Quantum effects in the current-voltage characteristic of a small Josephson junction

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We study a simple quantum-mechanical generalization of the resistively shunted junction model to describe the current-voltage characteristic of a single Josephson junction. An exact series representation is given for the nonlinear resistance, which is valid for arbitrary temperature and dissipation and includes both classical and quantum phase slips. For sufficiently high temperature, the quantum effects disappear and $\hbar$ cancels from the corresponding series. The resulting expression is equivalent to classical Brownian motion in a periodic potential with friction coefficient γ. We derive a rigorous expression for the leading quantum corrections to the classical result which are of order $\hbar^2$. In the overdamped limit they are equivalent to a renormalization of the barrier, whereas for $\gamma \to 0$ they are bounded below by the effect of an additional bias of order $\hbar\gamma$. An approximate continued fraction is derived for the resistance in the general case, which becomes exact in the classical high-damping limit. It is evaluated numerically and leads to reasonable qualitative results for small barriers, unless the temperature approaches zero. We discuss the possibility of seeing the quantum effects in junctions with small capacity, in particular the existence of a regime with negative differential resistance and the implications of our results to the quasireentrant behavior observed in thin superconducting granular films.

I. INTRODUCTION AND MODEL

Following the pioneering work by Caldeira and Leggett,[1] there has recently been a lot of interest in the quantum behavior of collective macroscopic variables like the phase difference φ in a Josephson device. Specific effects like tunneling[2] or level quantization[3] have been seen experimentally. Moreover, theoretical considerations suggest that even coherent oscillations of the magnetic flux in a superconducting quantum interference device[4] (SQUID) or the voltage in a current-driven Josephson junction,[5] which arise from a linear superposition of macroscopically different states, might be observable. In this work, we will examine the possibility of observing the quantum nature of the phase in a rather direct way, by looking at the current-voltage characteristic of a single current-biased Josephson junction. To this end, we will apply and extend a previously discussed model for quantum Brownian motion in a periodic potential[6] to the present problem. We will assume that in the limit where φ may be treated classically,[7] the system is described by the so-called resistively shunted junction model.[8] As a result, the dynamics of the phase is then equivalent to Brownian motion of a classical particle in an external potential and may, for instance, be formulated using a simple Langevin equation of the form

$$M\ddot{q}+\eta\dot{q}+\frac{\partial V(q)}{\partial q}=\xi^{cl}(t)\ . \tag{1.1}$$

Here the particle coordinate q has to be identified with the phase φ, the mass M is proportional to the junctions capacitance C through $M=(\hbar/2e)^2C$ and the friction coefficient $\eta=M\gamma$ is determined by the inverse relaxation time $\gamma^{-1}=RC$ with R an effective shunt resistance. The potential energy $V(q)$ in the presence of an external current I has the form (see Fig. 1)

$$V(q)=-V\cos(2\pi q/q_0)-Fq \tag{1.2}$$

with $V=\hbar I_c/2e$ the Josephson coupling energy, I_c the critical current, $q_0=2\pi$ the lattice constant and $F=\hbar I/2e$ the external force. Finally, $\xi^{cl}(t)$ is the usual Gaussian white noise with zero average and correlation

$$\langle \xi^{cl}(t)\xi^{cl}(0)\rangle = 2\eta T\delta(t) \tag{1.3}$$

(throughout the paper we will choose units such that $k_B=1$). In order to determine the current-voltage characteristic within the present model, it is necessary to calculate the nonlinear dc mobility

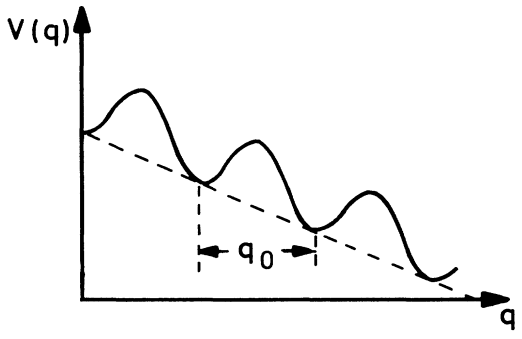

FIG. 1. Periodic potential $V(q)=-V\cos(2\pi q/q_0)-Fq$ with lattice constant q_0 and external force F.

$$\mu=\frac{\langle\dot{q}\rangle}{F} \tag{1.4}$$

of our Brownian particle for a given $F\neq 0$ with $\langle\dot{q}\rangle$ being the particle's steady-state velocity down the washboard potential. Obviously, for vanishing Josephson coupling $V=0$, the mobility is given by $\mu=\mu_0=1/\eta$ and is independent of both T and F. Using the Josephson relation $U=\hbar\dot{\varphi}/2e$ for the voltage drop U across the junction, this is equivalent to a simple Ohmic current-voltage characteristic $U=RI$. In the general case $V\neq 0$, the corresponding relation between a given external current and its associated voltage drop follows from

$$U=\frac{\mu}{\mu_0}RI\ . \tag{1.5}$$

In the Brownian motion analogy, the central quantity of interest which determines the junctions effective *resistance* is thus the normalized *mobility* μ/μ_0. It will vary between zero and one and generally depends on bias I, temperature T, friction η, and corrugation V. Although μ/μ_0 is not known analytically even for the purely classical problem, except in the limits of very large[9] or very small[10] damping η or to order V^2,[6] there are efficient numerical methods[11] which allow to determine the classical mobility μ^{cl} very accurately for arbitrary values of the parameters.

Our aim in the following is to understand the behavior of a *quantum* particle in the presence of a dissipative environment, which for sufficiently high temperatures is described by the classical equation (1.1). Specifically, we want to calculate the normalized mobility μ/μ_0, taking into account the quantum nature of the phase variable φ or, equivalently, the quantization of the electromagnetic field in the contact. The corresponding Brownian particle is thus able to tunnel through the potential barriers and will roll down the washboard potential for any finite external current $I<I_c$, even at $T=0$ where classical activation is impossible. The associated nonzero voltage will then lead to dissipation and, through the Josephson relation, is directly proportional to the number of quantum phase slips per time. Contrary to the classical limit, where the Langevin equation (1.1) completely specifies the problem, it is necessary in the quantum case to have a microscopic model of the bath coupling which is consistent with the given classical limit. As has first been pointed out by Caldeira and Leggett,[1] the knowledge of the classical limit allows a description in terms of a purely phenomenological model provided the environment interaction fulfills some rather general conditions. The model is characterized by the single friction parameter η which, at least in principle, is measurable by an experiment in the classical regime. It consists of a linear coupling of the Brownian particle to an infinite set of harmonic oscillators. Due to the periodicity of the phase or, equivalently, the discreteness of the number of Cooper pairs n as its conjugate momentum, both the bath and the external force can couple only to variables which are invariant under addition of 2π to the phase. A Hamiltonian which respects the symmetry $\varphi\rightarrow\varphi+2\pi$ can thus contain only terms like $\sin\varphi$, $\cos\varphi$, or $\dot{\varphi}$. Now, as long as quasiparticle excitations are neglected, the total number of Cooper pairs in both superconductors is conserved.[12] As a result, the bath oscillators may couple only to $\dot{\varphi}$, which is directly proportional to the charge Q on the capacitor. The physical picture of the shunted junction model implies that

$$Q=2en+Q_{\rm ext}-Q_n \tag{1.6}$$

splits into a contribution $2en$ which is transferred via pair tunneling, a second one $Q_{\rm ext}=\int_0^t I(t')dt'$ due to the external current and a term Q_n which takes into account the charge which flows dissipatively through the resistor in parallel to the junction. Assuming that Q_n may be represented as a weighted sum of harmonic oscillator coordinates, we are led to the Hamiltonian

$$H=\left[2en+\int_0^t dt'I(t')+\sum_\alpha c_\alpha x_\alpha\right]^2\Big/2C-V\cos\varphi+H_{\rm osc}\ , \tag{1.7}$$

where $H_{\rm osc}$ is the Hamiltonian of a harmonic oscillator bath with masses m_α and frequencies ω_α. As long as one is interested only in the dynamics of φ, it is sufficient to specify the weighted density of states which we define as

$$J(\omega)=\frac{\pi}{2}\left[\frac{\hbar}{2e}\right]^2\sum_\alpha\frac{c_\alpha^2\omega_\alpha}{m_\alpha}\delta(\omega-\omega_\alpha),\quad \omega>0\ . \tag{1.8}$$

In particular, the choice $J(\omega)=\eta\omega$ leads to the Langevin equation (1.1) by treating (1.7) as a classical Hamiltonian and taking the oscillators in thermal equilibrium at temperature T. By canonically quantizing the phase according to $[\varphi,n]=i$ and the additional oscillator bath in the standard way, (1.7) provides us with a simple model for a resistively shunted, current-driven Josephson junction which takes into account the quantum nature of φ. Since states which differ by adding 2π to the phase are identical, this is, in fact, a model for a driven dissipative quantum pendulum, rather than a particle in an extended periodic potential as in (1.2). It has been shown,[13] however, that in a case with $J(\omega)\sim\omega$ as $\omega\rightarrow 0$, the dissipative environment interaction destroys the indistinguishability of φ and $\varphi+2\pi$ in the following sense. As far as the distribution of $\dot{\varphi}$ is concerned, the Hamiltonian (1.7) is equivalent to the original Caldeira-Leggett model[1,14] for an *extended* coordinate $q=\varphi$

$$H_{\rm CL}=p^2/2M+V(q)+\sum_\alpha p_\alpha^2\Big/2m_\alpha+\frac{\lambda_\alpha}{2}(x_\alpha-q)^2\ , \tag{1.9}$$

with $p=M\dot{\varphi}$, $\lambda_\alpha=(\hbar/2e)c_\alpha\omega_\alpha$, and $V(q)$ given by (1.2). Thus for the calculation of the *voltage* $U=\hbar\dot{\varphi}/2e$, the simple washboard picture of Fig. 1 is valid even quantum mechanically,[15] since for any $\eta\neq 0$ coherence between paths with different winding numbers is destroyed completely.[13] For the particular case of ohmic dissipation $J(\omega)=\eta\omega$, the quantum description of a Josephson junction is therefore again equivalent to Brownian motion of a particle in an extended periodic potential. As a convenient measure of the dissipation strength we introduce the dimensionless parameter $\alpha=\eta q_0^2/2\pi\hbar$ which reduces to

$$\alpha=R_Q/R \tag{1.10}$$

in the Josephson context, with $R_Q=h/4e^2=6.45$ kΩ as a quantum unit of resistance. For an ideal junction, described by the tunneling Hamiltonian, the barrier of the periodic potential at zero external current is related to the BCS gap by $2V=\alpha\Delta_{\rm BCS}$.

This paper is organized as follows. In Sec. II, an exact expression for the mobility as an infinite series in powers of V^2, derived previously,[6] is reformulated in a way which resembles a grand canonical partition function of a classical one-dimensional Coulomb gas. At $T=0$ we derive a homogeneity relation for the mobility which shows, in particular, that the linear mobility μ_l for low friction $\alpha<1$ is equal to μ_0 for arbitrary V, i.e., independent of the barrier height. In Sec. III we first study the conditions which are necessary to obtain the classical limit, with μ^{cl} independent of $\hbar$. We then give a rigorous discussion of the leading quantum corrections to μ^{cl} which are of order $\hbar^2$. They arise from two different sources and their corresponding contributions are shown to be dominant at low or high damping, respectively. In the limit $\gamma\rightarrow\infty$ the quantum corrections are simply equivalent to a renormalization of the barrier. On the contrary, in the underdamped case $\gamma\rightarrow 0$ they have a much more complicated form and we can only give a lower bound in terms of an additional bias of order $\hbar\gamma$. In Sec. IV we introduce an approximation which allows to sum the series to all orders in V^2 in terms of a continued fraction. The approximation turns out to be exact in the classical large damping limit and to order V^2. Its numerical evaluation shows physically reasonable behavior for small barriers and not too low temperatures. Finally, in Sec. V we discuss the possibility of actually seeing quantum effects in the current-voltage characteristic of small Josephson junctions and show that for $\alpha<1$ and small enough temperature there is a regime with negative differential resistance. In addition, we mention briefly to which extent our results may be relevant to explain the quasireentrant behavior observed in thin superconducting granular films. A short derivation of the classical Fokker-Planck equation from Feynman's influence functional theory is given in the Appendix.

II. COULOMB GAS REPRESENTATION OF THE MOBILITY

As has been shown in previous work in collaboration with Fisher,[6] the normalized mobility in the periodic potential (1.2) is directly related to the corresponding quantity in an associated tight-binding model

$$\mu/\mu_0=1-\mu_{\rm TB}/\mu_0\ , \tag{2.1}$$

where $\mu_{\rm TB}$ may be calculated from

$$\mu_{\rm TB}=\lim_{t\rightarrow\infty}\frac{\langle x_s(t)\rangle}{Ft}\ . \tag{2.2}$$

Here $\langle x_s(t)\rangle$ is the average position of a particle moving on a discrete tight-binding lattice with hopping matrix element $V/2$ and lattice constant $\tilde{q}_0=q_0/\alpha$. The dual tight-binding model is again coupled to a dissipative oscillator bath at temperature T with a slightly modified spectral density[16]

$$J_\gamma(\omega)=\eta\omega\left[1+\left[\frac{\omega}{\gamma}\right]^2\right]^{-1}$$

and it is subject to the same external force F. Due to the change in the lattice constant, however, the dimensionless friction in the tight-binding model is given by

$$\tilde{\alpha}=\frac{\eta\tilde{q}_0^2}{2\pi\hbar}=1/\alpha\ . \tag{2.3}$$

Thus the mapping between the original periodic potential and its dual tight-binding model exchanges the regimes of weak and strong environment coupling. A simple pictorial representation of this mapping is obtained, by regarding the formal discrete tight binding lattice as the limit of a continuous periodic potential with a very large barrier.[17] Then both models may be represented in a single diagram of V versus α such that $V=\infty$ corresponds to a tight-binding model with zero hopping matrix element (see Fig. 2). It has been shown by Schmid[18] and others thereafter[6,19,20] that the ground state in the periodic potential is extended whenever $\alpha<1$ and localized if $\alpha>1$. The duality therefore exchanges both weak and strong corrugation as well as localized and extended behavior.

While we will not repeat the calculation leading to the duality, either for the partition function at $T=0$,[18] or in the extended version for the full dynamics at arbitrary T,[6] it is not difficult to understand at least its physical origin.[21] To this end, we consider the case of zero bias $F=0$ and the limit $\alpha\gg 1$, which corresponds to a very small friction $\tilde{\alpha}\ll 1$ in the tight-binding model. If $\tilde{\alpha}$ is neglected to lowest order, the unperturbed tight-binding Hamiltonian

$$H_{\rm TB}^{(0)}=-\frac{V}{2}\sum_n(|n\rangle\langle n+1|+{\rm H.c.}) \tag{2.4}$$

with $|n\rangle$ as the Wannier state localized around position $n\tilde{q}_0$, is diagonalized by introducing Bloch states $|k\rangle=N^{-1/2}\sum_n e^{ikn\tilde{q}_0}|n\rangle$ with quasimomentum k. This leads to a tight-binding band with energy

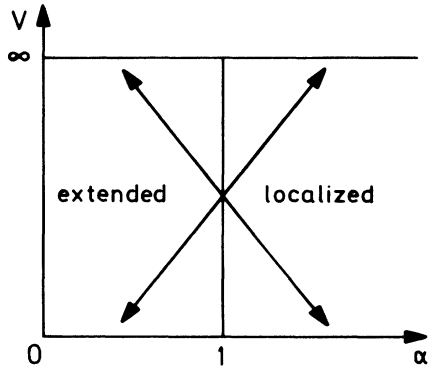

FIG. 2. The duality maps a continuous periodic potential with small amplitude V and dimensionless friction $\alpha\lesssim 1$ to a discrete tight-binding model with small hopping matrix element and $\alpha\gtrsim 1$. The ground state is extended whenever $\alpha<1$ and localized when $\alpha>1$.

$$E_{\mathrm{TB}}^{(0)}(k)=-V\cos k\tilde{q}_0 \tag{2.5}$$

which is identical with the potential in our original continuous model provided we identify $k\tilde{q}_0$ with $2\pi q/q_0$, i.e.,

$$k\leftrightarrow\eta q/\hbar\ . \tag{2.6}$$

This suggests that, up to a scale factor, the continuous coordinate q in the original model may be identified with the quasimomentum k in the dual tight-binding model, using an extended zone scheme. Thus the duality between the two models is basically a kind of Fourier transformation between real and momentum space.[22] With this picture in mind it is then easy to understand the inversion of the coupling $\tilde{\alpha}=1/\alpha$, i.e., the switching between extended and localized regimes in Fig. 2. For $\tilde{\alpha}\ll 1$ the tight-binding model is only very weakly coupled to its environment and thus k is a good quantum number. Correspondingly a particle in the continuous periodic potential with coordinate $q\sim k$ is strongly localized due to the suppression of quantum fluctuations by the strong dissipative coupling $\alpha\gg 1$. With increasing $\tilde{\alpha}$, however, α decreases and thus the quantum spread in k becomes larger, reflecting the fact that the quasimomentum becomes increasingly ill defined if the tight-binding model is subject to a growing strength of dissipation. Finally, for $\tilde{\alpha}>1$, k is completely delocalized and the quasimomentum picture has lost its meaning. This is obvious also from the dual model since the tight-binding ground state is localized in *real* space if $\tilde{\alpha}>1$.

The explicit calculation of $\langle x_s(t)\rangle$ may be done with the use of Feynmans influence functional formalism which is sketched very briefly in the Appendix. In this theory, the reduced density matrix of the particle is expressed as a double path integral over all possible paths $q_s(t')$ and $q_s'(t')$ on the tight-binding lattice. At time $t=0$ the paths start at $q_s=q_s'=0$ and then perform a kind of random walk with steps of size $\pm\tilde{q}_0$ and amplitude $\pm iV/2\hbar$ per unit time. Here the plus sign refers to jumps of q_s; the minus sign to those in q_s'. Due to the eliminated influence of the environment, each possible path is associated with an additional complex weighting factor which depends on the whole trajectory. Let n and n' be the number of jumps in $q_s(t')$ and $q_s'(t')$ respectively, and define

$$x_s=(q_s+q_s')/2,\quad y_s=q_s-q_s'\ . \tag{2.7}$$

In order to specify all the possible paths, we introduce variables η_l, $\zeta_l=\pm 1$ with $l=1,2,\ldots,n+n'$, which will be called charges in the following and times t_l with $0<t_1<t_2\ldots<t_{n+n'}<t$ such that[23]

$$x_s(t')=\frac{\tilde{q}_0}{2}\sum_{l=1}^{n+n'}\eta_l\Theta(t'-t_l)\ , \tag{2.8}$$

$$y_s(t')=\tilde{q}_0\sum_{l=1}^{n+n'}\zeta_l\Theta(t'-t_l)\ , \tag{2.9}$$

$\Theta(t)$ denoting the usual step function. As has been shown in Ref. 6, the influence functional due to the dissipative bath coupling allows only configurations with

$$\sum_{l=1}^{n+n'}\zeta_l=0 \tag{2.10}$$

and thus the charge variables ζ_l obey a neutrality condition. This implies that $n+n'=2N$ is even and that only diagonal elements of the tight-binding density matrix $y_s(t)=0$ are calculated. With these definitions the influence functional theory leads to the following expression for the generating function $\langle\exp(i\lambda x_s)\rangle$ in the steady state[6,24]

$$\langle\exp(i\lambda x_s)\rangle=\sum_{N=1}^{\infty}\left[\frac{V}{2\hbar}\right]^{2N}\sum_{\{\eta,\zeta\}}\int_0^t dt_{2N}\cdots\int_0^{t_2}dt_1\ , \tag{2.11}$$

$$\exp\left[i\sum_{l=1}^{2N}\eta_l\left[\frac{\pi}{2}\zeta_l+\lambda\frac{\tilde{q}_0}{2}-\frac{\pi}{\alpha}g_l\right]-i\bar{\epsilon}\gamma\sum_{l=1}^{2N}\zeta_l t_l-S_2\right]\ .$$

Here we have introduced the variables

$$g_l=-\sum_{l'=l+1}^{2N}\zeta_{l'}(1-e^{-\gamma(t_{l'}-t_l)}),\quad g_{2N}=0 \tag{2.12}$$

and $\bar{\epsilon}=\epsilon/\alpha\hbar\gamma$ as a dimensionless measure for the potential drop $\epsilon=Fq_0$ per period. Introducing ω_0 as the undamped oscillation frequency around the minima of the unbiased periodic potential, $\bar{\epsilon}$ may be written as

$$\bar{\epsilon}=\left[\frac{\omega_0}{\gamma}\right]^2 f \tag{2.13}$$

with $f=I/I_c$ as the ratio of the external current (or force) to its critical value where the barrier vanishes in the tilted cosine potential. Note that $\bar{\epsilon}$ is a purely classical quantity and, for a given q_0, is independent of the corrugation amplitude V. The contribution S_2, which is known to be strictly positive from the general properties of the influence functional, depends only on the charges ζ_l and is given by

$$S_2=-\frac{2}{\alpha}\sum_{l=2}^{2N}\zeta_l\sum_{l'=1}^{l-1}\zeta_{l'}Q[\gamma(t_l-t_{l'})]\ . \tag{2.14}$$

Here we have introduced the dimensionless integral

$$Q(x)=\int_0^{\infty}dy\frac{1-\cos(yx)}{y(1+y^2)}\coth(y/2\tilde{T}) \tag{2.15}$$

which depends on the reduced temperature

$$\tilde{T}=\frac{T}{\hbar\gamma}\ . \tag{2.16}$$

The function $Q(x)$ has a very simple physical meaning, being directly related to the mean-square displacement for quantum Brownian motion of a free particle. Indeed it has been shown[14] that for appropriate thermal initial conditions the model (1.9) with $V(q)=0$ leads to

$$\langle q^2(t)\rangle=\langle q^2(0)\rangle+\frac{\hbar^2(1-e^{-\gamma t})^2}{4\eta^2\langle q^2(0)\rangle}+\frac{2\hbar}{\pi\eta}Q(\gamma t) \tag{2.17}$$

for the spreading of an initial Gaussian wave packet of width $\langle q^2(0)\rangle$. The integral (2.15) cannot be expressed in

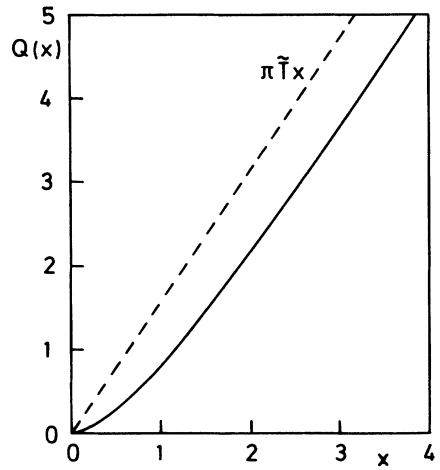

FIG. 3. The function $Q(x)$ defined in (2.15) for $\tilde{T}=\frac{1}{2}$. Asymptotically $Q(x)$ will approach the line $\pi\tilde{T}x$ if $\tilde{T}\neq 0$.

terms of elementary functions at arbitrary $\tilde{T}$ but is easily evaluated numerically (see Fig. 3). It defines a smooth and monotonically increasing function of $x\geq 0$ with $Q(x=0)=0$ and asymptotic behavior

$$\lim_{x\to 0} Q(x)=\frac{x^2}{2}\ln(1/x) \tag{2.18}$$

and

$$\lim_{x\to\infty} Q(x)=\pi\tilde{T}x, \quad \tilde{T}\neq 0 . \tag{2.19}$$

At zero temperature the behavior for large arguments is logarithmic

$$\lim_{x\to\infty} Q(x,\tilde{T}=0)=\ln x , \tag{2.20}$$

whereas, in the limit $\tilde{T}\to\infty$ we may replace $\coth(y/2\tilde{T})$ by $2\tilde{T}/y$ to obtain the "classical" result

$$Q^{cl}(x)=\pi\tilde{T}(x-1+e^{-x}) , \tag{2.21}$$

which, in contrast to (2.18), is proportional to x^2 in the limit $x\to 0$. Obviously we have $Q(x)>Q^{cl}(x)$ for any x.

The expression (2.11) may be simplified by performing the summation over the variables η_l which gives a factor

$$(-1)^N 2^{2N}\prod_{l=1}^{2N}\sin\left[\frac{\lambda q_0}{2\alpha}-\frac{\pi}{\alpha}g_l\right] .$$

Taking the derivative with respect to λ at $\lambda=0$ and inserting the resulting expression for $\langle x_s(t)\rangle$ into (2.2) we obtain

$$\frac{\mu_{\rm TB}}{\mu_0}=\frac{\pi}{\alpha\bar{\epsilon}}\lim_{L\to\infty}\frac{1}{L}\sum_{N=1}^{\infty}(-1)^N\tilde{V}^{2N}\int_0^L dx_{2N}\cdots\int_0^{x_2}dx_1 \tag{2.22}$$
$$\sum_{\{\zeta\}}\prod_{l=1}^{2N-1}\sin\left[\frac{\pi}{\alpha}g_l\right]\sin\left[\bar{\epsilon}\sum_{l=1}^{2N}\zeta_l x_l\right]\exp(-S_2) ,$$

where the summation over the $2N$ variables $\zeta_l=\pm 1$ is always understood to be restricted by the neutrality condition (2.10) and we have introduced $\tilde{V}=V/\hbar\gamma$,[25] $x_l=\gamma t_l$, and $L=\gamma t$. This brings (2.22) into a dimensionless form and shows that the reduced mobility μ/μ_0 is a function of the four dimensionless parameters α, $\tilde{V}$, $\tilde{T}$, and $\bar{\epsilon}$. The expression (2.22) will form the basis of all our subsequent discussion. Although it is generally of a rather complex form, it will enable us both to derive some exact results and a useful approximation which may be easily evaluated numerically.

A very interesting feature of the representation (2.22) is its formal similarity to a grand canonical partition function of a one-dimensional classical Coulomb gas. Indeed let us define

$$Q_N(L)=\sum_{\{\zeta\}}\int_0^L dx_{2N}\cdots\int_0^{x_2}dx_1\prod_{l=1}^{2N-1}\sin\left[\frac{\pi}{\alpha}g_l\right]\exp\left[i\bar{\epsilon}\sum_{l=1}^{2N}\zeta_l x_l-S_2\right] \tag{2.23}$$

as the canonical partition function of a classical neutral gas of $2N$ charges $\zeta_l=\pm 1$ located at positions $0<x_1<\cdots<x_{2N}<L$ on a line of length L. This is suggestive, because S_2 is equivalent to the potential energy of the charges due to a two-particle interaction $(2/\alpha)Q(x)$ which is linearly (or logarithmically at $T=0$) confining at large distances. The equivalent temperature of our gas is thus proportional to the dimensionless friction α and should not be confused with the real physical temperature T which, according to (2.15), determines the form of the interaction $Q(x)$. With $\sum_{l=1}^{2N}\zeta_l x_l$ as the dimensionless dipole moment, the bias $\bar{\epsilon}$ becomes equivalent, in this formulation, to an external imaginary electric field $i\bar{\epsilon}$ acting on the charges. Thus, apart from the unusual factor $\prod_{l=1}^{2N-1}\sin[(\pi/\alpha)g_l]$, $Q_N(L)$ is precisely identical with a classical partition function of a one-dimensional neutral Coulomb gas with imaginary external field. Defining

$$z=-\tilde{V}^2 , \tag{2.24}$$

(2.22) may then be written as

$$\frac{\mu_{\rm TB}}{\mu_0}=\frac{\pi}{\alpha\bar{\epsilon}}{\rm Im}\lim_{L\to\infty}\frac{1}{L}\sum_{N=1}^{\infty}z^N Q_N(L) , \tag{2.25}$$

and thus looks like the imaginary part of a grand canonical partition function with negative fugacity z. We will see in Sec. IV that this is not a merely formal definition but indeed, at least in a certain limit, it will enable us to sum the series exactly to all orders in a way, which is completely analogous to the calculation of the standard grand partition function of a one-dimensional Coulomb gas by Lenard.[26] Thus, although the series is oscillating

and Q_N contains an unusual term, our nonequilibrium quantum transport problem has, in a sense, been reduced to one in classical *equilibrium* statistical mechanics. It is therefore interesting to ask, whether it is possible to apply methods like the renormalization group to the present problem. As an example, we will determine in the following the $\bar{\epsilon}$ dependence of the nonlinear mobility at zero temperature from a simple argument concerning the behavior of the partition function $Q_N(L)$ under a change of length scale. It was mentioned above that the ground state of the tight binding model is localized if $\tilde{\alpha}>1$ and thus one expects the linear mobility to vanish at $T=0$. The proof for localization[6,20] was restricted, however, to lowest-order perturbation theory in the hopping matrix element. Similarly, the dynamical calculation for the mobility has only been performed to order V^2 by evaluating exactly the partition function Q_1 for a single neutral pair of charges [see Eq. (4.14) below]. This corresponds to the first term in a virial expansion of (2.25) and gives[6] $\mu_{\rm TB}\sim\tilde{V}^2\bar{\epsilon}^{2(\tilde{\alpha}-1)}$ if $\tilde{\alpha}>1$ and $\bar{\epsilon}\to 0$, which indeed implies a vanishing linear mobility. In order to extend the argument to arbitrary V, we observe that this result follows from a simple power counting argument since the interaction term $\exp(-S_2)$ at $T=0$ falls off as $(\Delta x)^{-2\tilde{\alpha}}$ for large separation $\Delta x\gg 1$ of the two charges. This suggests that, quite generally, the behavior of Q_N in the limit $\bar{\epsilon}\to 0$ may be obtained by considering values of the dimensionless distances

$$\Delta x_l = x_{l+1}-x_l \gg 1 \tag{2.26}$$

if $\tilde{\alpha}>1$ and $T=0$. Due to the logarithmic increase (2.20) of the interaction at large distances the contribution $\exp(-S_2)$ then takes the form

$$\exp(-S_2)\to\prod_{l=2}^{2N}\prod_{l'=1}^{l-1}(x_l-x_{l'})^{2\tilde{\alpha}\zeta_l\zeta_{l'}}\ . \tag{2.27}$$

Let us now perform a scale transformation where all the lengths are multiplied by a factor λ. As a result, (2.27) acquires an additional factor

$$\exp\left[2\tilde{\alpha}\ln\lambda\sum_{l=2}^{2N}\zeta_l\sum_{l'=1}^{l-1}\zeta_{l'}\right],$$

whereas, due to (2.26) the variables g_l are essentially unaffected and thus $\prod_{l=1}^{2N-1}\sin(\pi\tilde{\alpha}g_l)$ gives no contribution from the change in scale. Using the charge neutrality condition (2.10), it is straightforward to show that

$$\sum_{l=2}^{2N}\zeta_l\sum_{l'=1}^{l-1}\zeta_{l'}=-N \tag{2.28}$$

for arbitrary realizations of the $2N$ variables $\{\zeta_l=\pm 1\}$. Thus the interaction term (2.27) changes by a factor $\lambda^{-2N\tilde{\alpha}}$, which leads to the simple relation

$$Q_N(\lambda L,\bar{\epsilon})=(\lambda^{2(1-\tilde{\alpha})})^N Q_N(L,\lambda\bar{\epsilon}) \tag{2.29}$$

for the partition function with length λL. This implies immediately the homogeneity relation

$$\mu_{\rm TB}(\tilde{V},\bar{\epsilon})=\mu_{\rm TB}(\lambda^{1-\tilde{\alpha}}\tilde{V},\lambda\bar{\epsilon}) \tag{2.30}$$

for the mobility which, in the limit $\bar{\epsilon}\to 0$ should hold at $T=0$ and for $\tilde{\alpha}>1$. Although the argument given here is certainly not rigorous, and indeed fails if $\tilde{\alpha}<1$, the relation (2.30) is consistent with previous results and has some very interesting consequences. First of all, (2.30) tells us that the $T=0$ mobility depends only on the combination $u=\tilde{V}\bar{\epsilon}^{\tilde{\alpha}-1}$ but not on $\tilde{V}$ and $\bar{\epsilon}$ separately. In addition, the perturbation theory result shows that $\mu_{\rm TB}\sim u^2$ in the limit $u\to 0$. As a consequence, the linear mobility will vanish and the tight binding ground state be localized for *arbitrary* $\tilde{V}$ and $\tilde{\alpha}>1$, because $\bar{\epsilon}$ may always be chosen so small that $u\to 0$. However, we expect that with increasing hopping matrix element V the external bias below which the localization effects start to become important will *decrease*. Secondly, from the duality relation (2.1), we find the surprising result that the linear mobility at $T=0$ and $\alpha<1$ is equal to μ_0 for all V and thus is completely independent of the barrier height. This confirms the corresponding conjecture made in Ref. 6 and indicates that, within the present model, all cases with $V\neq 0$ scale towards the trivial fixed line $V=0$ if $\alpha<1$ and $T=0$. The linear effective resistance is thus the same as if no Josephson coupling were present at all. A very intuitive heuristic explanation for this effect has recently been given by Fisher[27] in terms of the resistively shunted junction picture. Consider the charge ΔQ_n which is transferred through the resistor if the phase changes by 2π. It is easy to see that $\Delta Q_n=2e\alpha$, whereas through the Josephson element charge will obviously flow only in units of $2e$. The capacitive energy, however, favors transporting the charge in the smallest possible units. Thus for $\alpha<1$ it is advantageous that all charge flows through the resistor and nothing through the Josephson junction (and vice versa if $\alpha>1$). Correspondingly, the linear effective resistance will be equal to R (or zero) independent of how much current could flow as Cooper pairs.

III. CLASSICAL LIMIT AND QUANTUM CORRECTIONS

In this section we start by showing that for sufficiently high temperatures the $\hbar$ dependence of the mobility is eliminated completely. It is then shown that there are two different types of quantum corrections which are determined explicitly to lowest order in $\hbar$. For very large damping their effect on the classical mobility is equivalent to a renormalization of the barrier and is due to the enhancement of the noise spectrum of the fluctuating environment force. On the contrary, in the underdamped limit $\gamma\to 0$ the corrections arise from the quantum effects of the Brownian particle itself and are shown to be bounded below by the effect of an additional bias of order $\hbar\gamma$.

To obtain the classical result from a quantum mechanical expression one may formally take the limit $\hbar\to 0$. Since $\hbar$ is neither zero nor dimensionless, a proper derivation should, however, be based only on calculating to lowest order in $\hbar\omega/T$ with ω a typical frequency in the problem. In the present case, taking $T\gg\hbar\gamma$, we may replace the function $Q(x)$ by its classical limit $Q^{cl}(x)$ given in (2.21). The contribution S_2 then takes the form

$$S_2^{cl}=-K\sum_{l=2}^{2N}\zeta_l\sum_{l'=1}^{l-1}\zeta_{l'}[(x_l-x_{l'})-1+e^{-(x_l-x_{l'})}]\ . \tag{3.1}$$

Here we have introduced the dimensionless purely classical quantity

$$K=\frac{2\pi\tilde{T}}{\alpha}=\frac{T}{V}\left[\frac{\omega_0}{\gamma}\right]^2 \tag{3.2}$$

which, for given q_0, is independent of the barrier height as is $\bar{\epsilon}$. Since α and $\tilde{V}$ are each of order $\hbar^{-1}$,[28] Planck's constant does drop out in the reduced mobility provided that the product of the sine functions in (2.22) may be replaced by the product of their arguments, i.e.,

$$\frac{\pi}{\alpha}|g_l|\ll 1 \ . \tag{3.3}$$

In order to find whether this condition indeed holds in the limit of high temperatures, it is convenient to introduce a function

$$g(t')=\sum_{l=1}^{2N}\zeta_l h(t'-t_l) \tag{3.4}$$

with $h(t)=e^{\gamma t}\Theta(-t)+\Theta(t)$. Thus $g(t')$ is a continuous version of the normalized trajectory $y_s(t')/\tilde{q}_0$, where the steps are rounded on a time scale γ^{-1}.[6] It is then straightforward to show that $g_l=g(t_l)$ and

$$S_2^{cl}=K\gamma\int_0^t dt' g^2(t') \ . \tag{3.5}$$

We will now argue that for sufficiently high temperatures the term $\exp(-S_2^{cl})$ suppresses the values of g_l such that (3.3) is obeyed. This is easy to see in the limit $\gamma(t_{l+1}-t_l)=\Delta x_l\gg 1$, where $g(t')$ is essentially a succession of step functions and

$$S_2^{cl}\approx K\sum_{l=1}^{2N-1} g_l^2\Delta x_l \ .$$

Thus $\exp(-S_2^{cl})$ restricts the magnitude of $|g_l|$ by a term of order $(K\Delta x_l)^{-1/2}$ and (3.3) is fulfilled if $E_0/T\ll\Delta x_l$. Here $E_0=(2\pi\hbar)^2/Mq_0^2$ is the quantum-mechanical energy necessary to confine a particle of mass M within a lattice spacing. In the Josephson case this reduces to (twice) the Coulomb energy $E_0=(2e)^2/C$ for a single Cooper pair. Thus, as long as the dimensionless distances Δx_l between two successive charges are kept larger than say of order one, the condition (3.3) is obeyed and the mobility will become independent of $\hbar$ if $T\gg\hbar\gamma$, E_0. What happens, however, if Δx_l is very small such that in a number of $N\gg 1$ steps $|g_l|$ may become large but S_2^{cl} is only of order one and therefore does not suppress this large but strongly localized fluctuation? To answer this, let us consider an extreme case of trajectory with

$$\zeta_1=\cdots=\zeta_N=1, \quad \zeta_{N+1}=\cdots=\zeta_{2N}=-1 \tag{3.6}$$

and with equidistant time steps Δx, i.e.,

$$x_l=x_1+(l-1)\Delta x, \quad l=1,\ldots,2N \ . \tag{3.7}$$

Now, if the total length $2N\Delta x$ of the fluctuation is of order 1 or larger, the maximal value of $|g_l|$ at $l=N$ is of order N. The corresponding S_2^{cl}, however, is given by $S_2^{cl}\sim KN^3\Delta x$ and thus such a configuration is suppressed very effectively. On the other hand, if $N\Delta x\ll 1$ we have for $N\gg 1$, $g_N=N^2\Delta x/2$ and $S_2^{cl}=Kg_N^2$. Thus, even in this case, a configuration with $g_N\gg 1$ has negligible weight. As a consequence, we conclude that whenever $T\gg\hbar\gamma$, E_0 we may replace S_2 by S_2^{cl} and $\sin[(\pi/\alpha)g_l]$ by $(\pi/\alpha)g_l$. Then, using $\pi\tilde{V}/\alpha=\frac{1}{2}(\omega_0/\gamma)^2$, the normalized mobility as given by (2.22), reduces to the classical expression

$$\frac{\mu_{\rm TB}^{cl}}{\mu_0}=\frac{1}{\bar{\epsilon}}\lim_{L\to\infty}\frac{1}{L}\sum_{N=1}^{\infty}(-1)^N\left[\frac{1}{2}\left[\frac{\omega_0}{\gamma}\right]^2\right]^{2N}\int_0^L dx_{2N}\cdots\int_0^{x_2}dx_1\sum_{\{\zeta\}}\prod_{l=1}^{2N-1}g_l\sin\left[\bar{\epsilon}\sum_{l=1}^{2N}\zeta_l x_l\right]\exp(-S_2^{cl}) \tag{3.8}$$

which evidently contains no $\hbar$ any more. The expansion parameter $\tilde{V}=V/\hbar\gamma$ has thus been replaced by the classical quantity $\frac{1}{2}(\omega_0/\gamma)^2$,[29] and the mobility now depends only on the three dimensionless quantities K [or $(\omega_0/\gamma)^2$], V/T, and $\bar{\epsilon}$. Introducing

$$z^{cl}=-\left[\frac{1}{2}\left[\frac{\omega_0}{\gamma}\right]^2\right]^2 \tag{3.9}$$

as the classical fugacity parameter, (3.8) may again be written in the form of a grand canonical partition function

$$\frac{\mu_{\rm TB}^{cl}}{\mu_0}=\frac{1}{\bar{\epsilon}}\,{\rm Im}\lim_{L\to\infty}\frac{1}{L}\sum_{N=1}^{\infty}(z^{cl})^N Q_N^{cl}(L) \ . \tag{3.10}$$

Here the canonical partition function for the $2N$ charges $\zeta_l=\pm 1$ located at positions x_l is defined as[30]

$$Q_N^{cl}(L)=\sum_{\{\zeta\}}\int_0^L dx_{2N}\cdots\int_0^{x_2}dx_1\prod_{l=1}^{2N-1}g_l\exp\left[i\bar{\epsilon}\sum_{l=1}^{2N}\zeta_l x_l-S_2^{cl}\right] \tag{3.11}$$

with neutrality condition $\sum_{l=1}^{2N}\zeta_l=0$ as before.

Before we proceed to discuss the quantum corrections to the classical mobility, we should show that our expression (3.8) is indeed equivalent to the standard Fokker-Planck or Langevin formalism. A direct proof for this is given in Sec. IV in the particular case of large damping. Generally, it is demonstrated in an indirect way in the Appendix. There it is

shown that Feynmans influence functional theory, and thus the exact quantum dynamics of our model, reduces to the Fokker-Planck equation under precisely the same approximations which lead from (2.22) to (3.8), namely the replacement of S_2 by S_2^{cl} and the expansion to lowest order in the off-diagonal components of the density matrix. Corresponding to those two approximations, there are two different sources for the quantum corrections to the classical mobility. First there is a contribution from expanding the product of the sine functions in (2.22) beyond the leading order. This gives rise to an additional factor $1-\Delta S_1$ in the expression (3.8) with

$$\Delta S_1=\frac{1}{6}\left[\frac{\pi}{\alpha}\right]^2\sum_{l=1}^{2N-1} g_l^2 \ . \tag{3.12}$$

Secondly there is a correction $S_2=S_2^{cl}+\Delta S_2$ which is due to the lowest order deviation of $Q(x)$ from $Q^{cl}(x)$. Expanding $\coth(y/2\tilde{T})\approx 2\tilde{T}/y+y/6\tilde{T}+\ldots$ in (2.15) we obtain

$$\Delta S_2=-\frac{E_0}{12T}\sum_{l=2}^{2N}\zeta_l\sum_{l'=1}^{l-1}\zeta_{l'}(1-e^{-(x_l-x_{l'})}) \ . \tag{3.13}$$

Taken together, the leading corrections of order $\hbar^2$ to the classical mobility may be obtained from the series

$$\frac{\mu_{\rm TB}}{\mu_0}=\frac{1}{\bar{\epsilon}}\lim_{L\to\infty}\frac{1}{L}\sum_{N=1}^{\infty}(z^{cl})^N\int_0^L dx_{2N}\cdots\int_0^{x_2}dx_1\sum_{\{\zeta\}}(1-\Delta S_1-\Delta S_2)\prod_{l=1}^{2N-1}g_l\sin\left[\bar{\epsilon}\sum_{l=1}^{2N}\zeta_l x_l\right]\exp(-S_2^{cl}) \ . \tag{3.14}$$

This result is exact, as is (2.22) or (3.8), but due to the fact that even the series for the classical mobility can generally not be evaluated in closed form, it does not seem very useful. It is, however, possible to obtain a rigorous and *explicit* result on the effect of quantum corrections in the two extreme cases $\gamma\to 0$ and $\gamma\to\infty$. Indeed in each of these limits only one of the two correction terms is relevant. This may be seen by taking the dimensionless sums in ΔS_1 and ΔS_2 to be of the same order, which leads to the crude estimate

$$\Delta S_1/\Delta S_2\sim K \ . \tag{3.15}$$

Thus the effect of ΔS_1 will be negligible in the strong damping limit $K\to 0$ whereas for very weak damping $K\to\infty$ it is ΔS_2 which will become irrelevant. Generally, both ΔS_1 and ΔS_2 will contribute to the quantum corrections and, since both are positive, the associated actual increase in the mobility will always be larger than the one obtained by neglecting either ΔS_1 or ΔS_2. Let us consider first the overdamped case which turns out to be rather simple. If, in addition to $K\ll 1$, we have also $\omega_0\ll\gamma$, then the fugacity z^{cl} is very small and our gas of charges will be dilute. Correspondingly the distances Δx_l are large compared to one and we may neglect not only ΔS_1 but also the exponential term in (3.13). Using the relation (2.28), we then find that

$$\Delta S_2\,|_{\gamma\gg\omega_0}=\frac{E_0}{12T}N \ . \tag{3.16}$$

Accordingly, the leading quantum corrections to the classical mobility in the overdamped limit $K\ll 1$ and $\omega_0\ll\gamma$ may be incorporated to all orders in V into a change of the fugacity $z^{cl}\to z^{cl}(1-E_0/12T)$. Thus they are rigorously equivalent to a reduction of the barrier[31]

$$\frac{2V}{T}\to\frac{2V}{T}-\frac{1}{12}\left[\frac{\hbar\omega_0}{T}\right]^2 , \tag{3.17}$$

which will lead to an enhancement of the mobility as expected. This exact result should perhaps be contrasted with the naive expectation that the quantum corrections would reduce the barrier by the zero-point energy $2V\to 2V-\hbar\omega_0/2$, which does not even have the correct $\hbar$ dependence. It should be pointed out that the result (3.17) has in fact implicitly[32,33] or explicitly[34] been found in several previous treatments dealing with quantum corrections to the classical decay rate τ^{-1} from a metastable well. This is not surprising since for high and even moderate[10] damping, there is a direct relation between the linear mobility μ_l and τ^{-1}. It is based on the general Einstein relation $D=\mu_l T$ for the diffusion constant and the incoherent site to site hopping picture which leads to $D=q_0^2/2\tau$ with τ^{-1} as the total rate for hopping to the right or left. The advantage of the present derivation of (3.17), however, is that the result has been shown to follow rigorously from a *dynamical* theory and turns out to be valid for arbitrary values of V/T and $\bar{\epsilon}$. Most importantly, though, we have seen that it is restricted to the limit of large damping, where μ^{cl} is given by the Smoluchowski approximation.[35]

Contrary to the result (3.17) which is due to the quantum enhancement of the classical noise spectrum, the quantum corrections in the opposite limit of small damping $K\to\infty$ have a completely different origin. As was pointed out above and in the appendix, there are *two* approximations necessary to obtain the classical Fokker-Planck equation from the exact quantum dynamics as contained in the influence functional formalism. One is the replacement of the quantum-mechanical fluctuating force $\xi(t)$ with spectrum $\eta\hbar\omega\coth(\hbar\omega/2T)$ by the classical $\xi^{cl}(t)$ defined in (1.3). The other consists of expanding to lowest order the off-diagonal components $y\neq 0$ of the density matrix. Since $(\pi/\alpha)g_l$ is directly related to y [see Eq. (3.4) above], ΔS_1 and ΔS_2 can thus precisely be identified with the effects of expanding in y to next than leading order and the inclusion of the leading deviations between $\xi(t)$ and $\xi^{cl}(t)$. According to (3.15), in the small damping limit $K\to\infty$, it is the former effect which gives the dom-

inant quantum corrections. Contrary to the overdamped case, we have found no explicit result for the influence of ΔS_1 on μ^{cl}, except in the limit $V \to 0$.[6] It is, however, possible to obtain at least a lower bound on the corrections associated with this term. To this end we note that from physical arguments $\Delta S_1 > 0$ will enhance the mobility compared to μ^{cl} and the enhancement will increase monotonically with ΔS_1. Thus, from the trivial inequality $\Delta S_1 \geq \frac{1}{6}(\pi/\alpha)^2 g_1^2$, we conclude that using only the $l=1$ term in ΔS_1 will definitely *underestimate* the quantum corrections to μ^{cl}. Moreover, in the limit $K \to \infty$, the fugacity of the Coulomb gas is very large and we therefore expect the dimensionless distances Δx_l to be small compared to one. Expanding the exponential in (2.12) then leads to

$$g_1 \mid_{K\to\infty} = -\sum_{l'=2}^{2N} \zeta_{l'}(x_{l'} - x_1) = -\sum_{l=1}^{2N} \zeta_l x_l \ . \tag{3.18}$$

Thus apart from an irrelevant sign, g_1 becomes identical with the dimensionless dipole moment which appears in the bias term $\sin(\bar{\epsilon}\sum_{l=1}^{2N}\zeta_l x_l)$. As a result, the expression for the *nonlinear classical* mobility to order $\bar{\epsilon}^2$ contains a factor $[1-\frac{1}{6}(\bar{\epsilon}g_1)^2]$ compared to μ_l^{cl}, whereas the *linear quantum* mobility to order $\hbar^2$ is bounded below by the identical expression with $\bar{\epsilon}$ replaced by π/α. Thus to order $\hbar^2$, we have the inequality[36]

$$\mu_l \mid_{K\to\infty} \geq \mu^{cl}(\epsilon = \pi\hbar\gamma) \mid_{K\to\infty} \ . \tag{3.19}$$

The nonlinear classical mobility for low damping and in the case $V \gg T$ has been determined by Nozieres and Iche.[10] It is given by

$$\mu^{cl}(\epsilon) \mid_{K\to\infty} = \mu_l^{cl} \mid_{K\to\infty} \left[1 + \left(\frac{\epsilon}{\delta} \right)^2 + \cdots \right] \tag{3.20}$$

with $\delta = 8(\gamma/\omega_0)V$ being the energy loss per cycle for motion at the barrier top energy and $\mu_l^{cl} \mid_{K\to\infty} = \mu_0(\pi/2)\exp(-2V/T)$[11]. Using (3.19) we therefore find

$$\mu_l \mid_{K\to\infty} > \mu_l^{cl} \mid_{K\to\infty} \left[1 + \left(\frac{\pi\hbar\omega_0}{8V} \right)^2 + \cdots \right] . \tag{3.21}$$

Thus, the quantum corrections are certainly finite as $\gamma \to 0$ and, most importantly, they are independent of the temperature as long as $V \gg T$.[37] This shows that the quantum corrections do behave qualitatively different for low and high damping, thus reflecting their fundamentally different physical origin. In particular, the corrections due to ΔS_1 are a genuine quantum effect of the particle itself, in contrast to ΔS_2 which arises from the quantum nature of the bath. They are absent in potentials which are at most quadratic and, apparently, also in descriptions which are based on equilibrium properties only. As a consequence, one has to be rather careful in drawing conclusions about quantum effects like tunneling or coherence from the study of purely harmonic models.

IV. APPROXIMATE CONTINUED FRACTION FOR THE QUANTUM MOBILITY

In this section we derive an approximate representation of the quantum mobility in terms of a continued fraction which becomes exact in the classical large damping limit. Although it is not *quantitatively* reliable beyond that, the numerical evaluation shows some reasonable qualitative results for small barriers and not too low temperature.

In view of the complexity of our general expression (2.22) it is obviously impossible to sum up the series for arbitrary values of α, $\tilde{V}$, $\tilde{T}$, and $\bar{\epsilon}$ either analytically or numerically. Thus it is desirable to find an approximation which is at least numerically accessible. Our basic assumption, whose validity and limits will be investigated later, is to consider the gas of charges as sufficiently dilute that the dimensionless distances Δx_l are large compared to one. Since the interaction potential $Q(x)$, however, *increases* with distance, a naive nearest-neighbor approximation is clearly impossible. Nevertheless, as has been shown by Lenard in the one-dimensional Coulomb gas problem with strictly linear potential, a nearest-neighbor problem may be obtained by introducing the variables

$$\nu_l = \sum_{l'=1}^{l} \zeta_l \tag{4.1}$$

with the properties

$$\mid \nu_l - \nu_{l-1} \mid = 1$$

and $\tag{4.2}$

$$\nu_{2N} = 0 \ .$$

The exact canonical partition function $Q_N(L)$ given in (2.23) may be expressed in terms of the ν_l by using

$$\sum_{l=1}^{2N} \zeta_l x_l = -\sum_{l=1}^{2N-1} \nu_l \Delta x_l \ , \tag{4.3}$$

$$g_l = \sum_{l'=l}^{2N-1} \nu_{l'} \left[e^{-(x_{l'} - x_l)} - e^{-(x_{l'+1} - x_l)} \right] , \tag{4.4}$$

and

$$S_2 = \frac{2}{\alpha} \sum_{l=1}^{2N-1} \nu_l^2 Q(\Delta x_l) + \frac{2}{\alpha} \sum_{l=2}^{2N-1} \nu_l \sum_{l'=1}^{l-1} \nu_{l'} \Lambda_{ll'} \ . \tag{4.5}$$

Here we have introduced the quantity

$$\begin{aligned} \Lambda_{ll'} = {} & Q(x_{l+1} - x_{l'}) - Q(x_{l+1} - x_{l'+1}) \\ & + Q(x_l - x_{l'+1}) - Q(x_l - x_{l'}) \ . \end{aligned} \tag{4.6}$$

Let us now assume that Δx_l is sufficiently large such that the different contributions in $\Lambda_{ll'}$ tend to cancel and terms like $e^{-(x_{l'} - x_l)}$ are negligible if $l' > l+1$. Then, with the approximations[38]

$$S_2 \approx S_2^a = \frac{2}{\alpha} \sum_{l=1}^{2N-1} \nu_l^2 Q(\Delta x_l) \tag{4.7}$$

and

$$g_l \approx g_l^a = \nu_l (1 - e^{-\Delta x_l}) \ , \tag{4.8}$$

the partition function (2.23) reduces to

$$Q_N^a(L)=\sum_{\{\nu\}}\int_0^L dx_{2N}\cdots\int_0^{x_2}dx_1\prod_{l=1}^{2N-1}\sin\left[\frac{\pi}{\alpha}\nu_l(1-e^{-\Delta x_l})\right]\exp\left[-i\bar{\epsilon}\nu_l\Delta x_l-\nu_l^2\frac{2}{\alpha}Q(\Delta x_l)\right] \tag{4.9}$$

which depends only on the distances $\Delta x_l=x_{l+1}-x_l$ and the variables ν_l defined in (4.1). Their introduction and the approximations (4.7) and (4.8) have thus turned the problem into one with nearest-neighbor interaction only. As a result, the integration over all possible locations x_l of the charges is an iterated convolution. It is therefore convenient to introduce the Laplace transform

$$Q_N(s)=\int_0^\infty dLe^{-sL}Q_N(L) \tag{4.10}$$

which, thermodynamically, is equivalent to going over from an ensemble with given length L to one with given pressure s. In this manner we find

$$Q_N^a(s)=\frac{1}{s^2}\sum_{\{\nu\}}\prod_{l=1}^{2N-1}a(s,\nu_l) \tag{4.11}$$

with

$$a(s,\nu_l)=\int_0^\infty dxe^{-sx}\sin\left[\frac{\pi}{\alpha}\nu_l(1-e^{-x})\right]\times\exp\left[-i\bar{\epsilon}\nu_l x-\nu_l^2\frac{2}{\alpha}Q(x)\right]. \tag{4.12}$$

It is now a trivial matter to take the limit $L\to\infty$. At finite temperature the linearly confining interaction $Q(x)$ renders $a(s=0,\nu_l)$ finite for all possible values of $\nu_l\neq 0$, α, and $\bar{\epsilon}$. If the temperature is zero, however, it is necessary that $\alpha<1$ in order to ensure the convergence of $\mathrm{Im}a(s=0,\pm 1)$ in the limit $\bar{\epsilon}\to 0$. Thus the coefficients $a(s=0,\nu_l)$ are well defined in all cases except $\alpha>1$ and $T=\bar{\epsilon}=0$. As a consequence, we find that $\lim_{s\to 0}Q_N^a(s)\sim s^{-2}$ or, equivalently, $\lim_{L\to\infty}Q_N^a(L)\sim L$ for any N.[39] The limit $L\to\infty$ can thus be interchanged with the summation on N and the approximate tight binding mobility is given by

$$\frac{\mu_{\mathrm{TB}}^a}{\mu_0}=\frac{\pi}{\alpha\bar{\epsilon}}\mathrm{Im}\sum_{N=1}^\infty z^N\sum_{\{\nu\}}\prod_{l=1}^{2N-1}a(\nu_l) \tag{4.13}$$

with $a(\nu_l)=a(s=0,\nu_l)$. To lowest order in the fugacity $z=-\tilde{V}^2$ the approximations (4.7) and (4.8) are exact and one obtains

$$\frac{\mu_{\mathrm{TB}}}{\mu_0}=\frac{2\pi\tilde{V}^2}{\alpha\bar{\epsilon}}\int_0^\infty dx\sin(\bar{\epsilon}x)\sin\left[\frac{\pi}{\alpha}(1-e^{-x})\right]\times\exp\left[-\frac{2}{\alpha}Q(x)\right]. \tag{4.14}$$

The corresponding mobility μ/μ_0 and, in particular, its nonmonotonic temperature dependence for $\alpha<1$ have been discussed in Ref. 6.[40] We emphasize that this naive perturbation theory in powers of $\tilde{V}^2$ is justified to *any* order by the fact that, at least within the present approximation and unless $\alpha>1$ and $T=\bar{\epsilon}=0$, we have $\lim_{L\to\infty}Q_N^a(L)\sim L$ for all N. This is in contrast to the statement[6] that $\langle x_s(t)\rangle$ would grow as t^N to order $\tilde{V}^{2N}$. Indeed the corresponding argument was based on keeping only a restricted class of configurations with all the charges bound in independent neutral pairs, i.e., $\nu_l=\pm 1$ if l is odd and zero otherwise. Within the present approximation, however, these configurations have zero weight since $a(\nu)$ vanishes if $\nu=0$.[41] In the general case we believe, although cannot prove, that the behavior $\lim_{L\to\infty}Q_N(L)\sim L$ is true also for the *exact* partition function (2.23) except perhaps at $\alpha>1$ and $T=\bar{\epsilon}=0$, where already the first term (4.14) diverges.

In order to obtain the approximate mobility to *all* orders in $\tilde{V}$, we are now left with solving the combinatorical problem of summing over all possible realizations of the variables $\{\nu_l\}$. To do this, it is convenient to introduce a graphical representation. The ν_l constitute a simple random walk starting at $\nu_0=0$ and ending at $\nu_{2N}=0$ which may be represented as a continuous graph of ν_l versus l with vertices at points with integer coordinates (see Fig. 4). Due to $a(\nu=0)=0$ only those configurations do contribute to (4.13) whose graph does not cross the ν axis.[42] Thus they split naturally into those with $\nu_l\geq 1$, and those with $\nu_l\leq -1$ for $l=1,\ldots,2N-1$. Since $a(-\nu)=-a^*(\nu)$, the contribution of the latter configurations is minus the complex conjugate of those with positive ν_l's which are therefore the only ones which need to be considered. Following Lenard, let us define for $r=1,2,\ldots$

$$G_N^{(r)}=\sum_{\{\nu_1=r,\ldots,\nu_{2N-1}=r\}}\prod_{l=1}^{2N-1}a(\nu_l) \tag{4.15}$$

as the sum over all configurations with $2N-1$ vertices which assume their minimal value $\nu=r$ at the first and the last vertex, not excluding $\nu=r$ in between. With the associated generating functions

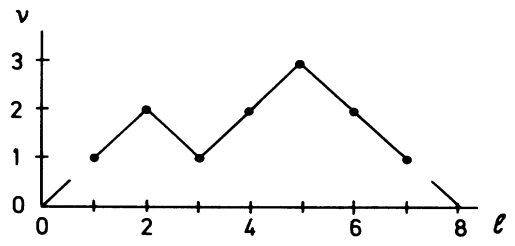

FIG. 4. Graphical representation of the configurations $\{\nu\}=\{\nu_1,\ldots,\nu_{2N-1}\}$ which form a random walk starting at $\nu_0=0$ and ending at $\nu_{2N}=0$ (in the figure $N=4$). Only those configurations contribute which do not cross the ν axis.

$$G^{(r)}(z)=\sum_{N=1}^{\infty} z^N G_N^{(r)} \tag{4.16}$$

our result (4.13) may then be written as

$$\frac{\mu_{\mathrm{TB}}^a}{\mu_0}=\frac{2\pi}{\alpha\bar{\epsilon}}\mathrm{Im}G^{(1)}(z) \ . \tag{4.17}$$

In order to calculate $G^{(1)}(z)$, we will set up a recursion relation which connects the generalized generating function $G^{(r)}(z)$ with $G^{(r+1)}(z)$. It is obvious that for $N=1$ we have

$$G_1^{(r)}=a(r)=\int_0^{\infty} dx \sin\left[\frac{\pi}{\alpha}r(1-e^{-x})\right] \times\exp\left[-i\bar{\epsilon}rx-\frac{2r^2}{\alpha}Q(x)\right] \ . \tag{4.18}$$

If $N\geq 2$, the graphs contributing to $G_N^{(r)}$ may be divided into k subgraphs such that $\nu_l=r$ at the beginning and the end of each subgraph but $\nu_l>r$ in between (see Fig. 5). Thus there are $k+1$ vertices with $\nu_l=r$ and the possible values of k are $k=1,2,\ldots,N-1$. Moreover let us denote by $2N_k-1$, $N_k=1,2,\ldots$ the number of vertices in the corresponding subgraph for which $\nu_l>r$. Since the total number of vertices is $2N-1$ we have

$$\sum_{k'=1}^{k} N_{k'}=N-1 \ . \tag{4.19}$$

With these definitions it is easy to see that for $N\geq 2$ the following relation holds

$$G_N^{(r)}=\sum_{k=1}^{N-1} a(r)^{k+1} \sum_{N_1,\ldots,N_k} G_{N_1}^{(r+1)}\cdots G_{N_k}^{(r+1)} \ , \tag{4.20}$$

where the sum over the variables N_k is restricted by (4.19). By going over to the generating function (4.16) this restriction is removed and the k sum then extends from 1 to ∞. Thus (4.20) leads to the simple recursion relation

$$G^{(r)}(z)=\frac{a(r)z}{1-a(r)G^{(r+1)}(z)} \ . \tag{4.21}$$

The approximate normalized mobility can therefore be written in the form

$$\mu^a/\mu_0=1+\frac{2\pi\tilde{V}^2}{\alpha\bar{\epsilon}}\mathrm{Im}\,A \ , \tag{4.22}$$

where A is the continued fraction

$$A=\cfrac{a(1)}{1+\tilde{V}^2\cfrac{a(1)a(2)}{1+\tilde{V}^2\cfrac{a(2)a(3)}{1+\cdots}}} \tag{4.23}$$

and the coefficients $a(r)$ are given by the integrals (4.18). In order to evaluate A numerically, we use the fact that the Nth approximant $A(N)$ to the continued fraction may be written as the ratio $A(N)=U(N)/V(N)$ of two polynomials $U(N)$ and $V(N)$. They satisfy the fundamental recursion relations

$$U(N)=U(N-1)+\tilde{V}^2a(N-1)a(N)U(N-2) \ , \tag{4.24}$$

$$V(N)=V(N-1)+\tilde{V}^2a(N-1)a(N)V(N-2) \ , \tag{4.25}$$

for $N=2,3,\ldots$ with initial conditions $U(0)=0$, $U(1)=a(1)$, and $V(0)=V(1)=1$. Thus, within the approximations (4.7) and (4.8), the calculation of the quantum mobility for arbitrary parameters is reduced to the evaluation of the integrals (4.18) and the recursion (4.24) and (4.25) assuming the procedure does converge.

Before we proceed to the numerical calculation, let us first discuss the range of validity of the approximations (4.7) and (4.8). They are obviously exact to order V^2, where (4.22) is equivalent to the result (4.14). Keeping only the leading term in the expansion in powers of V^2 is justified as long as μ/μ_0 stays close to one.[43] Thus the current-voltage characteristic is given correctly in all cases, where it deviates only slightly from the simple Ohmic line which characterizes a normal tunnel junction. Usually, however, one is interested in a situation where the Josephson coupling cannot be treated as a small perturbation but leads to a highly nonlinear characteristic with almost no voltage drop below the critical current I_c. The interesting feature about the result (4.22) is now, that it is *exact* to *all* orders in V, provided we take the classical limit $T\gg\hbar\gamma$, E_0 and high damping $\gamma\gg\omega_0$. In this limit the fugacity is very small and we may approximate

$$Q^{cl}(x)\,|_{\gamma\gg\omega_0}\approx\pi\tilde{T}x \tag{4.26}$$

by its large-x behavior and neglect all terms of order $\exp(-\Delta x_l)$, implying $g_l\approx\nu_l$. Since $\Lambda_{ll'}=0$ for a strictly linear $Q(x)$, the approximation (4.7) is exact and the integrals (4.18) reduce to

$$a^{cl}(r)\,|_{\gamma\gg\omega_0}=\frac{\pi}{\alpha}(i\bar{\epsilon}+Kr)^{-1} \ . \tag{4.27}$$

After a slight rearrangement our approximate result (4.22) then reads

$$\mu^{cl}/\mu_0\,|_{\gamma\gg\omega_0}=1+\frac{2}{f}\mathrm{Im}\cfrac{1/4}{T/V+if+\cfrac{1/4}{2T/V+if+\cdots}} \tag{4.28}$$

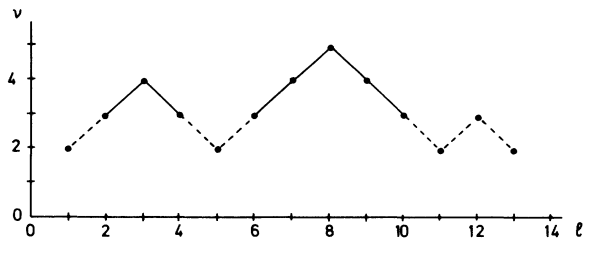

FIG. 5. A contribution to $G_7^{(2)}$ is split into $G_2^{(3)}G_3^{(3)}G_1^{(3)}$ times $(G_1^{(2)})^4$

which depends only on V/T and f, but not on K or $(\omega_0/\gamma)^2$ any more. The expression (4.28) is now indeed the correct result for the normalized classical mobility to all orders in V if the damping is large and thus the Fokker-Planck equation (A13) may be replaced by the Smoluchowski limit (A14). This follows easily by Fourier expanding the corresponding stationary solution with finite current.[11] We have thus shown that our approximate result (4.22) is exact in the classical high damping limit and, in addition, have explicitly demonstrated the equivalence of our representation (3.8) with the standard Fokker-Planck description in this case. In order to see whether (4.22) is reliable beyond that, we recall that the lowest-order quantum corrections in the large damping limit are rigorously equivalent to a renormalization of the barrier (3.17). This arises by retaining only the constant term in

$$\frac{2}{\alpha}\Delta Q(x)=\frac{E_0}{12T}(1-e^{-x})$$

for the leading corrections to $Q^{cl}(x)$. Within the approximation (4.7), however, this leads to a term

$$\Delta S_2^q \mid_{\gamma\gg\omega_0}=\frac{E_0}{12T}\sum_{l=1}^{2N-1}\nu_l^2 . \tag{4.29}$$

In contrast to its exact value (3.16), ΔS_2^q depends on the particular configuration and, for $N>1$, is equal to the correct result only for those configurations which do not contribute in the present approximation.[44] It will, in fact, always be much larger than (3.16) and therefore *overestimate* even the leading quantum corrections to the Smoluchowski result. Thus, unfortunately, the approximation (4.22) is quantitatively incorrect as soon as it is applied to the quantum regime. A similar conclusion holds even in the purely classical case if (4.22) is used beyond the Smoluchowski limit. Indeed there is an extremely simple, and obviously unnoticed, exact result for the leading γ^{-2} corrections to (4.28). It is only necessary to replace (4.26) by $Q^{cl}(x)\approx\pi\tilde{T}(x-1)$, but to neglect all further contributions to either Q^{cl} or $g_l\approx\nu_l$ since they are of order $\exp(-\Delta x_l)$, which is negligible if the fugacity is small. Using again the relation (2.28), we find that the additional constant $-\pi\tilde{T}$ in $Q^{cl}(x)$ gives rise to a change in the fugacity $z^{cl}\to z^{cl}(1+K)$. As a result, the corrections of order γ^{-2} are simply equivalent to a renormalization of the barrier

$$\frac{2V}{T}\to\frac{2V}{T}+\left[\frac{\omega_0}{\gamma}\right]^2 . \tag{4.30}$$

This result is valid for arbitrary V/T and f and, obviously, the normalized mobility will always be reduced through this effect.[45] Including the contribution from ΔS_1 and the result (3.17), the complete expression for the reduced mobility at high damping may be written as ($\epsilon=Fq_0$)

$$\mu/\mu_0=1+\frac{2}{\epsilon/2\pi T}\,\mathrm{Im}\,\frac{v^2}{(1+i\epsilon/2\pi T)c_1+\dfrac{v^2}{(2+i\epsilon/2\pi T)c_2+\cdots}} \tag{4.31}$$

with

$$v=\frac{V}{2T}+\frac{1}{4}\left[\frac{\omega_0}{\gamma}\right]^2-\frac{1}{48}\left[\frac{\hbar\omega_0}{T}\right]^2 \tag{4.32}$$

and

$$c_r=1+\frac{r^2}{6}\left[\frac{\pi}{\alpha}\right]^2 . \tag{4.33}$$

While (4.31) is the *exact* extension of the Smoluchowski-limit result (4.28) to lowest order in ω_0/γ and $\hbar\omega_0/T$, an argument as in (4.29) shows again that the approximation (4.22) fails to correctly describe even the leading barrier renormalization (4.30) unless we stay at order V^2. It is interesting to note that the temperature independent quantum corrections due to ΔS_1 which appear through the coefficients c_r in (4.33) are of order $(\pi/\alpha)^2=(\omega_0/\gamma)^2(\hbar\omega_0/2V)^2$ and thus become comparable to the quantum corrections in (4.32) only if ω_0/γ is of order one.

In spite of the short comings of our approximation (4.22) discussed above, its numerical evaluation shows physically reasonable behavior at least for small barriers and, if $\alpha>1$, not too low temperatures. For example, in Fig. 6 we have plotted the linear mobility μ_l as a function of temperature for $\alpha=2\pi$ and $\tilde{V}=1$. Although the approximation obviously breaks down for small temperatures where μ_l becomes negative and is thus unable to reproduce the correct value $\mu_l=0$ at $T=0$ associated with the localized ground state, it describes the expected monotonic increase in the mobility due to incoherent transport. More interesting are the results shown in Fig. 7 for $\alpha=\frac{1}{2}$ and two different values of the barrier. In the case

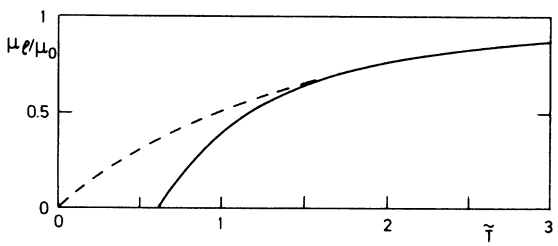

FIG. 6. Normalized linear mobility μ_l/μ_0 as a function of temperature for $\alpha=2\pi$ and $\tilde{V}=1$. At low temperatures the approximation (4.22) breaks down and the true expected behavior is indicated qualitatively by the dashed line.

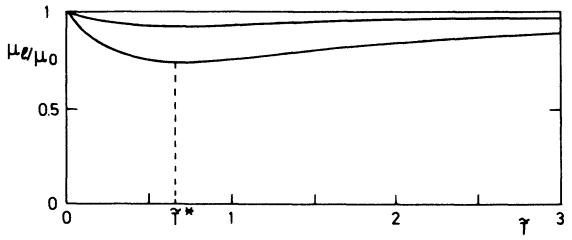

FIG. 7. Normalized linear mobility as a function of temperature for $\alpha=\frac{1}{2}$ and two different values of $\tilde{V}=1/\pi$ or $2/\pi$ (lower curve). The temperature T^* of the minimum is approximately $\tilde{T}^*\approx 0.7$ in both cases.

$\tilde{V}=1/\pi$ the result is basically given by the lowest-order perturbative expression (4.14), whereas, for $\tilde{V}=2/\pi$ the iteration converges after two or three steps. As expected, the nonmonotonic behavior of the mobility which is due to a crossover from coherent to incoherent transport,[6] is more pronounced at higher barriers. The temperature T^*, where the minimum occurs, however, remains essentially unchanged at these still rather small values of the barrier. In the limit $V\to 0$, T^* was given by[6]

$$T^*\mid_{V\to 0}=c(\alpha)E_0\ , \tag{4.34}$$

where $c(\alpha)$ approaches $\frac{1}{12}$ as $\alpha\to 0$ and zero as $\alpha\to 1^-$. A similar nonmonotonic behavior, as shown in Fig. 7, is found for the nonlinear mobility at $T<T^*$ and $\alpha<1$ as a function of bias $\bar{\epsilon}$. From the argument given in Sec. II, we expect that the position $\bar{\epsilon}^*$ of the minimum will shift down for larger barriers. By analogy, the same should be true for T^*.[46] Unfortunately, the quantitative change of T^* with V cannot be determined from our approximation (4.22), since again it turns out to yield negative values of the mobility if $\tilde{V}$ is increased beyond values of order 1. Thus, except in the classical high damping limit, it is reliable only as long as the Josephson coupling can be treated perturbatively. Crudely speaking, it breaks down when the barrier is larger than the zero-point energy and the discrete level structure within a well becomes important. Indeed for large barriers with many levels, it is clearly not appropriate to expand in the bare corrugation V but to truncate the system to a discrete tight-binding model first.[47] However, using the duality described in Sec. II, the perturbative results derived above for the continuous periodic potential, can then also be applied to explain the behavior of a case with small hopping matrix element. Thus, with appropriately redefined parameters, our expressions may be used even for a periodic potential with very large barriers, provided the temperature is smaller than T^*. Indeed, if the dimensionless coupling $\tilde{\alpha}$ in the tight-binding model is larger than 1, the duality predicts a linear mobility which vanishes at $T\to 0$, has a maximum at T^* and goes to zero again for large T. However, as has been argued above, the mobility for an actual continuous periodic potential should be completely monotonic in this case. Similarly, for $\tilde{\alpha}<1$, the tight-binding model fails to reproduce the nonmonotonic behavior expected in a situation with an extended ground state and can be used only if $T<T^*$. It is evident that the range of validity of the tight-binding picture, and therefore T^*, should *decrease* with increasing hopping matrix element V, in agreement with the argument given above.

V. DISCUSSION

In this work we have studied a simple model which describes the current-voltage characteristic of a resistively shunted Josephson contact and takes into account the quantum nature of the phase. We have shown that the effective nonlinear resistance is determined by the mobility of a quantum Brownian particle in an extended periodic potential. A scaling law for the nonlinear mobility at $T=0$ has been derived for small dimensionless friction $\alpha<1$, where the Josephson effect is destroyed by quantum fluctuations of the phase. We have given a rigorous discussion of the leading quantum corrections to the classical limit and have discussed their rather different form and origin for small or large damping. Finally a simple continued fraction representation for the quantum mobility, which is exact in the classical high damping limit, has been derived and numerically evaluated. It shows physically reasonable behavior for small Josephson coupling and sufficiently large temperature.

Concerning the possibility of observing experimentally the quantum effects of the phase in the current-voltage characteristic of a single Josephson junction, we recall that the necessary condition on temperature is $T\lesssim\hbar\gamma, E_0$. In most cases which have been investigated until recently, the dimensionless friction α was very large and the requirement $T\lesssim E_0$ is thus much more restrictive than $T\lesssim\hbar\gamma$. In fact even for capacitances as small as $C=10^{-14}$ F, the equivalent charging energy E_0 corresponds to a temperature of only 0.74 K. As a result, one is often in a regime where it is possible to see the quantum effects in the noise spectrum of the shunt resistor, but where the phase itself is still a classical variable. Indeed, in the limit $\alpha\gg 1$, the equivalent particle is strongly localized at $T=0$ and tunneling through the barriers is negligible. It therefore behaves very much like a classical particle subject to quantum noise and is then describable by the so-called quantum Langevin equation,[48] which is (1.1) with ξ^{cl} replaced by a frequency-dependent quantum noise ξ [see Eq. (A8) in the Appendix]. This situation has been investigated by Koch, van Harlingen, and Clarke,[49] who found that in junctions with $\alpha\simeq 10^4$ the measured voltage noise is in good agreement with the predictions of the quantum Langevin equation. However, in order to really test the quantum nature of the phase itself, for instance through tunneling, level quantization or, at least, the corresponding quantum corrections ΔS_1, it is necessary to go to lower temperature $T\lesssim E_0$ and smaller damping $\alpha\lesssim 1$. There have been no experiments in this regime looking specifically at quantum effects in the current-voltage characteristic but both, tunneling[2] and level quantization,[3] have been found in a situation with a strong bias $f=I/I_c\lesssim 1$, where the effective barrier is reduced to a value at which tunneling becomes observable even in contacts with a strong Josephson coupling. In particular, in the experiments on the decay rate of the

zero-voltage state close to I_c, the quantum corrections to the classical Arrhenius behavior have also been investigated. In one of them [Ref. 2(a)] the corrections associated with (3.17) seem to have been observed[50] even though they were not in the overdamped regime. On the contrary, similar experiments on a highly damped SQUID [Ref. 2(b)] did not find the quantum corrections to classical activation, although the conditions for seeing the effect (3.17) should have been fulfilled. It would therefore be of considerable interest to look at these questions in more detail, using junctions where all parameters may be accurately inferred from the classical regime. In particular, it seems worthwhile to study the quantum effects directly in the current-voltage characteristic and to investigate their behavior as a function of both temperature and damping. In this way one should be able, for instance, to see the deviations from (3.17) for smaller damping, where the quantum corrections are expected to be independent of T as long as $T \ll V$. In the absence of a complete theory for even the leading corrections to the classical result for arbitrary damping, such measurements would be of considerable interest.

The most spectacular consequence of the interplay between dissipation and the quantum nature of the phase is the existence of a localization transition at $T=0$. For $\alpha<1$ there is a finite linear resistance even at $T=0$ and the Josephson effect is destroyed by quantum fluctuations of the phase. As has been discussed in the preceding section, the mobility in this case will be nonmonotonic both as a function of bias and temperature. As a result, it is possible that with increasing external current the associated voltage *decreases*. Thus there will be a region with negative differential resistance, an effect which has also been discussed in the context of Bloch oscillations in a Josephson junction.[5] In the limit of a small Josephson coupling this may qualitatively be described within our approximation (4.22). The corresponding normalized current-voltage characteristic of $f=I/I_c$ versus U/RI_c is plotted in Fig. 8 for $\alpha=\frac{1}{2}$, $\tilde{V}=1.1$, and $\tilde{T}=0.3$, and clearly reproduces the expected behavior with negative differential resistance between $f\simeq 0.1$ and $f\simeq 0.25$. In view of the arguments given at the end of Sec. IV, we expect that with increasing Josephson coupling V both the necessary temperature $T<T^*$ and bias f where the effect may be observed will be smaller. For a possible experimental verifcation one should therefore try to use junctions where the Josephson coupling V is of the order of the charging energy E_0 or less. According to (4.34), even then the necessary temperature is $T<E_0/12$ which is 0.62 K for $C=10^{-15}$ F. A very interesting possibility to realize junctions with such tiny capacitances is to look at thin granular films. Below the bulk superconducting transition temperature, they can be modelled as a collection of locally superconducting islands linked together by small Josephson junctions. Recent experiments[51] on these systems have shown that, independent of the geometry, the sheet resistance $R_\square$ at low temperatures seems to depend only on the normal-state resistance R_n. If R_n is smaller than a value of order 6 kΩ, the film appears to be superconducting at low T, whereas otherwise $R_\square(T)$ shows a nonmonotonic behavior with a minimum around

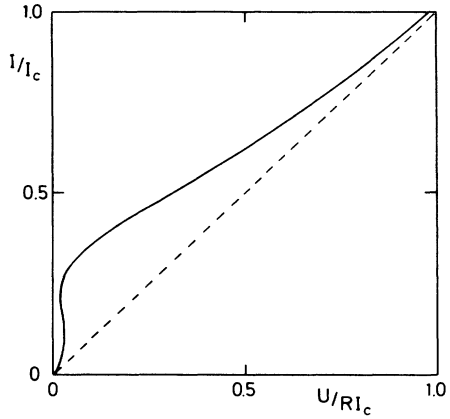

FIG. 8. The normalized current-voltage characteristic $f=I/I_c$ vs U/RI_c in the case $\alpha=\frac{1}{2}$, $\tilde{V}=1.1$, and $\tilde{T}=0.3$. Between $f\approx 0.1$ and $f\approx 0.25$ the differential resistance is negative. The dashed line corresponds to the ohmic characteristic of a normal tunnel junction.

$T^*\approx 3.1$ K for $R_n=10$ kΩ and a finite value of the resistance as $T\to 0$. This behavior is strikingly similar to the one obtained above for a *single* Josephson junction with very small capacitance and, as has been pointed out by Fisher,[52] it is thus tempting to try to understand these observations in terms of our present model. Apparently, this is possible only in the quasireentrant case, since the question of global phase coherence can clearly not be answered within a single junction model. Indeed, in a single junction there is always a finite resistance at $T\neq 0$ and it is only through collective effects that the whole array may become superconducting at nonzero temperature.[53] In the reentrant and thus resistive case, however, one may try to carry over the percolation argument which determines the resistance in the normal state in terms of the individual link resistances R_i. It amounts to replacing the whole random array by a *single* link with resistance $R=R_n$, such that the links with resistances $R_i\leq R$ form a percolating network. Within this approximation, the sheet resistance $R_\square(T)$ can be calculated in terms of our present model as

$$R_\square(T)=\frac{\mu_l}{\mu_0}R_n \ . \tag{5.1}$$

The nonmonotonic behavior of the reduced mobility with temperature shown in Fig. 7 thus translates directly into that of the sheet resistance and reproduces the experimental observations[51] in a qualitative way. Using the $V\to 0$ result (4.34) for the crossover temperature and $c(0.64)\approx c(0)/2$, the corresponding estimate for the junction capacity is $C=10^{-16}$ F. Within the present model this is in fact an upper bound, since T^* is expected to decrease with increasing V. Unfortunately, the actual quantitative behavior of the sheet resistance is in considerable discrepancy with (5.1). Indeed the single-junction model fails to reproduce the observed very rapid decrease of $R_\square(T)$ below the bulk superconducting temperature. Moreover, the measured sheet resistance seems not to ap-

proach its normal state value in the limit $T\to 0$ as predicted by $\mu_I(T=0,\alpha<1)=\mu_0$ for all V, but generally remains much smaller. This indicates that collective effects, which favor phase coherence and thus reduce the resistance, may play a major role even if global superconductivity is not established. The adequacy of a single junction picture for granular films is further questioned by the fact that no negative differential resistance behavior as in Fig. 8 has been observed in the nonlinear current-voltage characteristic of the quasireentrant samples.[54] Nevertheless it is clear that dissipation plays a dominant role in these systems and replaces the ratio V/E_0 in nondissipative theories[55] by the actually relevant parameter $\alpha=2\pi\hbar\gamma/E_0$, which is independent of capacity.

ACKNOWLEDGMENTS

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APPENDIX: CLASSICAL LIMIT FROM FEYNMAN'S INFLUENCE FUNCTIONAL THEORY

The central quantity in the Feynman-Vernon theory[56] for dissipative quantum systems is the reduced time-dependent density matrix

$$\rho(t)=\mathrm{Tr}_B\rho_{\mathrm{tot}}(t)\ , \tag{A1}$$

where ρ_{tot} is the total density matix of the system plus bath and Tr_B denotes the trace over the bath variables. It is convenient to define the coordinate representation of ρ as

$$\rho(x,y,t)=\langle x+y/2\,|\,\rho\,|\,x-y/2\rangle \tag{A2}$$

which determines the Wigner distribution function

$$f(x,p,t)=\int_{-\infty}^{\infty}\frac{dy}{2\pi\hbar}\rho(x,y,t)\exp(-ipy/\hbar) \tag{A3}$$

via a Fourier transform. With an appropriately factorized initial condition, the reduced density matrix at time t can be related to its value at $t=0$ by

$$\rho(x,y,t)=\int dx_0dy_0\rho(x_0,y_0)J(x,y,t;x_0y_0)\ . \tag{A4}$$

Here

$$J(x,y,t;x_0y_0)=\int_{x_0}^{x}Dx\int_{y_0}^{y}Dy\exp\left[\frac{i}{\hbar}\int_0^t[M\dot{x}\dot{y}-V(x+y/2)+V(x-y/2)]dt'\right]F[x(t'),y(t')] \tag{A5}$$

is given by a double path integral over all possible paths $x(t')$ from x_0 to x and $y(t')$ from y_0 to y. Writing the influence functional F as $F=\exp(i\Phi)$ the corresponding influence phase Φ for our model (1.9) is given by[57]

$$i\Phi[x,y]=-\frac{i}{\hbar}\eta\int_0^t dt'\dot{x}y-S_2[y]\ . \tag{A6}$$

Here $\exp(-S_2)$ can be expressed as an average

$$\exp(-S_2)=\left\langle\exp\left[\frac{i}{\hbar}\int_0^t dt'\xi(t')y(t')\right]\right\rangle_{\xi} \tag{A7}$$

over a Gaussian fluctuating force $\xi(t')$ with spectrum

$$\int_{-\infty}^{\infty}dt\langle\xi(t)\xi(0)\rangle\exp(i\omega t)=\eta\hbar\omega\coth\left[\frac{\beta\hbar\omega}{2}\right]\ . \tag{A8}$$

The contribution $S_2\geq 0$, which suppresses the off-diagonal components $y\neq 0$ of the density matrix, thus represents the fluctuating force which is exerted on the Brownian particle by the unperturbed environment motion. Similarly, the first term in (A6), which is odd under time reversal, describes the systematic influence of the bath and, classically, leads to a dissipative force $F_{\mathrm{diss}}=-\eta\dot{q}$.

In order to obtain the classical limit from the influence functional theory, two approximations are necessary:[48]

(1) expand to lowest order in the off-diagonal components, i.e.,

$$V(x+y/2)-V(x-y/2)\approx V'(x)y\ ; \tag{A9}$$

(2) replace the fluctuating force $\xi(t)$ by its classical limit $\xi^{cl}(t)$ with a white spectrum $2\eta T$, i.e.,

$$S_2\approx S_2^{cl}=\frac{\eta T}{\hbar^2}\int_0^t dt'y^2(t')\ . \tag{A10}$$

With these two approximations, the path integral $\int Dy$ is Gaussian and one obtains for the classical limit of the Wigner function (A3) the real single path integral

$$f(x,p,t)=\int_{-\infty}^{\infty}dx_0\int_{x_0}^{x}{}''Dx''f(x_0,M\dot{x}_0)\delta(p-M\dot{x})\times\exp\left[-\frac{1}{4\eta T}\int_0^t dt'[M\ddot{x}+\eta\dot{x}+V'(x)]^2\right] \tag{A11}$$

with measure

$$''Dx''=\lim_{\substack{N\to\infty\\ N\epsilon=t}}2\pi\left[\frac{M}{2\pi\epsilon}\right]^N\left[\frac{\pi}{\epsilon\eta T}\right]^{(N-1)/2}\prod_{j=1}^{N-1}dx_j\ . \tag{A12}$$

This is precisely the functional integral representation of the Fokker-Planck equation

$$\partial_t f(x,p,t)=-\frac{p}{M}\partial_x f+V'(x)\partial_p f+\gamma\partial_p(pf+MT\partial_p f)\ , \tag{A13}$$

which is equivalent to the classical Langevin equation (1.1). In the overdamped limit, the momentum p can be eliminated as a fast variable and one obtains the simpler Smoluchowski equation

$$\partial_t P(x,t)=\frac{1}{\eta}\partial_x[V'(x)P+T\partial_x P] \tag{A14}$$

for the coordinate distribution $P(x,t)$. Finally we mention that the so called quantum Langevin equation,[48] which is simply equation (1.1) with ξ^{cl} replaced by ξ, assumes that (A9) is valid but retains the frequency dependence of the noise spectrum (A8). Thus it is an equation for a *classical* particle subject to *quantum* noise. According to the discussion in Sec. III, such an equation will correctly describe the leading quantum corrections to the purely classical limit, provided that the effect of ΔS_1 is negligible compared to ΔS_2. From the order of magnitude estimate (3.15) this is true in the overdamped limit $K \ll 1$.

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[14]The simple form (1.9) arises because, without restriction of generality, one may choose $\lambda_\alpha=m_\alpha\omega_\alpha^2$ for convenience. As has been pointed out by V. Hakim and V. Ambegaokar, Phys. Rev. A **32**, 423 (1985), the Hamiltonian (1.9) has then a very intuitive physical interpretation, in which the Brownian particle has attached to it a collection of masses m_α via a harmonic potential with spring constants λ_α.

[15]Taken as *classical* Hamiltonians, (1.7) and (1.9) are trivially equivalent since their corresponding Lagrangians differ only by a total time derivative. Note that in (1.7) the canonical momentum $p=\hbar n$ is not equal to $M\dot{\varphi}=(\hbar/2e)Q$ but, according to (1.6), is a generalized momentum $p=M\dot{\varphi}-\hbar/2e(Q_{\rm ext}-Q_n)$.

[16]It should be noted that this particular form of the spectral density is fixed by the duality transformation and, except for some qualitative results, cannot be replaced by a different form of the cutoff.

[17]Note that this makes sense as a physical model only if the barrier is large and the temperature is small compared to $\hbar\omega_0$, ω_0 being the small oscillation frequency in the potential wells. On the contrary, the duality derived in Ref. 6 is exact at *arbitrary* temperature, even where the tight-binding model may have lost its physical meaning. See, also, the discussion at the end of Sec. IV.

[18]A. Schmid, Phys. Rev. Lett. **51**, 1506 (1983).

[19]S. A. Bulgadaev, Pis'ma Zh. Eksp. Teor. Fiz. **39**, 264 (1984) [JETP Lett. **39**, 315 (1984)].

[20]F. Guinea, V. Hakim, and A. Muramatsu, Phys. Rev. Lett. **54**, 263 (1985).

[21]This is based on unpublished arguments by M. P. A. Fisher.

[22]It should be noted, however, that the scale factor in (2.6) is proportional to η and the mapping is therefore possible only for a finite dissipation $\eta\neq 0$. This shows that the duality is not merely a transformation from Wannier to Bloch states and thus is more subtle than the one found by S. Aubry and G. André, Ann. Isr. Phys. Soc. **3**, 133 (1980).

[23]Note that these definitions differ slightly from those in Ref. 6, where *two independent* sets of n or n' ordered times were introduced but, instead, the two sets of charge variables each had only n or n' elements and not both $n+n'$ as here.

[24]In fact, (2.11) can only be used to calculate the first moment $\langle x_s(t)\rangle$, but it is straightforward to write down also the expression for the full generating function, see Ref. 6.

[25]Note that for the Josephson junction $\tilde{V}=I_c/2e\gamma$ is the number of Cooper pairs which may maximally flow through the contact within a relaxation time γ^{-1}. Usually this is a rather large number unless the capacitance is tiny. This is also evident for an ideal tunnel junction with $2V=\alpha\Delta_{\rm BCS}$, where $\tilde{V}=\pi\Delta_{\rm BCS}/E_0$ with $E_0=(2e)^2/C$ as (twice) the Coulomb energy associated with a single Cooper pair at the capacitor.

[26]A. Lenard, J. Math. Phys. **2**, 682 (1961).

[27]M. P. A. Fisher, Ph. D. thesis, University of Illinois, 1986.

[28]Note that in the Josephson context we have actually $\alpha\sim\hbar$ since $M\sim\hbar^2$ but the "classical" limit in this case has to be taken with M and V being kept finite.

[29]From a practical point of view, the series (3.8) is therefore useful only in the overdamped limit $\gamma\gg\omega_0$ but we emphasize that, at least formally, it is exact for arbitrary γ. We also note that in the literature on the resistively shunted junction model, the parameter $(\omega_0/\gamma)^2$ is often abbreviated as β_c.

[30]Note that the *cl* in $Q_N^{cl}(L)$ refers only to its relation to the classical mobility whereas *both* Q_N and Q_N^{cl} themselves are purely classical partition functions even if Q_N contains parameters which depend on $\hbar$.

[31]Note that $2V$ and ω_0 are the barrier and oscillation frequency in the *unbiased* periodic potential, but (3.17) is valid for arbi-

trary bias $\bar{\epsilon}$.

[32]I. Affleck, Phys. Rev. Lett. **46**, 388 (1981).

[33]P. G. Wolynes, Phys. Rev. Lett. **47**, 968 (1981); V. I. Mel'nikov and S. V. Meshkov, Pis'ma Zh. Eksp. Teor. Fiz. **38**, 111 (1983) [JETP Lett. **38**, 130 (1983)].

[34]P. Hänggi, H. Grabert, G. L. Ingold, and U. Weiss, Phys. Rev. Lett. **55**, 761 (1985).

[35]More precisely, we will see in Sec. IV that the approximation which led to (3.17) is equivalent, in the purely classical case, to the Smoluchowski result including the first leading corrections of order γ^{-2}. This provides us with a precise criterion for the limits of validity of the result (3.17). See Eq. (4.32) below.

[36]Equation (3.19) is an equality to order V^2 if $V \to 0$.

[37]In the opposite limit $V \ll T$, the corrections have been determined exactly in Ref. 6 and are found to vanish (as $T^{-5/2}$) if $T \to \infty$, as they should, even if ΔS_1 is temperature independent. Note that the quantity $W(K)$ defined there should approach $\frac{1}{8}$ as $K \to \infty$ and not $\frac{1}{4}$ as stated.

[38]This approximation, which is denoted by a superscript a in the following, is formally equivalent to the so-called "noninteracting blip approximation" in the dissipative two-state system which was introduced by S. Chakravarty and A. J. Leggett, Phys. Rev. Lett. **52**, 5 (1984). See, also, A. J. Leggett, S. Chakravarty, A. T. Dorsey, M. P. A. Fisher, A. Garg, and W. Zwerger, Rev. Mod. Phys. **59**, 1 (1987). It should be noted, however, that in contrast to the two state system both the fugacity and the variables ν_l may have arbitrary large values in the present case and therefore, the justification for this approximation which is given in the preceding papers cannot be carried over here.

[39]Note that for a standard thermodynamic partition function with *short* range interaction one expects instead $\lim_{L\to\infty} Q_N(L) \sim L^N$.

[40]The result (4.14) has been rederived using the Keldysh technique by U. Eckern and F. Pelzer, Europhys. Lett. **3**, 131 (1987). There it is also suggested to write $\mu/\mu_0 = 1 - V^2 \int \ldots + O(V^4)$ as $\mu/\mu_0 \approx (1 + V^2 \int \cdots)^{-1}$ which allows to use the lowest order $\tilde{V}^2$ expansion even if $\bar{\epsilon}$, $T \to 0$, and $\alpha > 1$. Within this approximation, however, the temperature T^*, where the mobility has a minimum is independent of V for given $\alpha < 1$.

[41]It is interesting to note that in the calculation of the standard Coulomb-gas partition function in Ref. 26 it is indeed precisely these configurations which contribute in the limit $s \to 0$. The fact that they are irrelevant in the present case arises from the unusual factor $\prod_{l=1}^{2N-1} \sin[(\pi/\alpha) g_l]$ which vanishes in the approximation (4.8) if $\nu_l = 0$.

[42]These are the so called irreducible configurations in Ref. 26.

[43]If $\alpha > 1$ and $T = \bar{\epsilon} = 0$, the ground state is localized for arbitrary weak corrugation V and thus the linear mobility has to vanish. The corresponding value of $\mu_{TB}/\mu_0 = 1$ is clearly not obtainable from perturbation theory.

[44]Note that even if $\Lambda_{ll'} = 0$ for a constant additional term $\Delta Q(x)$ in $Q(x)$, we have $\Delta S_2 \neq \Delta S_2^q$ since (4.5) is valid only if $Q(0) = 0$.

[45]Thus, keeping R and I_c constant, a smaller damping due to an increase in C will, at least to lowest order in ω_0/γ, always lead to a *decrease* in the voltage for any given current.

[46]Note that in Ref. 6 it was stated that T^* should *increase* with barrier height. This is incorrect for T^* but applies instead to the temperature above which there is classical behavior. The temperature T^*, however where the mobility has a minimum will change in the opposite direction. Thus for high barriers there will be an increasingly wide range of temperatures where the mobility increases with T but the behavior is still nonclassical.

[47]For a double well with Ohmic dissipation such a procedure has explicitly been performed by A. T. Dorsey, M. P. A. Fisher, and M. S. Wartak, Phys. Rev. A **33**, 1117 (1986).

[48]A. Schmid, J. Low Temp. Phys. **49**, 609 (1982).

[49]R. H. Koch, D. J. van Harlingen, and J. Clarke, Phys. Rev. Lett. **45**, 2132 (1980); Phys. Rev. Lett. **47**, 1216 (1981).

[50]In order to identify our corrections (3.17) for the mobility with those of the decay rate τ^{-1}, it is necessary to relate the decay rate from a metastable well at $f \neq 0$ to the equivalent hopping rate for diffusion in the *unbiased* case with appropriately modified barrier and oscillation frequency [see the discussion following (3.17)]. Thus, for $\tau^{-1}(f)$ (3.17) applies with the *effective* values of barrier and frequency at $f \neq 0$. On the contrary, the nonlinear mobility has to be calculated by renormalizing the *bare*, unbiased values of V and ω_0. This is a nontrivial point because V and ω_0^2 scale differently with f.

[51]B. G. Orr, H. M. Jaeger, A. M. Goldman, and C. G. Kuper, Phys. Rev. Lett. **56**, 378 (1986).

[52]M. P. A. Fisher, Phys. Rev. Lett. **57**, 885 (1986).

[53]The question of global phase coherence in an array of Josephson junctions, including dissipation, has been addressed in Ref. 27 and also by S. Chakravarty, G. L. Ingold, S. Kivelson, and A. Luther, Phys. Rev. Lett. **56**, 2303 (1986).

[54]B. G. Orr, H. M. Jaeger, and A. M. Goldman, Phys. Rev. B **32**, 7586 (1985).

[55]See, for instance, S. Doniach, in *Percolation, Localization and Superconductivity*, Ref. 4.

[56]R. P. Feynman and F. L. Vernon, Ann. Phys. (N.Y.) **24**, 118 (1963).

[57]Note that (A6) is translation invariant due to a choice for the initial density matrix, which does not single out a particular point in coordinate space (see Ref. 13).