimension $d \leq 8$, as deduced from the Parisi-Sourlas relation[17]

$$(1 - \theta)/\nu = 2 - d\ . \tag{13}$$

The new relation (12) generalizes Eq. (13) to fractal lattices and shows that the two exponents θ and ν are now independent in general. This is not really surprising, since the derivation of the Parisi-Sourlas relation depends on translational invariance, which is lost on fractals. The quantity δ appearing in Eq. (12) is a new fractal exponent, probably not reducible to previously defined quantities.

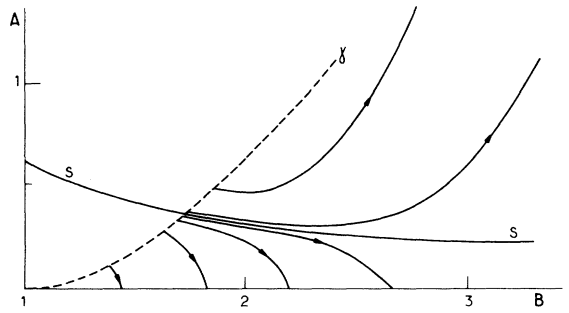

FIG. 3. Schematic flow diagram for branched polymers on the T-fractal lattice. Under the recursion relations [Eq. (5)], the set of initial conditions, represented by curve γ [Eq. (6)] iterates either to the B axis or to infinity. These two regions are separated by the separatrix S, whose asymptotic equation is given by $AB \simeq (\sqrt{5} - 1)/2$ for large A.

C. Branching Koch curve

As a second example we consider the animal problem on the branching Koch curve shown in Fig. 4. This lattice contains loops on all scales and one needs a third parameter to get a closed set of recursion equations. Beside two functions analogous to those described in Fig. 2, we have to define a third partial generating function C_r which enumerates configurations with two disconnected pieces of polymer, each one attached to an extremity of the lattice at stage r (Fig. 5). It is straightforward to write down the

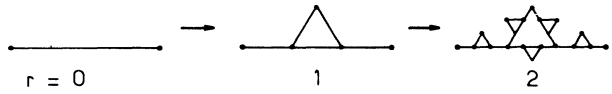

FIG. 4. First stages of the construction of the branching Koch curve.

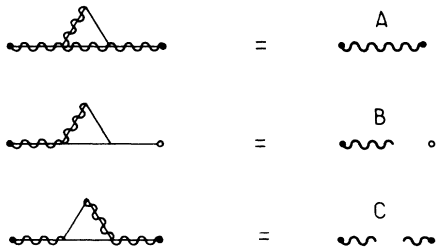

FIG. 5. Diagrams representing the restricted generating functions A_r, B_r, and C_r for the branching Koch curve, and examples of configurations belonging to each type.

recursion relations for these three functions (see Fig. 6 for an example):

$$\begin{aligned} A_{r+1} &= A^3(B^2+3AC+A^2) , \\ B_{r+1} &= B(1+AB+A^2(B+B^2)+3A^3C+A^4) , \\ C_{r+1} &= B^2+2AB^2(B+AC+AB) \\ &\quad +A^2C(B^2+8AC+2A^2) , \end{aligned} \tag{14}$$

with initial values $A_0=x$, $B_0=1$, $C_0=1$ [the iteration index r has been omitted on the right-hand side of Eq. (14) for clarity]. The expression of the global generating function $G(x)$ is cumbersome and will not be given here.

If the fugacity x is larger than a critical value $x_c=1/\mu=0.4598059\ldots$, A_r, B_r, C_r tend to infinity for large r and $G(x)$ is infinite. For $x=x_c$ the asymptotic behavior of the generating functions is given by

$$\begin{aligned} A_r &\sim \tilde{A} l^{-r} , \\ B_r &\sim \tilde{B} l^{r} , \\ C_r &\sim \tilde{C} l^{2r} , \end{aligned} \tag{15}$$

where $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and l are finite constants. Choosing as new variables $y=A$, $z=AB$, $t=A^2C$, and keeping only the dominant terms, one obtains the recursion relations

$$\begin{aligned} y_{r+1} &= yz^2 , \\ z_{r+1} &= z^3(1+z+z^2) , \\ t_{r+1} &= z^6(1+2z+3t) , \end{aligned} \tag{16}$$

which have a nontrivial fixed point

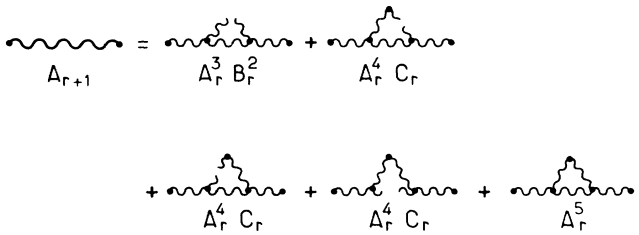

FIG. 6. Diagrammatic representation of the recursion relations for the restricted generating function A_r for the animal problem on the branching Koch curve.

$$(y^*,z^*,t^*)=(0.,\,0.682328,\,0.342244) ,$$

where z^* is the real root of $z^3+z-1=0$

$$z^*=[(\tfrac{31}{108})^{1/2}+\tfrac{1}{2}]^{1/3}-[(\tfrac{31}{108})^{1/2}-\tfrac{1}{2}]^{1/3} .$$

The linearized relations around the fixed point have only one eigenvalue larger than one, $\lambda=3.751185\ldots$. The geometric exponent is

$$\nu=\ln 3/\ln\lambda=0.830978\ldots .$$

At the critical point the dominant contribution to the global generating function, at iteration order R, is of the form

$$G_R(x_c)\sim\sum_{r=1}^{R}5^{-r}(A_rB_r^4+2B_r^3)\sim\sum^R[l^3/5]^r .$$

This yields finally, through the same scaling argument as above and noting that the constant l in Eq. (15) is equal to $(z^*)^{-2}$

$$\begin{aligned} (1-\theta)/\nu &= -6\ln z^*/\ln 3-D \\ &= \delta-D , \end{aligned} \tag{17}$$

where $D=\ln 5/\ln 3$ is the fractal dimension of the branching Koch curve. The numerical values of the exponents are

$$\theta=0.482606\ldots ,\quad \delta=2.0876\ldots .$$

The detailed calculation of the leading singularity of $G(x)$ gives the same results.

An important remark is that the effect of loops can be studied in detail, by distinguishing explicitly the configurations containing loops, at every stage of iteration. The generating function $A_r(x)$, for instance, splits into two parts, $A_r^{(1)}$ and $A_r^{(2)}$, corresponding respectively to the configurations with and without loops. One obtains a recursion system of six equations that generalizes Eqs. (14). The asymptotic behavior of the generating functions without loops on any scale remains the same as that of the full generating functions, as given by Eqs. (15) and (16), though the critical fugactiy $\bar{x}_c$ is different. So, lattice animals *with* and *without* loops belong to the same universality class, and this type of constraint has no influence on the large-scale behavior of branched polymers. This is in agreement with the standard renormalization-group arguments given for Euclidean lattices, but here the result is completely rigorous. It is also worth noticing that the exponent δ is again quite close to the value $\delta=2$ of Euclidean lattices: this near coincidence is puzzling, but it does not persist for other fractals studied below.

III. SELF-INTERACTING BRANCHED POLYMERS: THE ANIMAL-GLOBULE COLLAPSE TRANSITION ON FRACTAL LATTICES

A. Formulation of the problem

We now consider the effect of interactions between the monomer units, in addition to the excluded volume constraint which takes into account the hard-core repulsion.

In a poor solvent monomers belonging to the same branched polymer will experience an effective attractive interaction and it is well known that a collapse transition may occur if this self-interaction is strong enough.[9]

This phenomenon can be studied on a lattice using the two-variable generating function $G(x,w)$

$$G(x,w)=\sum_{N,P}\Omega(N,P)x^N w^P=\sum_N Z_n x^N , \tag{18}$$

where $\Omega(N,P)$ is the number of different configurations per site of a branched polymer having N monomers and P pairs of nearest neighbors and $Z_N=Z_N(w)$ is the partition function of N-link animals. The interaction strength w is related to the temperature T by the usual Boltzmann factor $w=\exp(\epsilon/k_BT)$, $\epsilon>0$ being the attractive energy associated with a pair of nearest-neighbor bonds (monomers) and k_B is the Boltzmann constant. At infinite temperature the energies of different configurations play no role and the generating function $G(x,w=1)$ reduces to the usual generating function of unweighted animals [Eq. (1)]. In particular, the free energy per site $f(T)$ is given by[7]

$$\begin{aligned} f(T)&=-k_BT\lim_{N\to\infty}N^{-1}\ln Z_N \\ &=k_BT\ln[x_c(w)] , \end{aligned} \tag{19}$$

where $w=\epsilon/k_BT$ and $x_c(w)$ is the radius of convergence of the series in the right-hand side of Eq. (18).

For simplicity we restrict in the following the attractive interactions to bonds within first-order units of the fractal lattice. It can be shown explicitly for several models of linear polymers on fractals that this restriction has no effect on the critical exponents.[13] This result is expected to be general, by analogy with the universality properties of phase transitions in Euclidean spaces—it is clear anyway that if a transition exists with restricted interactions it will also exist with the full set of interactions (the converse might not hold in some cases). The restriction has the important technical advantage that the recursion relations do not contain explicitly the interaction strength w, which appears only in the initial values of the generating functions. An analysis of the recursion relations established for *non*-self-interacting polymers is then sufficient also for self-interacting polymers: One has just to look for other fixed points of these relations that may be reached from more general initial conditions. Of course, it remains necessary to use the full equations with interactions on all scales if one wants to obtain the precise form of the phase boundaries, of the specific heat or of the compressibility. This is not our purpose here; we are only interested in the existence of the collapse transition and its general features.

B. Collapse transition on the two-dimensional Sierpinski gasket

We did not find a collapse transition on the branching Koch curve studied in Sec. II, for any finite value of the temperature. This is probably due to its quasilinear character (no transition is possible on the T fractal since it has no loops). The simplest lattice on which we did find a transition is the two-dimensional (2D) Sierpinski gasket (or equivalently the three-simplex lattice).[18] A closed set of recursion relations may be obtained with six restricted generating functions (see Fig. 11 of Sec. IV). We will not repeat here the technical details which were given in a previous paper,[15] but for completeness and as a basis for comparison we quote the main results.

Three different regimes were identified:

(i) For high temperatures (i.e., for small interaction parameter, $w<w_t\simeq 5.5$), the polymer is in a swollen phase and its large scale properties are the same as for the random-animal problem ($w=1$). The global generating function has a power-law singularity when $x\to x_c(w)$ and the critical exponents are $\nu=0.71655$, $\delta=2.2370$, which corresponds to an exponent $\theta=0.5328$

(ii) When the temperature is lowered a collapse transition occurs for $w=w_t$. The corresponding fixed point is a tricritical one, with two eigenvalues larger than one, in complete analogy with the situation on the square lattice for instance.[7] The geometrical exponent is $\nu_t=0.63250$, extremely close to the value $\nu_c=0.63093$ in the collapsed globule phase. The thermal critical exponent α controlling the singular part of the free energy

$$f_{\text{sing}}(w)\sim |w-w_t|^{2-\alpha} \tag{20}$$

is negative: $\alpha=-4.0269$, so the specific heat is very smooth at $w=w_t$.

(iii) At low temperatures ($w<w_t$), the polymer is in a collapsed phase, with $\nu_c=1/D$, $D=\ln 3/\ln 2$ being the fractal dimension of the gasket. The monomer density per site is finite in that regime.

The most interesting aspect of these results, apart from the existence of the transition itself, is their similarity with recent numerical findings on the square lattice. An accurate transfer-matrix study of the collapse of branched polymers on that lattice[7] gives $\nu_t=0.509\pm 0.003$, much closer to the compact value $\frac{1}{2}$ than previously believed. This suggests that the phenomenon is not accidental and may receive a general interpretation.

C. Three-dimensional Sierpinski gasket

The problem of branched polymers is much more difficult on the three-dimensional (3D) gasket (equivalently, the four-simplex lattice, Fig. 7). We have to define eleven restricted generating functions to obtain a closed

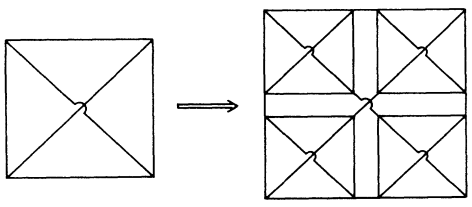

FIG. 7. Basic iteration step for the four-simplex lattice, shown here in projection (in 3D space each elementary unit is a tetrahedron).

set of recursion equations and these functions are depicted in Fig. 8, using a notation that generalizes the one used by Dhar for SAW on the four simplex.[16] The number of polymer configurations to consider is so large that one has to use computer enumeration to sort them out and the resulting relations take up too much space to be given here.[19]

The initial values of the eleven functions are for the self-interacting case:

$$A_1=x,\quad B_1=x^2w^4,\quad C_1=1\ ,$$
$$D_1=xw^2,\quad E_1=w,\quad F_1=xw^5\ ,$$
$$G_1=w^3,\quad H_1=w^6,\quad I_1=x^2(x+3xw)\ , \tag{21}$$
$$J_1=x^2w^3(x+3w)\ ,$$
$$K_1=x^3(x^3+6x^2w+15xw^2+16x^3w^3)\ .$$

For a given interaction strength w, there exists a critical value $x_c(w)$ above which the iterations diverge—this value determines the free energy $f(T)$ via Eq. (19). For $x=x_c(w)$ we find that the recursion equations have the following asymptotic behavior as r goes to infinity:

$$A_r\sim A^*,\quad B_r\sim B^*,\quad C_r\sim C^*q^r,\quad D_r\sim D^*q^r\ ,$$
$$E_r\sim E^*q^{2r},\quad F_r\sim F^*q^{2r},\quad G_r\sim G^*q^{3r},\quad H_r\sim H^*q^{4r}\ , \tag{22}$$
$$I_r\sim I^*q^{-r},\quad J_r\sim J^*,\quad K_r\sim K^*q^{-2r}\ ,$$

where A^*, B^*, . . . , K^* and $q(>1)$ are finite and may *a priori* depend on w.

To determine the location of the relevant fixed points and the critical exponents we introduce new, nondiverging variables

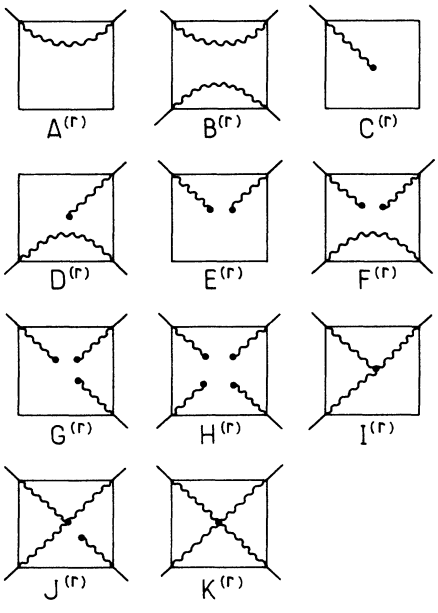

FIG. 8. Diagrammatic representation of the 11 partial generating functions for the four-simplex lattice.

$$a=A,\quad b=B,\quad c=CI,\quad d=DI\ ,$$
$$e=EK,\quad f=FK,\quad g=GDK,\quad h=HK^2\ , \tag{23}$$
$$i=I^2/K,\quad j=J,\quad k=K\ .$$

Other choices are possible and give the same results (in practice, some choices may prove more convenient to locate the fixed points numerically, but we did not investigate that question in detail). The search for the fixed points in terms of the new variables is not straightforward, however, because the resulting system of 11 nonlinear algebraic equations has a large number of fixed points and a direct application of the multidimensional Newton method fails. It is necessary to iterate the complete system of equations, in order to identify approximately the nature and location of the relevant fixed points. Their coordinates are then determined very accurately by Newton's method.

We find in this way three relevant fixed points for the large-scale behavior of self-interacting branched polymers. In terms of the reduced variables defined in Eq. (23):

(i) The fixed point $(a,b,c,\ldots,k)^*=(0.18459,\ 0.00322,\ 0.06673,\ 0.00463,\ 0.01710,\ 0.00071,\ 0.00057,\ 0.00011,\ 0.15776,\ 0.02138,\ 0)$ is reached for large values of the temperature—i.e., all initial conditions with $w<w_t\simeq 2$ lead to that fixed point. This is the random-animal fixed point corresponding to the swollen phase of the polymer. Linearizing around the fixed point one finds only one relevant eigenvalue: $\lambda=3.14069$. The geometrical exponent is then $\nu=\ln 2/\ln\lambda=0.60566$. The global generating function has again a power-law singularity and one can obtain the corresponding exponent along the same lines as given in Sec. II, using finite-size scaling. An independent calculation of the dominant singularity of $G(x)$ as $x\to 1/\mu[=x_c(w=1)=0.129263]$ gives the same values of the exponents: $\theta=0.75667$ and $\delta=2.40177$.

(ii) The fixed point which is reached for low values of the temperature ($w>w_t$) has most reduced variables equal to zero, except four of them:

$$(b,f,h,j)^*=(0.00999,\ 192^{-2/3},\ \tfrac{1}{48},\ 0.16334)\ .$$

These nonvanishing generating functions describe the polymer configurations having all vertices occupied (configurations B, F, H, J, K in Fig. 8). Linearization about this fixed point leads to only one relevant eigenvalue: $\lambda=4$, corresponding to $\nu_c=\frac{1}{2}=1/D$, $D=2$ being the fractal dimension of the 3D Sierpinski gasket. This is, therefore, a compact fixed point describing the collapsed phase of the polymer, with a finite density per site.

(iii) A third fixed point $(a,b,c,\ldots,k)^*=(0.03728,\ 0.01592,\ 0.01870,\ 0.00491,\ 0.02239,\ 0.02378,\ 0.00257,\ 0.01496,\ 0.03059,\ 0.13771,\ 0.)$ is reached only for $w=w_t$ and has two eigenvalues greater than one: $\lambda_1=3.94050$ and $\lambda_2=1.32094$. This tricritical point corresponds to the collapse transition of the branched polymer. The average gyration radius scales as N^{ν_t} with $\nu_t=\ln 2/\ln\lambda_1=0.50546$, once again quite close to the value $\nu_c=\frac{1}{2}$ in the collapsed phase. If one looks at the location of the fixed points in phase space, it is clear that the tricritical and the compact points are close to each other, since only

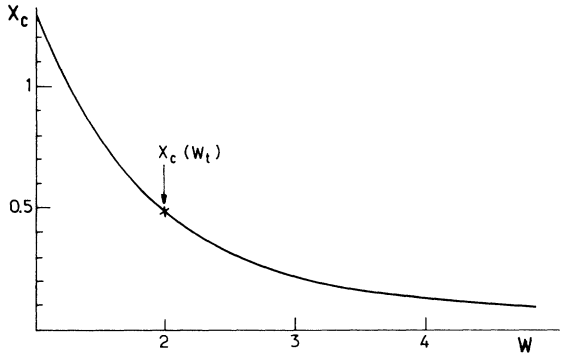

FIG. 9. Critical fugacity x_c for branched polymers on the four-simplex lattice, as a function of the self-interaction parameter w. The collapse transition occurs for $w=w_t\simeq 2$.

the variable j^* is large at both points, all other coordinates being small. The random-animal point is very far apart, the largest variables being a^*, i^*, and c^*, all others being much smaller.

The free energy per site $f(T)$ of the branched polymer is singular at $w=w_t$, and the thermal critical exponent is given by

$$\alpha=2-\ln\lambda_1/\ln\lambda_2=-2.9267 \ .$$

The transition is weak, but less so than on the 2D Sierpinski gasket.

In Fig. 9 we plot the radius of convergence x_c of $G(x,w)$ as a function of the interaction strength w. This gives the "phase diagram" for polymers in solution in a fractal environment. The figure also gives the internal energy per site $\bar{E}$ of the polymer, through the simple relation

$$\bar{E}=-wd(\ln x_c)/dw \ . \tag{24}$$

A final remark is that our model of self-interacting branched polymers is, in fact, more general than the cases studied above. In particular, it includes as a special case the problem of linear polymers: If one suppresses all terms containing the generating functions G, H, I, J, and K of Fig. 8, one obtains the recursion relations for *linear* polymers on the 3D Sierpinski gasket, which have themselves three relevant fixed points, including a tricritical point corresponding to a collapse transition.[13] So, by introducing appropriate weights for the configurations containing branch points, it would be possible to study the crossover between the two types of polymers. Other fixed points of the recursion system might be reached if other constraints or more complicated interactions are included.

IV. ESSENTIAL SINGULARITY FOR POLYMERS ON A MODIFIED GASKET

A. Construction of the recursion relations

The last fractal lattice we study is a close relative of the 2D Sierpinski gasket of Sec. II and belongs to a whole family of generalized gaskets introduced by Given and Mandelbrot.[20] Its recursive construction is described in Fig. 10; it corresponds to a linear scale factor $b=3$ and has fractal dimension $D=\ln 6/\ln 3$. A first account of the results obtained for that lattice has been presented in a previous letter,[21] and we give here only the main points with more comments and some additional results.

The six partial generating functions which appear in the closed set of recursion relations are depicted in Fig. 11. They are exactly the same as for the 2D Sierpinski gasket, since they depend only on the different ways parts of the branched polymer may be connected to the three vertices. The relations themselves are much more intricate—one of them contains 154 terms—and have to be obtained by computer enumeration.[19] The initial values of the partial generating functions, in the presence of self-interactions, are

$$\begin{aligned} &A_1=1, \quad B_1=x, \quad C_1=xw^2 \ , \\ &D_1=w, \quad E_1=w^3, \quad F_1=x^3+3wx^2 \ . \end{aligned} \tag{25}$$

B. Random-animal phase

Iterating these relations for low values of w, one observes that only a few terms remain dominant in the region close to the critical fugacity $x\simeq x_c$ (for the case without self-interactions, $w=1$, $x_c^{-1}=\mu=5.14165$). The asymptotic form of the recursion system is then

$$A'=2A^2B^2 \ , \tag{26a}$$

$$B'=A^2B^2F+3AB^2 \ , \tag{26b}$$

$$C'=3A^2B^4+A^3B^2F \ , \tag{26c}$$

$$D'=4A^3B^2 \ , \tag{26d}$$

$$E'=6A^4B^2 \ , \tag{26e}$$

$$F'=2B^6+6AB^4F \ , \tag{26f}$$

where A' and A, for instance, stand for A_{r+1} and A_r.

A physical interpretation for the origin of this reduced system comes from the examination of the diagrams associated with the terms that appear on the right-hand side of Eqs. (26). These are shown in Fig. 12: In all of them the polymer goes through the central site and has a branch point there, it is clear that at a given order of iteration taking advantage of the higher connectivity of the central site allows many more different conformations for the same number of monomers, and this effect occurs on all scales. Mathematically, this is related to the notion of the "ramification order" R that measures the number

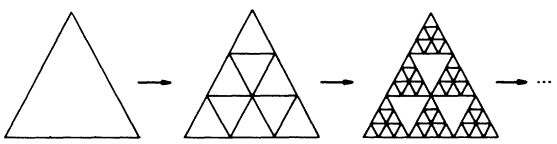

FIG. 10. First stages of the construction of the modified $b=3$ gasket.

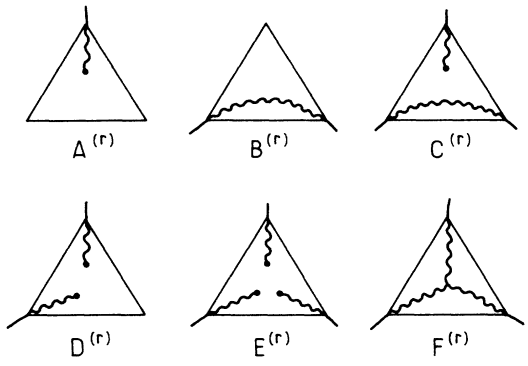

FIG. 11. Diagrammatic representation of the restricted generating functions for the modified gasket. They are identical to the generating functions for the Sierpinski gasket.

of bonds one has to cut to disconnect a finite region of the lattice (when the limit of vanishingly small elementary units is taken).[22] R depends on the point considered and it may be shown that for the Sierpinski gasket its minimum value is $R_{\min}=3$ and its maximum value $R_{\max}=4$, while for the modified $b=3$ gasket $R_{\min}=3$ but now $R_{\max}=6$. The important qualitative difference is that the Sierpinski gasket fulfills the "quasihomogeneity" condition $R_{\max}=2R_{\min}-2$,[22] while the modified gasket does not possess that property—intuitively, its topological environment may vary a lot from point to point.

The recurrence system, Eqs. (26), is in fact easier to study than one might think at first sight and its solution presents several new features. Introducing new variables $y=AF/B^2$, $z=EF/C^2$, $s=CE/A^2$, $t=AE/D^2$, $u=FA^{\sqrt{3}}$, one finds that the random-animal fixed point has coordinates

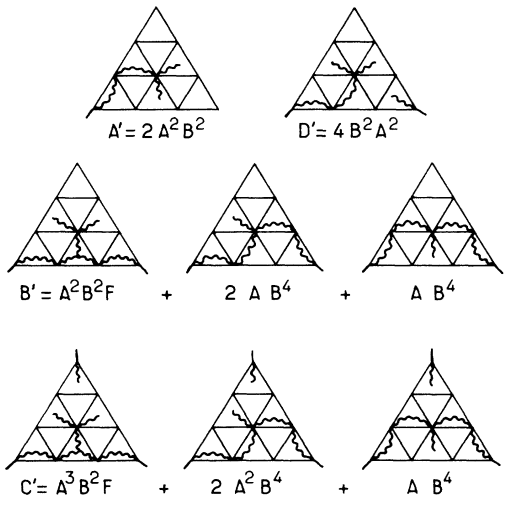

FIG. 12. Graphical interpretation of the dominant terms in some of the recursion relations for the modified gasket [Eq. (26)]. Note that in all diagrams the polymer has a branch point at the central site.

$$(y,z,s,t,u)^*=(1,3,3,\tfrac{3}{4},2^{-3+\sqrt{3}}) \tag{27}$$

in agreement with a direct search by iteration of the full system. Note that the variable u contains the generating function A elevated at an *irrational* power, while all variable changes used in the previous examples were rational. Linearizing around this fixed point we find an eigenvalue larger than one: $\lambda=3+\sqrt{3}$, so the geometric exponent is

$$\nu=\ln b/\ln\lambda=\ln 3/\ln(3+\sqrt{3})=0.7068\ldots .$$

This value is a little lower than $\nu=0.7165$ for the 2D Sierpinski gasket, in agreement with the intuitive expectation that the $b=3$ gasket is "closer" to a 2D Euclidean lattice, for which $\nu\simeq 0.64$.

The most important novel result concerns the global generating function $G(x)$. The dominant contribution when $x\to x_c$ is of the form

$$G(x)=\sum_r 6^{-r}[A_{r-1}(x)]^3 . \tag{28}$$

To estimate this quantity note that inserting relations (27) for the fixed point into the recursion system (26) gives an equation for A_r

$$A_{r+1}\sim 2^{\sqrt{3}-2}A_r^{3-\sqrt{3}} \quad (x=x_c) \tag{29}$$

so that

$$\ln A_r(x_c)\sim a_r(3-\sqrt{3})^r , \tag{30}$$

where a_r is a slowly varying function of r. This type of behavior is not compatible with a standard finite-size-scaling form for $G_L(x_c)$. Rather, it implies that $\ln G$ has a scaling form

$$\ln G_L(x_c)\sim L^{\psi/\nu} \tag{31}$$

and that $G(x)$ has an essential singularity of the form

$$G(x)\sim\exp[c(x_c-x)^{-\psi}] \tag{32}$$

where a new exponent appears

$$\psi=\ln(3-\sqrt{3})/\ln(3+\sqrt{3})=0.15273 .$$

This behavior of $G(x)$ may be traced back to an unusual asymptotic form of the number of distinct animals of mass N (Ref. 23):

$$\bar{\Omega}_N\sim\mu^{N+c'N^{\omega}} \tag{33}$$

with

$$\omega=\psi/(1+\psi)=0.13249 .$$

This is in contrast with the "standard" form with only a power-law correction to the dominant μ^N term, as given by Eq. (2). In fact, subdominant exponential terms of the form (33) are not excluded by any general theorem: Their absence in Euclidean systems is expected because the properties of polymers can be related to those of magnetic systems and the existence of the critical exponents is then justified by field theory. This argument fails on fractal lattices due to the loss of translational invariance, and our results show that in some cases the exponent θ may still

be defined, while in other cases the singularity of $G(x)$ is of a different nature. The important point is that the argument may also fail for disordered systems, since they do not possess translational invariance, and the singular behavior discovered for a particular fractal may be far more general.

We have noticed earlier that the $b=3$ gasket differs deeply from the Sierpinski gaskets due to its lack of quasihomogeneity. It is plausible that this property is necessary for the existence of an essential singularity, but it is not sufficient since the Koch curve is also nonquasihomogeneous. The study of other lattices will be necessary to clarify the conditions for essential singularities to appear. An important case is that of the fractal incipient infinite cluster at a percolation threshold. This fractal appears to have finite ramification order, as the gaskets, and it would be very interesting to see if it is (statistically) quasihomogeneous or not, and if there are indications of an essential singularity for branched polymers placed on percolation clusters.

C. Compact phase

In the opposite limit, when the self-interaction parameter w is large (low temperatures), the polymer is found in a collapsed state, as for the standard gasket. Here again, only a few terms of the recurrence relations are dominant and only three of the six partial generating functions, namely C, E, and F are relevant—they correspond to the densest configurations, where all three vertices are occupied. The recursion then reduces to

$$\begin{aligned} C' &= 142C^3F^3 + 18CEF^4 , \\ E' &= 2E^2F^4 + 77C^2EF^3 + 171C^4F^2 , \\ F' &= 60CE^2F^3 + 564C^3EF^2 + 468C^5F . \end{aligned} \tag{34}$$

The analysis of these relations yields the coordinates of the compact fixed point

$$EF/C^2 = 3, \quad E^2/C^7 = (14^7 3^{13} 10^3)^{1/5}$$

and linearization around that point gives only one relevant eigenvalue $\lambda_m = 6$, so the geometric exponent is $\nu_c = \ln 3/\ln 6 = 1/D$, as expected for a compact phase.

We did not locate the collapse transition temperature accurately enough to discover what terms are relevant in that region and find the corresponding tricritical point.

V. CONCLUSION

We have presented results for branched polymers on a variety of fractal lattices. Thanks to the simple topological structure of these lattices, the large scale properties of the polymers can be obtained *exactly* by a real-space renormalization-group method. Many of these properties are similar to those of polymers on periodic lattices, in particular the existence of a collapse transition in the presence of self-interactions, when the temperature is lowered. This is not always the case, however, and the existence of the critical exponent θ is shown to depend on the lattice considered: For a modified fractal gasket we find an essential singularity in the polymer generating function, instead of the power-law singularity predicted by field theory for Euclidean spaces. We suggest that such an essential singularity may also exist for polymers in sufficiently nonhomogeneous systems.

A technical aspect of our work needs some comments: The equations we derive for the polymer problem, though exactly solvable for the asymptotic properties, are quite complicated in some cases and it does not seem practical to go much further in that direction. This makes it desirable to have a systematic and well-controlled approximation scheme. This might be based on the empirical observation that in general only a few variables have large values at the critical points. The other variables are necessary to write a closed set of equations and are relevant in the renormalization-group sense, but they have much smaller values and might be treated as a perturbation. From that point of view the fractal lattices are an excellent testing ground to validate approximation methods before applying them to more difficult problems, such as random media.

Finally, it would be very instructive to study the *dynamical* properties of polymers on fractals, in view of the recent interest for the dynamics of polymers placed in porous materials.[24] It is not clear to us how to adapt the present methods to these problems: The main difficulty is to formulate a closed recursion system, analogous to the one we have obtained for static properties.

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