

## Condensate fraction in superfluid <sup>4</sup>He

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Recently, a relationship between the chemical potential and the condensate fraction  $n_0(T)$  has been derived for all temperatures in the superfluid region. An analysis of liquid <sup>4</sup>He chemical-potential data yields  $n_0(T=0)=0.062$ , and  $n_0(T)$  is in excellent agreement with the empirical results of Svensson, Sears, and Griffin.

### I. INTRODUCTION

In 1938, London<sup>1</sup> proposed that the lambda transition in liquid <sup>4</sup>He might be related to the phenomenon of Bose-Einstein condensation, but only in the past ten years have condensate fractions,  $n_0(T)$ , been determined experimentally. Measurements of  $n_0(T)$  from inelastic high-momentum neutron scattering,<sup>2-4</sup> and neutron<sup>5</sup> and x-ray<sup>6</sup> diffraction experiments are shown by the solid circles and squares in Fig. 1. Analyses of these data carried out by Sears and Swensson indicate that  $n_0(0) \cong 0.13$  at absolute zero.<sup>7,8</sup> However, by reconsidering the procedure of extracting the condensate fraction from inelastic neutron scattering studies, Griffin<sup>9</sup> has used the values represented by the solid squares<sup>3,4</sup> in Fig. 1 to come up with new ones,

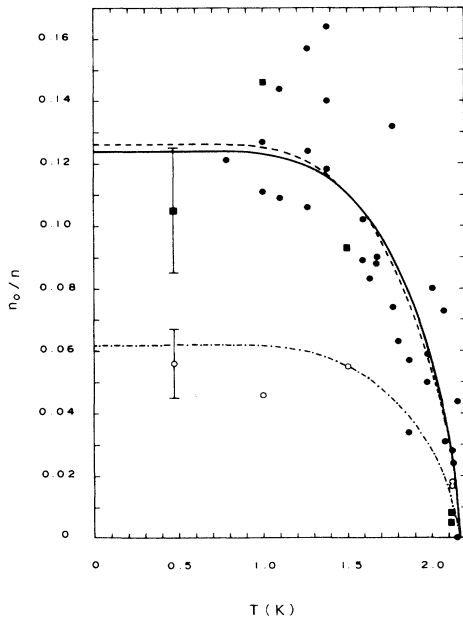


FIG. 1. Condensate fraction vs temperature. Dashed curve: empirical Eq. (1.1) (from Ref. 14),  $n_0(0)=0.126$ , Solid curve: Eq. (2.7)  $n_0(0)=0.124$ . Dashed-dotted curve: Eq. (2.7),  $n_0(0)=0.062$ . Solid circles and squares are experimentally determined values in superfluid <sup>4</sup>He from Refs. 2-6. Open circles are the measurements corresponding to the solid squares, as recalculated in Ref. 9. A typical experimental error is indicated at  $T=0.47$  K.

as shown by the open circles in the same figure. Although higher accuracy is still needed in both the experiments and the analysis procedure, one can now be rather confident about the existence of the condensate in superfluid <sup>4</sup>He, as first advanced by London more than four decades ago.<sup>1</sup>

From the theoretic side the first calculation of the condensate fraction at absolute zero is due to Penrose and Onsager<sup>10</sup> who found  $n_0=0.08$  by assuming a plausible ground-state wave function. More recent calculations have yielded values of  $n_0(0)$  that vary from<sup>11</sup> 0.09 to<sup>12</sup> 0.119, depending on the interatomic-force and calculation method.<sup>13</sup>

An empirical relationship between  $n_0(T)$  and the superfluid fraction  $n_s(T)$  were derived by Svensson, Sears, and Griffin,<sup>14</sup> for all temperatures  $T \leq T_\lambda$ , namely,

$$n_s(T) = \frac{n_0(T)[2 - n_0(T)]}{n_0(0)[2 - n_0(0)]} \quad (1.1)$$

A least-squares fit of Eq. (1.1) to the data<sup>2-6</sup> yields  $n_0(0)=0.126$  and corresponds to the dashed curve in Fig. 1, where use has been made of the measured values of  $n_s(T)$  obtained by Maynard.<sup>15</sup>

Recently, a theory based on generalized Ward identities has shown that, in addition to  $n_s(T)$ , the condensate fraction depends also on the atomic interaction and the chemical potential.<sup>16</sup>

The aim of the present paper is to further investigate the results of this theory. In Sec. II we compare it with the empirical approach. By fitting superfluid <sup>4</sup>He chemical potential data, we show in Sec. III that  $n_0(T)$  turns out to be close to the new values found by Griffin.<sup>9</sup> We conclude with a discussion in Sec. IV.

### II. COMPARISON WITH THE EMPIRICAL APPROACH

In order to compare the microscopic and empirical formulations of the condensate fraction we first outline briefly the former.

The broken gauge symmetry of the interacting Bose system is accomplished by the following prescription regarding the zero-mode amplitude,<sup>17</sup>

$$b_0 \equiv (1 - \xi)^{1/2} \alpha_0 + (\xi n'_0)^{1/2}, \quad (2.1)$$

such that the ensemble averages satisfy the condition

$$\langle \alpha_0 \rangle = \langle \alpha_0^\dagger \rangle = 0. \quad (2.2)$$

Hence, it follows that

$$\langle b_0^\dagger b_0 \rangle = (1 - \xi) \langle \alpha_0^\dagger \alpha_0 \rangle + \xi N'_0. \quad (2.3)$$

The real parameter  $0 \leq \xi \leq 1$  is interpreted as the probability of finding particles with negligible fluctuations in the zero-momentum state, the annihilation and creation operators keep their usual commutation relation,  $[\alpha_0, \alpha_0^\dagger] = 1$ , and  $N'_0$  is a finite fraction of the total number of particles. Thus,  $N_0 = \xi N'_0$  is the number of particles in the condensate. The gauge symmetry-breaking parameter  $\xi$  renders the continuity equation with a source term that contributes to the associated Ward identities. In the shielded potential approximation the relevant identities give<sup>16</sup>

$$n_s(T) = n_0(T) [1 - 2U_0 n(T) \mu(T)^{-1}] \quad (2.4)$$

and

$$\mu(T) = \xi U_0 n(T) [n_0(T) - 2], \quad (2.5)$$

where  $\mu(T)$  is the chemical potential,  $n(T)$  the total density, and  $U_0$  the interaction constant. Since  $n_s(0) = 1$ , it follows from Eqs. (2.4) and (2.5) that

$$\xi = \frac{n_0(0)}{[1 - n_0(0)][1 - \frac{1}{2}n_0(0)]}. \quad (2.6)$$

Hence,  $n_0(0)$  specifies  $\xi$  and conversely; either one serves as an adjustable parameter and we choose  $n_0(0)$  on account of the empirical expression (1.1). Thus, from Eqs. (2.4)–(2.6) we finally have

$$n_s(T) = \frac{n_0(T) \{2 - n_0(0)[1 - n_0(0) + n_0(T)]\}}{n_0(0)[2 - n_0(T)]}. \quad (2.7)$$

This result is to be compared with Eq. (1.1). By fitting least-squares to the data,<sup>2–6</sup> Eq. (2.7) yields  $n_0(0) = 0.124$  and the solid curve in Fig. 1.<sup>18</sup> One can easily see that  $n_0(0)$  furnished by Eq. (2.7) cannot exceed that from Eq. (1.1). As  $n_0^3(0) \ll 1$ , we can neglect terms of the third power in Eq. (2.7), so that

$$n_s(T) \simeq \begin{cases} \frac{n_0(T)[2 - n_0(0)]}{n_0(0)[2 - n_0(T)]} & (n_0^3 \ll 1), \\ \frac{n_0(T)[2 + n_0(T)]}{n_0(0)[2 + n_0(0)]} & (n_0^3 \ll 1). \end{cases} \quad (2.8)$$

I, for a moment, denote by  $n'_0$  and  $n''_0$  the fractions, respectively, given by Eqs. (1.1) and (2.8). For the same  $n_0(0)$  in both equations I have

$$\frac{n'_0(T)}{n''_0(T)} = \frac{[2 - n_0(0)]^2}{[2 - n'_0(T)][2 - n''_0(T)]} \leq 1, \quad [n'_0(0) = n''_0(0)]. \quad (2.10)$$

Hence, by least-squares fitting one must have

$n'_0(0) \geq n''_0(0)$ . In particular, the two fittings above show that  $n'_0(0) - n''_0(0) = 0.002$ , a difference that is 1 order of magnitude smaller than the experimental error (0.03).<sup>8</sup>

Finally, as the critical temperature  $T_\lambda$  is approached, both Eqs. (1.1) and (2.7) imply that  $n_s$  and  $n_0$  have the same critical exponent in the neighborhood of  $T_\lambda$ . In contrast, according to Josephson<sup>19</sup>  $n_0 \propto (T_\lambda - T)^{2\beta}$  and  $n_s \propto (T_\lambda - T)^{2\beta - \eta\nu'}$  near  $T_\lambda$ . However the exponent  $\eta\nu'$  is believed to be small ( $\sim 0.02$ ).<sup>14</sup>

### III. CHEMICAL POTENTIAL AND CONDENSATE FRACTION

The resolution of the available measurements, as seen in Fig. 1, precludes an accurate direct determination of the condensate fraction. Since the chemical potential of superfluid <sup>4</sup>He is known,<sup>15</sup> it is then natural to make use Eq. (2.5) in an attempt to find out  $n_0(T)$ . Moreover, this approach serves as a further test of the theory.

By integrating some thermodynamic relations whose variables were measured experimentally, Maynard<sup>15</sup> has determined the chemical potential per unit mass of superfluid <sup>4</sup>He in 0.05 K steps from 1.20 to 2.10 K, and in 1 bar steps from the saturated vapor pressure (SVP) to 25 bar, within an accuracy of 0.01 Jg<sup>-1</sup>. The values of Maynard at SVP are plotted in Fig. 2.

Now I denote by  $\bar{\mu}(T)$  the chemical potential per unit mass. Substituting (2.6) in (2.5) and dividing by the mass  $m$  I obtain

$$\bar{\mu}(T) = -g \frac{n_0(0)[2 - n_0(T)]}{[1 - n_0(0)][1 - \frac{1}{2}n_0(0)]}, \quad (3.1)$$

where  $g \equiv nU_0/m$ . Strictly, the quantity  $g$  depends on the temperature through  $n(T)$ , but since the density of liquid <sup>4</sup>He varies by less than 1% below the lambda point, one may consider  $g$  as  $T$  independent. The adjustment of Eq. (3.1) to Maynard's data at SVP (Fig. 2) involves two parameters,  $n_0(0)$  and  $g$ , and goes as follows.

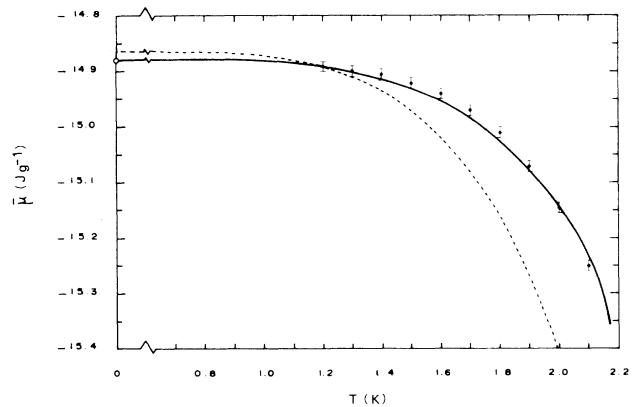


FIG. 2. Chemical potential per unit mass vs temperature at SVP. Dashed curve: Eq. (3.1), for  $n_0(0) = 0.124$  and  $\gamma = 0.33920$ . Solid curve: Eq. (3.1) for  $n_0(0) = 0.062$  and  $\gamma = 0.15825$ . Solid circles are the experimentally determined values from Ref. 15. Open circle is the value obtained in Ref. 22.

TABLE I. The parameters  $\gamma$  and  $g$ , and the values of  $\bar{\mu}(T=0)$  and  $\mu(T=0)$ , for  $n_0(T=0)=0.062$  and  $0.124$ .

$n_0(0)$	$\gamma$	$g$ ( $\text{Jg}^{-1}$ )	$\bar{\mu}$ ( $\text{Jg}^{-1}$ )	$\mu$ (K)
0.062	0.158 25	112.54	-14.878	-7.160
0.124	0.339 20	52.506	-14.865	-7.154

Firstly, for a series of  $n_0(0)$  trials the parameter  $g$  is so chosen that the value furnished by Eq. (3.1) at SVP and  $T=1.2$  K coincides with Maynard's, i.e.,<sup>15</sup>

$$\bar{\mu}(\text{SVP}, 1.2 \text{ K}) = -14.891 \text{ Jg}^{-1}. \quad (3.2)$$

The choice of this point ( $T=1.2$  K) is due to the observation that  $\bar{\mu}(T)$  becomes approximately constant around  $T=1.2$  K. This is expected since the relevant mass densities of superfluid  $^4\text{He}$  display negligible variation for  $T < 1$  K.

Secondly,  $n_0(0)$  is determined by the best fit to Maynard's values in the range  $T > 1.2$  K. One notices that  $n_0(0)$  specifies the temperatures dependence of  $\bar{\mu}(T)$  through  $n_0(T)$ , Eq. (2.7), once the value at  $T=1.2$  K is made to agree with (3.2) by means of  $g$ .

In reference to the adjustment of  $g$ , we proceed as follows. The hard-sphere parameter ( $a=2.556$  Å) gives  $nU_0 \cong 8.5$  K but for reasons that will become clear shortly, we choose instead  $nU_0 = \Delta_r = 8.5738$  K, which is the roton energy gap at SVP and  $T=1.2$  K.<sup>20</sup> In addition, we interpret  $m$  as an effective mass, namely  $m = \gamma m_{\text{He}}$ , where  $m_{\text{He}}$  is the mass of the  $^4\text{He}$  atom. Therefore, we have

$$g = \Delta_r / \gamma m_{\text{He}}, \quad \Delta_r / m_{\text{He}} = 17.810 \text{ Jg}^{-1}, \quad (3.3)$$

and the adjustment of  $g$  to the prescribed value (3.2) is achieved in terms of  $\gamma$ .

In Table I we find the values associated with the two cases:  $n(0)=0.062$  and  $0.124$ , which are represented by the solid and dashed curves in Fig. 2, respectively. The former is the result yielded by a least-squares fit, constrained by (3.2). The last column displays the chemical potential,  $\mu = \bar{\mu} m_{\text{He}}$ , at  $T=0$ .

The best fit provides two important results. First, notice the remarkable agreement between the corresponding  $\gamma$  value and the roton effective mass at  $T=1.2$  K, viz.,  $m_r = 0.15824 m_{\text{He}}$ .<sup>20</sup> Since for a fixed density, the ratio

$\Delta_r / m_r$  is invariant for all temperatures in the superfluid region,<sup>21</sup> the best fit subjected to the condition (3.2) is achieved by

$$g = \Delta_r / m_r. \quad (3.4)$$

Second, the value of the chemical potential of superfluid  $^4\text{He}$  at  $T=0$  is known to be  $\mu(0) = -7.16$  K.<sup>22</sup>

#### IV. DISCUSSION

The preceding section indicates that the condensate fraction of superfluid  $^4\text{He}$  is  $n_0=0.062$  at  $T=0$ . This result calls to mind the value  $n_0=0.06$  estimated by Kerr, Pathak, and Singwi,<sup>23</sup> and, particularly, the new values for the condensate fractions as reanalyzed more recently by Griffin.<sup>9</sup> Hence, I go back to Eq. (2.7) and determine the corresponding  $n_0(T)$ , which is represented by the dashed-dotted curve in Fig. 1. Likewise, the empirical formula (1.1) yields condensate fractions that differ by less than 0.001 from those furnished by Eq. (2.7). Consequently, for such a low  $n_0(0)=0.062$ , Eqs. (1.1) and (2.7) become indistinguishable in Fig. 1, and they agree reasonably well with Griffin's recalculated values of  $n_0(T)$ .

One notices, incidentally, that a small condensate fraction implies that  $N'_0$ , introduced in the prescription (2.1), is close to the superfluid particle number, rather than the number of particles in the condensate. Accordingly, I let  $n'_0$  be the density fraction associated with  $N'_0$ . Thus,  $n_0 = \xi n'_0$ , and from Eqs. (2.4) and (2.5), I have

$$n'_0(T) = [n_s(T) - n_0(T)] [1 - \frac{1}{2} n_0(T)]. \quad (4.1)$$

Recalling that  $n_0/n_s = \xi/(1+\xi)$ , as  $T \rightarrow T_\lambda^-$ ,<sup>16</sup> the ratio  $n'_0(T)/n_s(T)$  varies from  $0.94$  ( $T=T_\lambda$ ) to  $0.91$  ( $T=0$ ), whereas in dilute Bose systems, where  $\xi=1$  and  $n_0(0)=0.438$ ,<sup>16</sup> this ratio decreases from  $0.50$  ( $T=T_\lambda$ ) to  $0.44$  ( $T=0$ ).

I finally remark that the  $g$  expression in terms of the Landau parameters,<sup>24</sup> Eq. (3.4), may be justified on account of the dominant contribution that rotons make to all properties of superfluid  $^4\text{He}$  above 1 K. In contrast, twice as large a  $n_0(0)$  ( $=0.124$ ), implying less than half a  $g$  ( $=52.506 \text{ Jg}^{-1}$ ), give only a variation of 0.08% for the chemical potential in the phonon dominated region, i.e.,  $\mu(T < 0.8 \text{ K}) = -7.154$  K. As seen in Fig. 2, the parameters  $n_0(0)$  and  $g$  play a crucial role in the roton region only.

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