Acoustic localization in one dimension in the presence of a flow field

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(Received 7 November 1986)

The effect of a uniform flow field on one-dimensional localized acoustical excitations is investigated by use of both the usual Feynman diagrammatic perturbation theory and the Berezinskii formalism. While leading-order perturbation theory suggests that localization is destroyed by a flow, or at least drastically modified, the Berezinskii method proves that localization suffers no essential modifications. The reason for the breakdown of the usual perturbative approach is discussed.

I. INTRODUCTION

Recently there have been a number of papers on the problem of Anderson localization in disordered nonelectronic systems.[1–13] In the case of acoustic systems, the best candidates for the observation of Anderson localization seem to be experiments that use third sound in superfield helium films as the excitations to be localized.[7,8] In the proposed experiments the disorder is introduced by modifying the substrate; the resulting configuration may be effectively one or two dimensional. In the one-dimensional case, parallel, identical strips are produced either by roughening the substrate, which changes the effective speed of sound, or by using a second material having a different van der Waals constant, which alters the film thickness.[8] In the two-dimensional proposal, the disorder is generated by the addition of dust particles to an otherwise smooth substrate. The scattering is due to the helium puddles formed by capillary condensation around the particles.[7]

Several reasons moved us to suggest a superfluid helium film as a convenient medium for the investigation of acoustic localization. First, the intrinsic attenuation, which corresponds to the electron-phonon interaction in electron-localization problems, is expected to be very small at sufficiently low temperatures. Second, the interaction between the third-sound excitations can also be made small by controlling their amplitude. Finally, the scatters are macroscopic and, at least in the one-dimensional configurations, can be made identical to each other. Therefore, we expect the theoretical model to furnish very detailed predictions.

It is possible to create a stable uniform flow field in a superfluid film.[14] Since the flow destroys time-reversal invariance a substantial modification of the localization phenomena may be expected to occur. In a separate publication[15] we present a field theory that describes the crossover between orthogonal (in the absence of flow) and unitary (for strong flows) localization: in two dimensions the localization length l_l is predicted to increase from $e^{(E_0/E)^2}$ to $e^{(E_0/E)^4}$ as the flow is turned on. (Here E is the excitation frequency and E_0 is a fixed frequency scale.) In this paper we discuss the effects of flow in one-dimensional systems. The methods used here, although not as general as those in Ref. 15 have the advantage of permitting an explicit evaluation of the relevant coefficients. We note that, although our results have general validity for one-dimensional (1D) fluids, we will refer frequently to the superfluid films, which we believe to be the most appropriate experimental systems. For this application we assume that the temperature is low enough that we can neglect the normal component.

Previously,[5,7,8] we employed a perturbative expansion in terms of Feynman diagrams to evaluate the average of the squared Green's function. Localization was then discussed using the procedure of Vollhardt and Wölfle.[16] In this procedure the maximally crossed diagrams (MCD's) act as a starting point for the formulation of a self-consistent theory (SCT). For $d=2,3$ this procedure is reasonable, since in these dimensions a well-defined coupling-constant expansion exists. In one-dimensional systems the situation is less clear: explicit evaluation of diagrams in $d=1$ shows that it is impossible to simply order the Feynman diagrams in a coupling-constant expansion. As a consequence, no simple systematic expansion is possible with this approach. On the other hand, a systematic technique for diagram summation due to Berezinskii[17] gives for the electronic problem a result which is similar to that of the SCT. We will see below that the same is true of the localization of the third-sound waves in the absence of flow.

The Berezinskii formalism thus shows the correctness of the SCT in certain 1D problems. However, when a uniform flow is introduced the results of these calculational schemes no longer coincide. While the usual leading-order perturbation theory suggests the disappearance of the localized states, the Berezinskii technique predicts that the flow will cause no qualitative changes in the localization picture. The rigor of the Berezinskii solution points to a breakdown in the usual Feynman-diagram perturbative method.

Strictly speaking, the usual perturbative approach also breaks down in $d=2,3$. However, the breakdown is less severe than in $d=1$. For $d=2$, leading-order perturbation theory suggests that localization is completely destroyed by the flow. Elsewhere[15] we have shown that in reality localization still exists in the presence of a flow, although it is substantially weaker. The perturbation theory is qualitatively correct in predicting that there are large modifications to the localization.

This paper is organized as follows. In Sec. II we present the physical model and derive the various Green's functions needed in the calculation. The Feynman diagram and Berezinskii calculations are presented in Secs. III and IV, respectively. Finally, the results of these methods are compared and discussed in Sec. V. In the Appendix the Berezinskii technique is used to calculate the effects of weak dissipation on the localized excitations.

II. FORMULATION OF THE PROBLEM

A. The differential equation

We will use the symmetric white-noise model, which results from the introduction of white-noise randomness at the level of the Euler's equation. Let the speed of sound be

$$c^2(\mathbf{x})=c^2+\psi(\mathbf{x}) , \tag{2.1}$$

where c is the speed of third-sound waves on a uniform substrate, or, more generally, the speed of sound in the homogeneous fluid, and $\psi(\mathbf{x})$ is a Gaussian random variable whose correlator is

$$\langle \psi(\mathbf{x}_1)\psi(\mathbf{x}_2)\rangle_{\rm av}=\mu\delta(\mathbf{x}_1-\mathbf{x}_2) . \tag{2.2}$$

Combining the hydrodynamic equations, we obtain the following differential equation for the density fluctuations $\rho(\mathbf{x},t)$:

$$\left[\frac{\partial^2}{\partial t^2}-c^2\nabla^2-\nabla\cdot[\psi(\mathbf{x})\nabla]+2(\mathbf{u}\cdot\nabla)\frac{\partial}{\partial t}+(\mathbf{u}\cdot\nabla)^2\right]\rho(\mathbf{x},t)=0. \tag{2.3}$$

Here $\mathbf{u}$ is the velocity of the uniform flow field. An alternative equation can be written if we introduce the randomness directly in the wave equation for the velocity potential. In this case, the density in Eq. (2.3) should be replaced by the velocity potential and the random term would have the "asymmetric" form $\psi(\mathbf{x})\nabla^2$. This asymmetric form is probably suitable to model a Si substrate containing a distribution of roughened regions.[8] In these regions the effect of the roughening is taken into account using an effective index of refraction.[18] The symmetric and asymmetric models yield the same results.

We choose the initial conditions used previously,[1,5,8]

$$\rho(\mathbf{x},t\le 0)=0 \tag{2.4a}$$

and

$$\frac{\partial\rho}{\partial t}(\mathbf{x},t=0)=f(\mathbf{x}) . \tag{2.4b}$$

for a given initial perturbation $f(\mathbf{x})$.

B. The Green's functions

In what follows we restrict ourselves to one-dimensional systems. We start by defining the Laplace transform of the density fluctuations,

$$\rho_E(x)=\int_0^\infty e^{i(E+i\eta)t}\rho(x,t)dt . \tag{2.5}$$

The Green's functions $G_E^\pm$ associated with the differential equation for $\rho_E(x)$ satisfy

$$\left[(c^2-u^2)\frac{\partial^2}{\partial x_1^2}+(E\pm i\eta)^2\pm 2ui(E\pm i\eta)\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_1}\psi(x_1)\frac{\partial}{\partial x_1}\right]G_E^\pm(x_1,x_2)=-\delta(x_1-x_2) . \tag{2.6}$$

The superscript minus indicates that the direction of the flow has been reversed. Note that $G^+_{-E+i\eta}(x_1,x_2)=G^-_{E-i\eta}(x_1,x_2)$: the system is invariant under the simultaneous inversions of the direction of flow and of the arrow of time. The free Green's functions (i.e., those in the absence of scatterers) are easily seen to be

$$G^+_{E,0}(x_1,x_2)=(i/2cE)[\Theta(x_1-x_2)\exp(ik_1\,|\,x_1-x_2\,|\,)+\Theta(x_2-x_1)\exp(ik_2\,|\,x_1-x_2\,|\,)] \tag{2.7a}$$

and

$$G^-_{E-\omega,0}(x_1,x_2)=-(i/2cE)[\Theta(x_1-x_2)\exp(-ik_4\,|\,x_1-x_2\,|\,)+\Theta(x_2-x_1)\exp(-ik_3\,|\,x_1-x_2)] . \tag{2.7b}$$

The "external frequency" ω has been added for convenience. The various k_i's are defined by

$$k_1=E/(c+u),\quad k_2=E/(c-u) , \tag{2.8a}$$

$$k_3=(E-\omega)/(c-u),\quad k_4=(E-\omega)/(c+u) . \tag{2.8b}$$

In our study of localization we are interested in the long-time limit. Since this corresponds to $\omega\to 0$, we have neglected the ω dependence in the factors in front of the exponentials in Eq. (2.7b).

We can go to the wave-number representation by writing

$$G_E^\pm(p_1,p_2)\equiv\langle p_1\,|\,G_E^\pm\,|\,p_2\rangle=\frac{1}{2\pi}\int dx_1dx_2e^{-ip_1x_1}e^{ip_2x_2}G_E^\pm(x_1,x_2) . \tag{2.9}$$

In particular, we obtain

$$G^\pm_{E,0}(p)=[(c^2-u^2)p^2-(E\pm i\eta)^2\pm 2up(E\pm i\eta)]^{-1} . \tag{2.10}$$

The averaged Green's function can be written in terms of a self-energy as usual. It is easy to compute this self-energy to the lowest order in the disorder. The imaginary part, the one important for localization, is

$$\sigma_E(p)=[(c+u)^{-2}+(c-u)^{-2}](\mu E/4c)p^2 . \tag{2.11}$$

The real part of the self-energy, on the other hand, renormalizes the speed of sound. This renormalized speed of sound will be called $\tilde{c}$.

III. LOCALIZATION: THE USUAL FEYNMAN DIAGRAM APPROACH

A. The squared Green's function

The discussion of the localization properties of the system is based on the analysis of the averaged squared Green's function. First, we make a Laplace transform and introduce the internal frequency E using the convolution property,[5]

$$P_\omega(x,x')=\int_0^\infty dt \exp[i(\omega+i\eta)t]G^2(x,t\mid x') \equiv \int_{-\infty}^{\infty}\frac{dE}{2\pi}P_{E,\omega}(x,x') . \tag{3.1}$$

Since in this section we need only G^+, it is convenient to omit the superscript and leave the total frequency dependence explicitly in the index. The energy density propagator $\langle P_E(k,\omega)\rangle_{\rm av}$ is the Fourier component of the average energy density corresponding to the E component of a pulse excited at $\mathbf{x}'$:

$$\langle P_E(k,\omega)\rangle_{\rm av}=\int d(x-x')\exp[-ik(x-x')] \times\langle P_{E,\omega}(x,x')\rangle_{\rm av} \equiv\int dp\,\Phi_p(k,\omega\mid E) . \tag{3.2}$$

Noting that the averaged Green's function is diagonal in wave-number space, and employing the techniques used previously,[5,8] it is possible to show that Φ_p satisfies the Bethe-Salpeter equation

$$\left[2E\omega-2(c^2-u^2)kp-2u(\omega p+kE)+\Sigma_{E_+ +i\eta}(p_+)-\Sigma_{-E_- +i\eta}(-p_-)\right]\Phi_p(k,\omega\mid E) =\Delta\langle G_p\rangle_{\rm av}\left[1+\int dp_2 U_{pp_2}(k,\omega\mid E)\Phi_{p_2}(k,\omega\mid E)\right] . \tag{3.3}$$

Here $\Sigma_E(p)$ is the self-energy, U_{pp_2} the irreducible four-point vertex function, $p_\pm=p\pm k/2$, and $E_\pm=E\pm\omega/2$. The function $\Delta\langle G_p\rangle_{\rm av}$ is defined by

$$\Delta\langle G_p\rangle_{\rm av}=\langle G_{E_+ +i\eta}(p_+)\rangle_{\rm av}-\langle G_{-E_- +i\eta}(-p_-)\rangle_{\rm av} \simeq(i\pi/\tilde{c}E)[\delta(p-p_>)+\delta(p-p_<)] . \tag{3.4}$$

The last line is valid when ω and k are small and the disorder is weak. This form selects the wave numbers,

$$p_>=E/(\tilde{c}+u) \tag{3.5a}$$

and

$$p_<=-E/(\tilde{c}-u) . \tag{3.5b}$$

The first step in a perturbative solution to Eq. (3.3) consists of evaluating the self-energies to lowest order in the disorder and including only the lowest-order diagram in U_{pp_2}, i.e., $U_{pp_2}\simeq U^B_{pp_2}=(\mu/2\pi)p^2p_2^2$. In this approximation (the Boltzmann limit), Eq. (3.3) can be solved. Substituting the solution in Eq. (3.2), we find,

$$\langle P^B_E(k,\omega)\rangle_{\rm av}=\frac{(\tilde{c}^2-u^2)}{E^2\tilde{c}^3}\frac{\pi}{-i\omega+D_B(E,u)k^2} . \tag{3.6}$$

It is important to note that the energy density propagator has the same form as in the absence of flow. The size of the Boltzmann diffusion coefficient D_B decreases when the flow speed is increased:

$$D_B(E,u)=2\tilde{c}(\tilde{c}^2-u^2)^3/\mu E^2 . \tag{3.7}$$

B. The maximally crossed diagrams

We next consider the MCD's, whose contribution is usually taken as the starting point for a self-consistent theory of localization. The MCD's can be thought of as representing the constructive interference of conjugate waves travelling in opposite directions along the same closed path.[19] The probability of return of a wave to the neighborhood of the source is enhanced by these coherence effects, which are therefore especially relevant to the localization properties of the medium. In the absence of flow the MCD's give rise to infrared divergences in the limit $|\mathbf{l}|=|\mathbf{p}+\mathbf{p}_2|\to 0$. This corresponds to the interference of waves having momenta of the same magnitude but opposite signs.

It is interesting to study how the MCD's are modified by the flow. We begin by writing an integral equation for the function $U^M_{pp_2}$, which represents the contribution of the MCD's plus the "Boltzmann" diagram $U^B_{pp_2}$:

$$U^M_{pp_2}(0,\omega\mid E)=(\mu/2\pi)p(p-l)p_2(p_2-l) +(\mu/2\pi)\int dq\,pq(p-l)(q-l)G_{E_+ +i\eta}(q) \times G_{-E_- +i\eta}(q-l)U^M_{qp_2}(0,\omega\mid E) . \tag{3.8}$$

This equation is obtained by rotating and relabelling the bottom line of the irreducible four-point vertex function.[16] Since the k dependence disappears from the intermediate propagators in the MCD's, these represent nonsingular functions of k and it suffices to consider $U^M_{pp_2}$

$(k=0,\omega \mid E)$.

Let us define a pair of auxiliary functions:

$$U_p(l,\omega)=p_2^{-1}(p_2-l)^{-1}U^M_{pp_2}(0,\omega \mid E) , \quad (3.9)$$

and

$$I(l,\omega)=\int dp\, p^2(p-l)^2 G_{E_+}(p)G_{-E_-}(p-l) . \quad (3.10)$$

It is easy to see that $U_p(l,\omega)$ satisfies the integral equation

$$U_p(l,\omega)=(\mu/2\pi)p(p-l) \times \left[1+\int dq\, q(q-l)G_{E_+}(q)G_{-E_-}(q-l) \times U_q(l,\omega)\right] . \quad (3.11)$$

After some algebra, we find

$$U_p(l,\omega)=(\mu/2\pi)p(p-l)[1-(\mu/2\pi)I(l,\omega)]^{-1} . \quad (3.12)$$

The integral $I(l,\omega)$ can be evaluated using residues. The result is algebraically involved, but we observe that the relevant denominators become of the order of μ when $\omega \to 0$ and $l \to -2uE/(\tilde{c}^2-u^2)$. Therefore, it is convenient to separate the wave number l into two pieces,

$$l=-2uE(\tilde{c}^2-u^2)^{-1}+l' . \quad (3.13)$$

We retain the result for $I(l,\omega)$ to lowest order in μ, ω, and l', inserting it into Eq. (3.11). We finally arrive at the following result for the contribution of the MCD's to the irreducible four-point vertex function:

$$U^M_{pp_2}(0,\omega \mid E) = \frac{\mu^2 E^2 p(p-l)p_2(p_2-l)C_+^4(4\pi\tilde{c}^3C_-^3C_u)^{-1}}{-i\omega+\tilde{D}(E,u)l'^2+2\mu E^2u^2C_+^2(\tilde{c}C_-^3C_u)^{-1}} , \quad (3.14)$$

where

$$\tilde{D}(E,u)=2\tilde{c}C_-^5(\mu E^2 C_u)^{-1} , \quad (3.15a)$$

$$C_\pm=\tilde{c}^2\pm u^2 , \quad (3.15b)$$

and

$$C_u=\tilde{c}^4+u^4+6\tilde{c}^2u^2 . \quad (3.15c)$$

The presence of a uniform flow field brings about the following modifications to the four-point vertex function.

(1) The maximum contribution occurs for $l=E[(\tilde{c}+u)^{-1}-(\tilde{c}-u)^{-1}]$. This means that the constructive interference essential for localization occurs between excitations whose wave numbers are Doppler shifted.

(2) In the absence of flow, the coefficient of l^2 coincides with $D_B(E,0)$. If $u\neq 0$ the coefficient of l'^2 is smaller than $D_B(E,u)$ by a factor of $C_-^2C_u^{-1}$.

(3) The most important consequence of flow is that the hydrodynamic pole has been cutoff by a term of the order of μu^2. If the MCD's were the only set of diagrams responsible for localization, we would conclude that the flow field destroys localization. The same conclusion would be reached if we accepted the premise that the $\omega \to 0$ behavior if the MCD's is typical of that of the other sets of diagrams contributing to localization.

One of the main points of this paper is to establish the failure under certain conditions of the conventional perturbation plus SCT technique. It is relevant to point out that this failure not only occurs "in principle" but also leads to substantially wrong results "in practice." If one was to accept the results of our calculation, the cutoff in the singularity of the MCD's would imply a lifetime $\tau \sim \tilde{c}^7/2\mu E^2u^2$ of the localization effects. Since it may be easily seen that this time can be much shorter than other relevant times in the superfluid system, it is clear that the theory in this section would predict important, experimentally observable effects of the flow. In Sec. IV we will see that this conclusion is contradicted by the rigorous Berezinskii calculation.

We end this section pointing out that an analogous calculation carried out using the asymmetric model mentioned in Sec. II leads to exactly the same results.

IV. LOCALIZATION: THE BEREZINSKII TECHNIQUE

A. The essential diagrams

Berezinskii[17] carried out a systematic low-density analysis of localization in the one-dimensional electronic problem. In his model, as in ours, the scatterers are described using a Gaussian-noise representation. In this section we apply the formalism he developed to our third-sound-wave problem.

The centerpiece of Berezinskii's technique is the introduction of diagrams that are explicitly ordered in coordinate space. In the electronic problem each line represents a free Green's function $G_{E,0}(x_i,x_j)$ connecting scattering vertices at x_i and x_j. For given values of x_i and x_j we have either $x_i > x_j$ or $x_j > x_i$ and only one of the terms in the electronic equivalent to our Green's functions (2.7) survives. The surviving term can be broken up into two factors, each containing only x_i or x_j, which are associated with the corresponding scattering vertices.

An important difference between the electronic and sound-wave problems is that in the sound-wave case the random functions $\psi(x_i)$ are acted upon by differential operators. These operators can be moved in front of the free Green's functions by repeated integrations by parts. Since only terms containing an even number of random functions $\psi(x_i)$ appear, the final sign after all differential operators are shifted is positive. The net result is that all the free Green's functions appear differentiated once with respect to each of their arguments. There is no

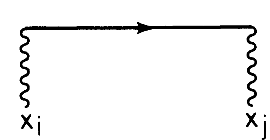

FIG. 1. The solid line represents a differentiated Green's function connecting two ordered vertices. To each wavy line representing the interaction with the random field we associate a factor of μ.

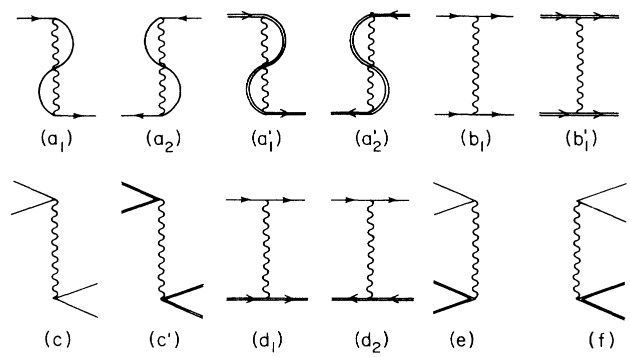

FIG. 2. Some of the essential internal vertices. The single (double) lines correspond to $G^{+}_{E,0}$ ($G^{-}_{E,0}$). The primed vertices are obtained by replacing the single lines by double ones.

differentiation with respect to the external points x and x', which are the arguments of the total Green's functions. For example, the solid line in Fig. 1 represents

$$\frac{\partial^2}{\partial x_i \partial x_j} G^{+}_{E,0}(x_i,x_j) = \left[\frac{i}{2cE}\right] k_2^2 \exp[ik_2(x_j - x_i)] \ . \quad (4.1)$$

We associate the factor

$$(i/2cE)^{1/2} k_2 \exp[-ik_2 x_i]$$

to the vertex at x_i and the factor

$$(i/2cE)^{1/2} k_2 \exp[ik_2 x_j]$$

to the vertex at x_j.

As shown by Berezinskii, we must select a group of "essential" vertices, which are to be explicitly evaluated. It is important to notice that the presence of flow introduces an asymmetry and the direction of the arrows becomes relevant. This can be seen explicitly from Eqs. (2.7). We have to evaluate 20 essential vertices, most of which are represented in Fig. 2. Those that are not represented are the six type (b) and (b') that can be obtained from other combinations of arrow directionality, and the two which are obtained rotating the upper arrows in (d_1) and (d_2). Arrow directions turn out to be irrelevant for vertices (c), (c'), (e), and (f). The evaluation of the vertices is elementary, except for the fact that the two end points of the "inner" line in the (a) vertices must be made to coalesce.

The following values are obtained for the vertices (we have factored out $\mu/4c^2E^2$):

(a_1): $-(k_1^2+k_2^2)(k_2^2/2)$ (a_2): $-(k_1^2+k_2^2)(k_1^2/2)$

(a_1'): $-(k_3^2+k_4^2)(k_3^2/2)$ (a_2'): $-(k_3^2+k_4^2)(k_4^2/2)$

(b_1): $--k_2^4$ (b_2)=(b_3): $-k_1^2k_2^2$

(b_4): $-k_1^4$ (b_1'): $-k_3^4$

(b_2')=(b_3'): $-k_3^2k_4^2$ (b_4'): $-k_4^4$

(c): $-k_1^2k_2^2$ (c'): $-k_3^2k_4^2$

(d_1): $k_2^2k_3^2$ (d_2): $k_2^2k_4^2$

(d_3): $k_1^2k_3^2$ (d_4): $k_1^2k_4^2$

(e): $k_1k_2k_3k_4\exp[2ic\omega x_1/(c^2-u^2)]$

(f): $k_1k_2k_3k_4\exp[-2ic\omega x_1/(c^2-u^2)]$.

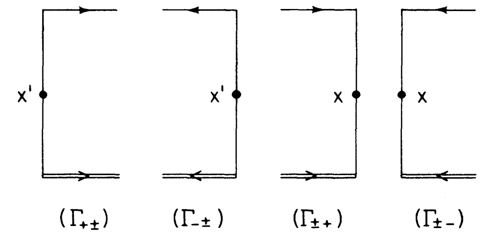

FIG. 3. External vertices.

Note that only (e) and (f) depend on the spatial position x_1 of the vertex.

The external vertices, represented in Fig. 3, can be immediately evaluated to yield:

$$\begin{aligned} &\Gamma_{+,\pm}: \ (2cE)^{-1}\exp[-i\omega x'/(c-u)] \ , \\ &\Gamma_{-,\pm}: \ (2cE)^{-1}\exp[i\omega x'/(c+u)] \ , \\ &\Gamma_{\pm,+}: \ (2cE)^{-1}\exp[i\omega x/(c-u)] \ , \\ &\Gamma_{\pm,-}: \ (2cE)^{-1}\exp[-i\omega x/(c+u)] \ . \end{aligned} \quad (4.2)$$

We are considering here the case $x'<x$. The first subscript in Γ indicates arrow direction at the "departure" vertex x', while the second subscript indicates arrow direction at the "arrival" vertex x.

In an ordered Berezinskii diagram, the integrations have to be carried out over all the internal variables x_i varying in a region of the form

$$-\infty < x_1 \le \cdots \le x_l \le x' \le x_{l+1} \le \cdots \le x_c \le x \le x_{c+1} \le \cdots \le x_r < \infty \ . \quad (4.3)$$

We can assign to the region between neighboring vertices x_i and x_{i+1} the numbers g_i and g_i' which correspond to the number of single and double lines in the interval between x_i and x_{i+1}, respectively. With each vertex x_i we can now associate a pair of numbers: $\Delta g_i = g_i - g_{i-1}$ and $\Delta g_i' = g_i' - g_{i+1}'$. As shown by Berezinskii, diagrams having vertices for which $\Delta g_i \ne \Delta g_i'$ turn out to give small contributions (due to phase cancellations) and can be neglected. The 20 internal vertices we have termed "essential" are those for which $\Delta g_i = \Delta g_i'$.

B. The basic equations: Formulation

Following Berezinskii,[17] we derive the basic equations by first dividing all diagrams into three regions. They will be called left-hand, central, and right-hand parts, and correspond to integrals over the variables $(x_1,\ldots,x_l)$, $(x_{l+1},\ldots,x_c)$, and $(x_{c+1},\ldots,x_r)$, respectively. We will call $\tilde{R}_m(x)$ the total contribution of all the right-hand parts corresponding to $g=g'=2m$ immediately to the right of x, and $Z_{m'm}(x',x)$ the total contribution of the central parts corresponding to $g=g'=2m'+1$ to the left of x' and $g=g'=2m+1$ to the right of x.

A few terms in the diagrammatic equation for $\tilde{R}_m$ are represented in Fig. 4. We must include in the summation all the different allowed positions for the vertices. Counting carefully the various possibilities, we arrive at the following differential equation for the right-hand part:

$$-\frac{d\tilde{R}_m}{dx}=(\mu m^2/4c^2E^2)(k_1k_2k_3k_4\{\exp[2ic\omega x/(c^2-u^2)]\tilde{R}_{m-1}+\exp[-2ic\omega x/(c^2-u^2)]\tilde{R}_{m+1}\}$$
$$+[(k_1^2+k_2^2)(k_3^2+k_4^2)-2(k_1^2k_2^2+k_3^2k_4^2)-(k_1^4+k_2^4+k_3^4+k_4^4)/2]\tilde{R}_m)\ . \tag{4.4}$$

The vertices (a)–(d) contribute to the coefficient of $\tilde{R}_m$ in the right-hand side of Eq. (4.4), while vertices (e) and (f) give the coefficients of $\tilde{R}_{m-1}$ and $\tilde{R}_{m+1}$, respectively. We can neglect the ω dependence in the coefficients of the right-hand side and extract the x dependence writing

$$\tilde{R}_m(x)=\exp[2ic\omega xm/(c^2-u^2)]R_m\ . \tag{4.5}$$

In this way we obtain the following recursion relation for R_m:

$$8ic^3(c^2-u^2)\omega R_m+m\mu E^2(R_{m-1}+R_{m+1}-2R_m)=0\ , \tag{4.6}$$

subject to the boundary condition $R_0=1$.

An analogous procedure leads to the differential equation for the central part

$$\frac{dZ_{m'm}}{dx}=\left[\frac{i\omega}{c-u}\right]Z_{m'm}+\frac{\mu E^2}{4c^2(c^2-u^2)^2}\{m^2\exp[-2ic\omega x/(c^2-u^2)]Z_{m',m-1}$$
$$+(m+1)^2\exp[2ic\omega x/(c^2-u^2)]Z_{m',m+1}-[m^2+(m+1)^2]Z_{m'm}\}\ . \tag{4.7}$$

To construct this equation we have kept m' fixed and used the external vertex (c) in Fig. 3. If we use the external vertex (d), the coefficient of $Z_{m'm}$ in the first term of the right-hand side of Eq. (4.7) is $[-i\omega/(c+u)]$. The difference between both cases turns out to be irrelevant. Note that diagrams containing the (d) vertex correspond to terms of the form $Z_{m'm}R_{m+1}$.

The flow velocity appears in Eqs. (4.6) and (4.7) only in the form (c^2-u^2). It is important to note that all the "anomalous" terms [i.e., those containing the factors $(c+u)^{-4}$ and $(c-u)^{-4}$] cancel exactly. These cancellations imply that the presence of the flow field will cause only minor modifications to the solution of the problem.

Defining $T_m=\frac{1}{2}(R_m+R_{m+1})$ and adding the contributions resulting from the four types of diagrams corresponding to the end-point vertices is Fig. 3, we obtain the following equation for averaged squared Green's function

$$\langle P_{E,\omega}(x,x')\rangle_{\rm av}=(cE)^{-2}\sum_{m=0}^{\infty}\sum_{m'=0}^{\infty}\exp[-2ic\omega(m'x'-mx)/(c^2-u^2)]T_{m'}Z_{m',m}T_m\ . \tag{4.8}$$

The energy density propagator can now be written as

$$\langle P_E(k,\omega)\rangle_{\rm av}=\int dx\,\exp[-ik(x-x')]\langle P_{E,\omega}(x,x')\rangle_{\rm av}$$
$$=(4/\mu)(cE)^{-2}\sum_{m=0}^{\infty}T_m[Q_m(\omega,k)+Q_m(\omega,-k)]\ , \tag{4.9}$$

where

$$Q_m(\omega,k)=(\mu/4)\sum_{m'=0}^{\infty}\int_{x'}^{\infty}dx\,\exp[-ik(x-x')]\exp[2ic\omega(mx-m'x')/(c^2-u^2)]Z_{m'm}(x',x)T_{m'}\ . \tag{4.10}$$

An equation for the function $Q_m(\omega,k)$ can be obtained if we use partial integration to evaluate

$$(\mu/4)\sum_{m'=0}^{\infty}\int_{x'}^{\infty}dx\,\exp[-ik(x-x')]\exp[2ic\omega(mx-m'x')/(c^2-u^2)]\frac{dZ_{m'm}}{dx}(x',x)T_{m'}\ .$$

Defining

$$\kappa=(\mu E^2)^{-1}c^2(c^2-u^2)^2k \ , \tag{4.11a}$$

and

$$W=(\mu E^2)^{-2}8c^3(c^2-u^2)\omega \ , \tag{4.11b}$$

we obtain

$$i\{[m+(c+u)/2c]W-\kappa\}Q_m+(m+1)^2[Q_{m+1}-Q_m]-m^2[Q_m-Q_{m-1}]+E^{-2}c^2(c^2-u^2)^2T_m=0 \ . \tag{4.12}$$

Note that Eqs. (4.6) and (4.12) are coupled through the inhomogeneity in Eq. (4.12).

C. The basic equations: Solutions

We have to solve our equations for R_m and Q_m. To obtain a convenient approximation we first note that the analysis of localization is performed by looking at the long-time behavior of the averaged squared Green's functions. In the Laplace-transform representation we are using, this is equivalent to looking at the $W\to 0$ limit. Therefore, as shown by Berezinskii, only large values of m, are important and the solution to Eq. (4.6) can be seen to have the form[17]

$$R_m(W)=R(p)=2p^{1/2}K_1(2p^{1/2}) \ , \tag{4.13}$$

where K_1 is a modified Bessel function. The solution has been expressed in terms of the continuous variable $p=-imW$.

Under the same conditions we can rewrite Eq. (4.12) as a differential equation,

$$\frac{d}{dp}\left[p^2\frac{dQ(p)}{dp}\right]-(p+i\kappa)Q(p)+2(c/E)^2(c^2-u^2)^2p^{1/2}K_1(2p^{1/2})=0 \ . \tag{4.14}$$

Gogolin and co-workers[20] have investigated in detail the solutions to this equation. They noticed that the substitution $z=2p^{1/2}$ transforms it into an inhomogeneous Bessel equation for the function $zQ(z)$. The solution satisfying the appropriate boundary conditions is

$$Q(z)=4c^2(c^2-u^2)^2(zE^2)^{-1}\times\left[K_\lambda(z)\int_0^z d\xi\xi I_\lambda(\xi)K_1(\xi)+I_\lambda(z)\int_z^\infty d\xi\,\xi K_\lambda(\xi)K_1(\xi)\right] , \tag{4.15}$$

where K_λ and I_λ are modified Bessel functions. This solution can be substituted into the continuous version of Eq. (4.9), yielding

$$\langle P_E(k,\omega)\rangle_{\rm av}=(4i/\mu WE^4)(c^2-u^2)^2[f(\kappa)+f(-\kappa)] \ , \tag{4.16}$$

with

$$f(\kappa)=4\int_0^\infty zdzK_1(z)I_\lambda(z)\int_z^\infty \xi d\xi K_\lambda(\xi)K_1(\xi) \ . \tag{4.17}$$

After doing some juggling with the integration contours,[20] the following result is obtained for the Laplace and Fourier antitransform of Eq. (4.16):

$$\langle P_E(x,x'\,|\,t)\rangle_{\rm av}=\left[\frac{\pi}{8}\right]^2\frac{\mu}{c^5(c^2-u^2)}\times\int_0^\infty d\eta\eta\frac{(1+\eta^2)^2}{(1+\cosh\pi\eta)^2}\sinh(\pi\eta)\times\exp[-\tilde{\mu}(1+\eta^2)\,|\,x-x'\,|\,/4] \ , \tag{4.18}$$

where $\tilde{\mu}=\mu E^2[2c(c^2-u^2)]^{-2}$. At long distances, $\mu\,|\,x-x'\,|\,/4\gg 1$, we can evaluate the η integral approximately:

$$\langle P_E(x,x'\,|\,t)\rangle_{\rm av}\simeq[\mu\pi^{7/2}/1024c^5(c^2-u^2)]\times(l_l/\,|\,x-x'\,|\,)^{3/2}\times\exp[-\,|\,x-x'\,|\,/l_l] \ . \tag{4.19}$$

The new length we have introduced,

$$l_l=4/\tilde{\mu}=16c^2(c^2-u^2)^2[\mu E^2]^{-1} \ , \tag{4.20}$$

is clearly seen to be the localization length for our excitations.

The Berezinskii technique thus tells us that the localization of third-sound waves in a superfield film is not substantially modified by the presence of a uniform flow field. The only predicted effect is a shortening of the localization length of $O((u/c)^2)$. In this connection, it should be pointed out that Gogolin[21] has performed a detailed

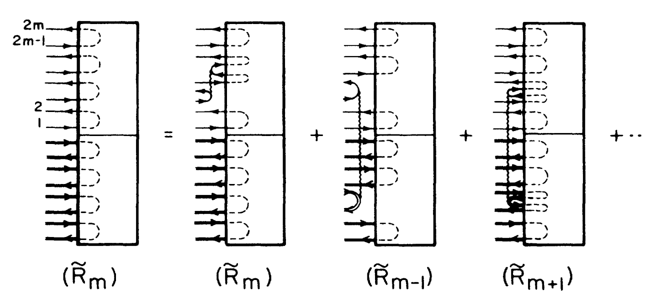

FIG. 4. Diagrams for the construction of the $\tilde{R}_m$ equation. The $\tilde{R}_m$ functions are represented by rectangles and we have drawn only three of the essential vertices: (c), (e), and (f).

analysis of the electronic probability distribution for the one-dimensional localized state. This analysis is immediately applicable to the problem we are considering here. In particular, from Eq. (4.18) it is easy to calculate the first two moments, P_0 and P_1, of $\langle P_E(x,x'\mid t)\rangle_{av}$. The ratio between them is $P_1/P_0=l_l/2$; this indicates that, exactly as in the electronic case, the dimension of the localized state is half the value l_l obtained from the asympotitc form (4.19). This property is not modified by the presence of the uniform flow field.

It should be also noticed that in the limit $u\to 0$, the localization length given by Eq. (4.20) is approximately equal to twice that obtained using the self-consistent diagram approach. This can be seen by taking $u\to 0$ in Sec. III and formulating a self-consistent theory. The localization length l_l is also seen to be equal to $2(l_1+l_2)$, where l_1 and l_2 are forward and backward coherence lengths resulting from the average of the Green's function.

V. DISCUSSION

In Ref. 15 it is shown that for $d\geq 2$ the breaking of time-reversal invariance leads to a substantial modification in the structure of the localized states. Here we have studied the influence of a uniform flow field on the acoustic localization in one-dimensional disordered configurations; specific application has been made to the problem of third-sound waves on inhomogeneous substrates. The perturbative scheme that includes leading-order ladder and maximally crossed Feynman diagrams leads to the conclusion that localization effects are drastically modified by the flow field, since the hydrodynamic pole in the MCD's is cut off by a term proportional to $\mu E^2 u^2$. This conclusion is contradicted by the Berezinskii formalism, which tells us that the flow causes no spectacular changes in the localized states. Since the Berezinskii approach is exact, the perturbative approach must be inadequate.

What is wrong the Feynman diagram formulation in one dimension? It is straightforward to show that there are diagrams other than the MCD's which contribute to one-dimensional localization to the same order in the scatterer density. It is also important to point out that although the singularities in D due to the MCD's are cut off by the flow, there are certainly contributions to D from the ladder diagrams that lead to singular behavior. The Berezinskii technique shows that there are so many diagrammatic contributions to localization in $d=1$, that the fact that the MCD's are cut off is of no importance. A direct proof of this statement via perturbation theory would be extremely difficult; however, its validity is shown by the fact that the Berezinskii technique takes automatically into account all the relevant diagrams, not only the MCD's.

Since the two predictions are so different, it should not be difficult to verify our conclusions experimentally. We believe the van der Waals striped substrate described in Ref. 8 to be the most suitable to reach relatively high values of u/c while keeping the flow field uniform. The "index of refraction" (Ref. 8) substrates could generate a microscopically complicated flow field pattern which might affect localization, especially in the high-frequency regime.

In the absence of flow, there is agreement between the two methods used in this paper. Therefore, the assumption of the self-consistent theory that the MCD's give a truly representative contribution of all the diagrams relevant to localization is correct when time-reversal invariance holds. It only fails when time-reversal invariance is broken, as it is in this system for nonzero u.

Let us also remark the Berezinskii approach can be used in combination with a pseudosphere approximation[22] to make quantitative predictions for systems in which the disorder is described using models other than the Gaussian noise model. The pseudosphere approximation permits to evaluate the correlation factor μ in terms of the physical parameters characterizing the system. This is achieved by equating the imaginary parts of the self-energies corresponding to the Gaussian noise model and the model under consideration. The procedure is complicated in the presence of flow, but can be carried out easily for $u=0$. In the case of the van der Waals model described in Ref. 8 it yields the correct position for the transmission resonances and essentially the correct size for the correlation length. For the index of refraction model[8] the results are similar, except that the type-II resonances are not predicted by the pseudosphere approximation

Finally, elsewhere, one of us[23] has shown that in the presence of a finite electric field the self-consistent perturbation approach is in qualitative agreement with the Berezinskii method in $d=1$. This can be understood by noting that a finite electric field modifies both the Cooperon (MCD's) and ladder contributions in the same way. As a consequence, the self-consistent approach is representative of the underlying physics, and qualitative agreement should be expected.

ACKNOWLEDGMENTS

We thank D. Belitz for critically reading the manuscript and J. Machta for many stimulating discussions. This work was supported by the National Science Foundation under Grants Nos. DMR 83-51473, DMR 83-09449, and DMR 83-17442.

APPENDIX

In this Appendix we use the Berezinskii formalism to describe the attenuation of the localized excitations in a weakly dissipative superfluid film. The mechanisms for the dissipation of third sound in thin superfluid films are not completely understood.[24] Keeping this in mind, we have chosen to analyze the effects of two reasonable dissipation models in order to obtain some qualitative information about what can be expected for a real system. We start by setting, for simplicity, $u=0$ in our modified wave equation, Eq. (2.3). The dissipation is introduced through the addition of an extra term $\Lambda(\rho)$. The two forms we have selected for this extra term are

$$\Lambda(\rho)=-\Gamma_1\frac{\partial^2}{\partial x^2}\frac{\partial\rho}{\partial t}\ . \qquad (A1)$$

This form corresponds to regular hydrodynamic dissipation.

$$\Lambda(\rho)=\Gamma_2\frac{\partial\rho}{\partial t}\ . \tag{A2}$$

This form was used to describe the damping of third sound in thin helium films on vitreous quartz.[25] A typical experimental value is $\Gamma_2 \sim 1\ \text{sec}^{-1}$. It was also used as a phenomenological description of damping in coupled-layered films.[26]

The analysis proceeds in the same way as in Sec. IV. The main difference is that the direction of the arrows becomes irrelevant and thus we have to consider only six essential vertices. The solution is simplified neglecting terms of $O(\Gamma_1^2 E^2/c^4)$ [or of $O(\Gamma_2^2/E^2)$]. This should be an excellent approximation for the very weakly dissipative systems and the intermediate frequencies (2×10^3Hz $\lesssim E \lesssim 2\times10^5$ Hz) of interest.

Instead of Eq. (4.19), we obtain,

$$\langle P_E(x,x'\mid t)\rangle_{\text{av}} \sim e^{-\Gamma_1(E/c)^2 t} e^{-|x-x'|/l_l}\ , \tag{A3}$$

and

$$\langle P_E(x,x'\mid t)\rangle_{\text{av}} \sim e^{-\Gamma_2 t} e^{-|x-x'|/l_l}\ . \tag{A4}$$

The localization length is given by

$$l_l = 16c^6/\mu E^2\ . \tag{A5}$$

These results are valid for times $t \gtrsim \Gamma_1^{-1}(c/E)^2$ and $t \gtrsim \Gamma_2^{-1}$, respectively.

We note that the temporal decay of the localized excitations has the same form as the attenuation of the propagating modes. This is not surprising since the damping processes have been associated with the film itself or with its interaction with the substrate but not with the scatterers. A different result may be obtained when the dissipation is directly related to the scattering centers. This would be the case if an internal mode is excited in the scatterer itself or if the scatterers are subject to a slow random motion (this would occur if they form a suspension in a fluid).

Equation (A5) tells us that the localization length itself is not affected by the damping. We also note that, due to the assumed weakness of the damping, we refer to the excitations as being "localized," even if they will have died out at long times.

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