Proof of the nonexistence of (formal) phase transitions in polaron systems. II

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In recent literature it was frequently claimed that free polarons should undergo a (formal) phase transition from a mobile to a localized state, if the electron-phonon coupling parameter α exceeds some critical value α_c. For $\alpha=\alpha_c$, the formal free energy should be nonanalytic in α. In this paper we prove that for nonoptical polarons no such transition exists for finite temperatures. Our results can be generalized to the case of lower spatial dimensions as well as to polarons in homogeneous electromagnetic fields. We include some comments on related problems; in particular, the partition function of an electron moving in a Gaussian random potential is considered.

I. INTRODUCTION AND STATEMENT OF THE PROBLEM

In a recent paper[1] we discussed the question of a possible (formal) phase transition in a system of optical polarons. The final answer was negative. To prove this, we considered the diagonal element of the reduced density matrix and used functional-integration techniques to calculate the (formal) free energy.

The aim of this paper is to generalize the previous results to nonoptical polarons and mixed couplings, lower-dimensional polarons, and anisotropic electron-phonon interactions. Finally, we turn to polarons in an external electric or magnetic field and to some related problems. Interesting enough, all these cases can fairly be reduced to our treatment of optical polarons in Ref. 1.

If an electron couples to several phonon branches (branching index j, $1\leq j\leq r$, wave vector $\mathbf{k}$, dispersion $\omega_{\mathbf{k}j}$) with coupling functions $g_{\mathbf{k}j}=\sqrt{\alpha_j}\tilde{g}_{\mathbf{k}j}$, where α_j is the dimensionless electron-phonon coupling parameter, the partition function Z and the formal free energy F depend on $\boldsymbol{\alpha}:=(\alpha_j)$ and the inverse formal temperature β.

Using the techniques discussed in Ref. 1 we deduce

$$Z(\boldsymbol{\alpha},\beta)=1+\sum_{n=1}^{\infty}\sum_{j_1,\ldots,j_n=1}^{r}\alpha_{j_1}\cdots\alpha_{j_n}Z_n(j_1,\ldots,j_n,\beta)\,, \tag{1}$$

where

$$Z_n(j_1,\ldots,j_n,\beta):=\frac{1}{n!}\frac{1}{(2\pi)^{3n}}\times\int_0^{\beta\hbar\omega}\cdots\int_0^{\beta\hbar\omega}d\tau_1\cdots d\tau_{2n}\int\cdots\int d^3k_1\cdots d^3k_n\times\exp\left[-\beta\hbar\omega\sum_{j,l=1}^{n}A_{jl}\left[\frac{\tau_1}{\beta\hbar\omega},\ldots,\frac{\tau_{2n}}{\beta\hbar\omega}\right]\mathbf{k}_j\mathbf{k}_l\right]\times\prod_{i=1}^{n}|\tilde{g}_{\mathbf{k}_ij_i}|^2\overline{G}(\hbar\omega_{\mathbf{k}_ij_i},\tau_{2i-1}-\tau_{2i})\,. \tag{2}$$

In Eq. (2) all variables are dimensionless. We have chosen $\hbar\omega$ and $\sqrt{\hbar/m\omega}$ as units of energy and length, ω being an arbitrary (e.g., LO phonon) frequency. As for A and $\overline{G}$, see Ref. 1. Direct inspection of (1) and (2) shows that $Z(\boldsymbol{\alpha},\beta)$ can be represented as a power series in the coupling parameters α_j, the coefficients being positive functions in β (for $0<\beta$). Inserting familiar types of couplings $\tilde{g}_{\mathbf{k}j}$ into (2), one may easily prove that $Z_n(j_1,\ldots,j_n,\beta)$ defines an analytic function in β for $0<\text{Re}\beta<\infty$. Consequently, the following statement is true:

Statement 1. If the power series (1) converges for all complex α_j, $Z(\boldsymbol{\alpha},\beta)$ is an holomorphic function for all complex α_j and β, $0<\text{Re}\beta<\infty$.

As $Z(\boldsymbol{\alpha},\beta)$ is positive for real α_j,β with $0\le\alpha_j$, $0\le\beta<\infty$, we deduce under the condition of Statement 1:

Statement 2. $F(\boldsymbol{\alpha},\beta)-F(\mathbf{0},\beta)$ is a real analytic function in α_j,β in the specified domain $0\le\alpha_j$, $0<\beta<\infty$.

In the subsequent sections we prove under various circumstances that (1) does converge for all complex α_j.

II. NONOPTICAL POLARONS AND MIXED COUPLINGS

Let the couplings and the dispersions fulfill the following conditions: For all $j=1,..,r$ there exist positive constants Q, C, and R, depending merely on β, such that

$$\int_0^{\beta\hbar\omega}\int_0^{\beta\hbar\omega} d\tau\, d\tau' \int d^3k \; |\tilde{g}_{\mathbf{k}j}|^2 \overline{G}(\hbar\omega_{\mathbf{k}j},\tau-\tau') \exp\left[\left(\frac{|\tau-\tau'|}{2}-\frac{(\tau-\tau')^2}{2\beta\hbar\omega}\right)k^2\right]\Theta(Q-k)\le R<\infty \tag{3}$$

and

$$k^2\,|\tilde{g}_{\mathbf{k}j}|^2\overline{G}(\hbar\omega_{\mathbf{k}j},\tau-\tau')\le C<\infty \tag{4}$$

for all $k>Q$ and $\tau,\tau'\in[0,\beta\hbar\omega]$. Under these circumstances, we shall prove that (1) converges for all α_j.

Splitting the domain of the $\mathbf{k}$ integrations, we obtain from (2):

$$\begin{aligned} Z_n(j_1,\ldots,j_n,\beta)=&\frac{1}{n!}\frac{1}{(2\pi)^{3n}}\\ &\times\int_0^{\beta\hbar\omega}\cdots\int_0^{\beta\hbar\omega}d\tau_1\cdots d\tau_{2n}\sum_{s=0}^{n}\binom{n}{s}\int\cdots\int d^3k_1\cdots d^3k_n\Theta(k_1-Q)\cdots\Theta(K_{n-s}-Q)\\ &\times\Theta(Q-K_{n-s+1})\cdots\Theta(Q-K_n)\\ &\times\exp\left[-\beta\hbar\omega\sum_{j,l=1}^{n}A_{jl}\mathbf{k}_j\mathbf{k}_l\right]\\ &\times\prod_{i=1}^{n}|\tilde{g}_{k_ij_i}|^2\overline{G}(\hbar\omega_{\mathbf{k}_i},\tau_{2i-1}-\tau_{2i})\ . \end{aligned}\tag{5}$$

We have to add the prescription that the product $\Theta(k_1-Q)\cdots\Theta(k_{n-s}-Q)$ has to be omitted for $s=n$, whereas $\Theta(Q-k_{n-s+1})\cdots\Theta(Q-k_n)$ has to be omitted for $s=0$. Estimating successively the maximum in the exponential, one finds

$$-\sum_{j,l=1}^{n}A_{jl}\mathbf{k}_j\cdot\mathbf{k}_l\le-\sum_{j,l=1}^{n-s}A_{jl}\mathbf{k}_j\cdot\mathbf{k}_l+\sum_{j=n-s+1}^{n}A_{jj}k_j^2\ . \tag{6}$$

A_{jj} is explicitly given by (see Ref. 1)

$$A_{jj}\equiv A_{jj}\left(\frac{\tau_1}{\beta\hbar\omega},\ldots,\frac{\tau_{2n}}{\beta\hbar\omega}\right)=-\frac{(\tau_{2j}-\tau_{2j-1})^2}{2(\beta\hbar\omega)^2}+\frac{1}{2\beta\hbar\omega}|\tau_{2j-1}-\tau_{2j}|\ . \tag{7}$$

To get an upper bound on $Z_n(j_1,\ldots,j_n,\beta)$ we insert (6) and (7) into (5) and, after having employed our conditions (3) and (4), we omit the first $n-s$ Θ functions. This yields

$$\begin{aligned} Z_n(j_1,\ldots,j_n,\beta)\le&\frac{1}{n!}\frac{1}{(2\pi)^{3n}}\sum_{s=0}^{n}\binom{n}{s}\int_0^{\beta\hbar\omega}\cdots\int_0^{\beta\hbar\omega}d\tau_1\cdots d\tau_{2n-2s}\int\cdots\int\frac{d^3k_1}{k_1^2}\cdots\frac{d^3k_{n-s}}{k_{n-2}^2}C^{n-s}R^s\\ &\times\exp\left[-\beta\hbar\omega\sum_{j,l=1}^{n-s}A_{jl}\mathbf{k}_j\cdot\mathbf{k}_l\right]\ . \end{aligned}\tag{8}$$

Within the s summation, the term for $n=s$ has to be replaced by R^n. The remaining integrations are well known from the case of optical polarons. Applying our estimations from Ref. 1 we finally obtain

$$Z_n(j_1,\ldots,j_n,\beta)\le\frac{\sqrt[4]{n}}{\sqrt{n!}}\frac{1}{(2\pi)^{3n}}[220(\beta\hbar\omega)^{3/2}C+R]^n\ . \tag{9}$$

Therefore the power series (1) is clearly convergent for all α_j.

The familiar couplings and dispersions satisfy the conditions (3) and (4). We mention optical polarons, where

$$\omega_{\mathbf{k}} \geq \omega > 0 \ , \tag{10}$$

$$\tilde{g}_{\tilde{k}} \sim \frac{1}{k} \tag{11}$$

holds true. In the case of acoustical (piezoelectric) polarons, we have

$$\omega_{\mathbf{k}} = ck \ , \tag{12}$$

$$\tilde{g}_{\mathbf{k}} \sim k^{-1/2}\Theta(k_0 - k) \ . \tag{13}$$

In (13), k_0 is a cutoff wave number (see Whitfield, Gerstner, and Tharmalingam[2]). In the theory of deformation-potential scattering, we find

$$\omega_{\mathbf{k}} \geq \omega > 0 \ , \tag{14}$$

$$\tilde{g}_{\mathbf{k}} \sim \Theta(k_0 - k) \ , \tag{15}$$

in the optical case (see Harrison[3]), whereas

$$\tilde{g}_{\mathbf{k}} \sim \sqrt{k}\,\Theta(k_0 - k) \tag{16}$$

is proposed for the acoustical case $\omega_{\mathbf{k}} = ck$ (Whitfield and Shaw[4]).

We may summarize our results of this section as follows: If an electron interacts simultaneously with an arbitrary number of phonon branches which are of the type (10)–(16), the free energy is real analytic in the coupling parameters α_j and the inverse temperature β, $0 < \beta < \infty$.

We finally add some remarks on previous work: To the best of our knowledge the analytical properties of the ground-state energy $E_0(\boldsymbol{\alpha})$ are not clear up to now, if the dispersion $\omega_{\mathbf{k}j}$ has a zero at a certain value of $\mathbf{k}$ and j. By variational calculations Peeters and Devreese,[5] Shoji and Tokuda,[6] Toyozawa,[7] Sumi and Toyozawa,[8] and Hashimoto and Tokuda[9] find nonanalytical behavior of the ground-state energy in several polaron models involving acoustical phonons. On the other hand, Spohn[10] has recently examined the localization transition of polarons. Taking a short-range coupling, he proves that the polaron is delocalized for any coupling parameter α.

III. LOWER-DIMENSIONAL POLARONS

In recent years, considerable attention was paid to the case of lower-dimensional polarons, because polaron effects have been observed in low-dimensional systems like, e.g., p-type InSb MOS (metal-oxide-semiconductor) structures (see Horst, Merkt, and Kotthaus[11]). Another example concerns the interaction of an electron with the surface modes of a thin liquid helium film, which was discussed by Jackson and Platzman.[12] In d dimensions the form of the Fröhlich Hamiltonian remains as it is in three dimensions except that now all vectors and vector operators are d dimensional. Of course the functional dependence of $\tilde{g}_{\mathbf{k}j}$ from $\mathbf{k}$ changes. We shall concentrate on dispersionless longitudinal optical phonons, where

$$\omega_{\mathbf{k}} = \omega > 0 \tag{17}$$

and assume the coupling to be

$$|g_{\mathbf{k}}|^2 = A_d k^{1-d} \tag{18}$$

with finite A_d for $d \geq 1$. We note that Peeters, Xiaoguang, and Devreese[13] have tried to prove Eq. (18) under severe assumptions concerning the electron-phonon interaction. In fact, they found (18) for $d > 1$, whereas A_1 did not exist. We are not going to rediscuss this point here but note that the assumptions in Ref. 13 may be questioned. Our attitude is here to use Eq. (18) as a model coupling for all $d \geq 1$.

All steps of our proof in Ref. 1 can be easily generalized to d-dimensional Fröhlich polarons, estimating the $\mathbf{k}$ integrations by introducing d-dimensional polar coordinates. In the case $d = 2$ one parametrizes the coupling according to Sak[14] as follows:

$$|g_{\mathbf{k}}|^2 = \frac{\sqrt{2}\pi\alpha}{k} \ . \tag{19}$$

For the formal partition function we obtain

$$Z(\alpha,\beta) \leq 1 + \frac{\pi^{3/2}(\beta\hbar\omega)^{3/2} I_0(\beta\hbar\omega/2)\alpha}{4\sinh(\beta\hbar\omega/2)} + \sum_{n=2}^{\infty} 0.427\frac{\sqrt[4]{n}}{\sqrt{n!}}[4\times 3^{-3/2}\pi^{-1/2}\Gamma^2(\tfrac{1}{4})(\beta\hbar\omega)^{3/2}\coth(\beta\hbar\omega/2)\alpha]^n \ , \tag{20}$$

where $I_0(x)$ is a modified Bessel function of first kind. In (20), the first coefficient is exact. In the case $d = 1$ we choose

$$|g_{\mathbf{k}}|^2 = \alpha \ . \tag{21}$$

Then we can deduce

$$Z(\alpha,\beta) \leq 1 + \frac{\pi^{1/2}(\beta\hbar\omega)^{3/2} I_0\left[\frac{\beta\hbar\omega}{2}\right]\alpha}{2^{3/2}\sinh\left[\frac{\beta\hbar\omega}{2}\right]} + \sum_{n=2}^{\infty} 0.854\frac{\sqrt[4]{n}}{\sqrt{n!}}\left[4\times 3^{-3/2}(\beta\hbar\omega)^{3/2}\coth\left[\frac{\beta\hbar\omega}{2}\right]\alpha\right]^n \ , \tag{22}$$

where the first coefficient is exact. The generalization to nonoptical polarons and mixed couplings is straightforward.

Summarizing, we can repeat our conclusion, that the free energy $F(\boldsymbol{\alpha},\beta) - F(0,\beta)$ is real analytic in α_j,β for $0 \leq \alpha_j$, $0 < \beta < \infty$.

As far as previous work is concerned, we note that the proof of Fröhlich,[15] which guarantees analyticity of the ground-state energy in the coupling parameter in the case $\omega_{\mathbf{k}} \geq \omega > 0$, is valid for arbitrary spatial dimensions. On the other hand, Farias, Studart, and Hipolito[16] as well as Bodas and Hipolito[17] claim that formal phase transitions exist, considering two-dimensional optical polarons. Their results are artifacts of the (variational) approximation. We add that in a recent paper, Xiaoguang, Devreese, and Peeters[18] have already questioned the conclusions in.[16,17] Matsuura[19] finds a nonanalyticity in the ground-state energy in a case of mixed lower-dimensional couplings. The work of Spohn[10] holds for arbitrary spatial dimension, too.

IV. ANISOTROPIC COUPLINGS

In systems with anisotropic energy bands the Fröhlich Hamiltonian must be modified. We restrict ourselves to dispersionless optical polarons, because the results can readily be generalized to non-optical cases. According to Gerlach and Schliffke[20] the formal partition function is given by

$$Z(\alpha,\beta) = \langle \exp(-S_I) \rangle_{S_0} \; , \tag{23}$$

where, now,

$$S_I[\mathbf{R}] = -\frac{\alpha}{\sqrt{2}} \int_0^{\beta\hbar\omega} d\tau \int_0^{\beta\hbar\omega} d\tau' \frac{\bar{G}(\hbar\omega,\tau-\tau')}{|\sqrt{m}\,[A][M]^{-1/2}[\mathbf{R}(\tau)-\mathbf{R}(\tau')]|} \; . \tag{24}$$

In (24), the strictly positive 3×3 matrix $[M]$ describes the anisotropic energy bands, m is a mean value of the eigenvalues of M. $[A]$ is a nondegenerate symmetric 3×3 matrix, which represents anisotropic electron-phonon coupling. Consequently, there exists a positive $u<\infty$ such that

$$S_I[\mathbf{R}] \geq -\frac{\alpha u}{\sqrt{2}} \int_0^{\beta\hbar\omega} d\tau \int_0^{\beta\hbar\omega} d\tau' \frac{\bar{G}(\hbar\omega,\tau-\tau')}{|\mathbf{R}(\tau)-\mathbf{R}(\tau')|} \; . \tag{25}$$

Clearly, the anisotropic polaron problem is reduced by (25) to the isotropic case. We remark that Fröhlich's analyticity proof for the ground state also applies to this case.

V. HOMOGENEOUS MAGNETIC FIELD

If a polaron is exposed to a homogeneous magnetic field $\mathbf{B}=(0,0,B)$, the Fröhlich Hamiltonian has to be generalized in a standard way. The same holds true for the functional-integral formulation of the renormalized formal free energy $F(\boldsymbol{\alpha},\beta,B)-F(\mathbf{0},\beta,B)$. As for details we refer to Peeters and Devreese.[21] $F(\boldsymbol{\alpha},\beta,B)-F(\mathbf{0},\beta,B)$ in turn can be derived from a partition function Z where now

$$Z(\boldsymbol{\alpha},\beta,B) = \langle \exp(-S_I-S_m) \rangle_{S_0} / \langle \exp(-S_m) \rangle_{S_0} \; . \tag{26}$$

The only new quantity, generated by the $\mathbf{B}$ field, is

$$S_m[\mathbf{R}] := -i\gamma \int_0^{\beta\hbar\omega} d\tau\, R^1(\tau)\dot{R}^2(\tau) \; , \tag{27}$$

where we have used the Landau gauge $\mathbf{A}(x,y,z)=[0,Bx,0)]$. In (27), we defined

$$\gamma := \frac{|e|\,|B|}{m\omega} = \frac{\omega_c}{\omega} \; . \tag{28}$$

ω_c is the cyclotron frequency, R^1, R^2 are the first two components of the three-dimensional vector $\mathbf{R}$, and e is the elementary charge. Since the action S_m is purely imaginary, we obtain immediately

$$Z(\boldsymbol{\alpha},\beta,B) \leq \frac{\langle \exp(-S_I) \rangle_{S_0}}{|\langle \exp(-S_m) \rangle_{S_0}|} = \frac{\sinh(\beta\hbar\omega_c/2)}{\beta\hbar\omega_c/2} Z(\boldsymbol{\alpha},\beta,0) < \infty \; . \tag{29}$$

Note that (29) gives a lower bound on the formal free energy. Combining Eq. (29) and our results for $Z(\boldsymbol{\alpha},\beta,0)$, derived in Sec. II, the dominated convergence theorem assures us that $Z(\boldsymbol{\alpha},\beta,B)$ can be represented as a power series in α_j, which converges for all complex α_j—provided we insert the familiar couplings. In particular, we find

$$Z(\boldsymbol{\alpha},\beta,B) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\langle (-S_I)^n e^{-S_m} \rangle_{S_0}}{\langle e^{-S_m} \rangle_{S_0}} = 1 + \sum_{n=1}^{\infty} \sum_{j_1,\ldots,j_n=1}^{r} \alpha_{j_1} \cdots \alpha_{j_n} Z_n(j_i,\ldots,j_n,\beta,B) \; , \tag{30}$$

where $Z_n(j_1,\ldots,j_n,\beta,B)$ can be calculated according to the prescription in Ref. 22. We obtain

$$Z_n(j_1,\ldots,j_n,\beta,B)=\frac{1}{n!}\frac{1}{(2\pi)^{3n}}\times\int_0^{\beta\hbar\omega}\cdots\int_0^{\beta\hbar\omega}d\tau_1\cdots d\tau_{2n}\int\cdots\int d^3k_1\cdots d^3k_n\times\prod_{s=1}^{n}|\tilde{g}_{\mathbf{k}_s j_s}|^2\overline{G}(\hbar\omega_{\mathbf{k}_s j_s},\tau_{2s-1}-\tau_{2s})\times\exp\left[-\sum_{i=1}^{3}\sum_{j,l=1}^{r}B_{jl}^{i}(\tau_1,\ldots,\tau_{2n})k_j^ik_l^i\right]\times\cos\left[\sum_{j,l=1}^{r}D_{jl}(\tau_1,\ldots,\tau_{2n})k_j^1k_l^2\right]. \quad (31)$$

In (31), we introduced

$$B_{jl}^1(\tau_1,\ldots,\tau_{2n}):=B_{jl}^2(\tau_1,\ldots,\tau_{2n}):=\frac{1}{2\gamma}[-\overline{G}(\hbar\omega_c,\tau_{2j-1}-\tau_{2l})-\overline{G}(\hbar\omega_c,\tau_{2j}-\tau_{2l-1})+\overline{G}(\hbar\omega_c,\tau_{2j-1}-\tau_{2l-1})+\overline{G}(\hbar\omega_c,\tau_{2j}-\tau_{2l})]\ , \quad (32)$$

$$B_{jl}^3(\tau_1,\ldots,\tau_{2n}):=\beta\hbar\omega A_{jl}\left[\frac{\tau_1}{\beta\hbar\omega},\ldots,\frac{\tau_{2n}}{\beta\hbar\omega}\right]. \quad (33)$$

Furthermore, D_{jl} is given by (32), if we replace $\overline{G}$ by a function D, defined as

$$D(\gamma,\tau):=\left[1+\frac{\sinh[\gamma(|\tau|-\beta\hbar\omega/2)]}{\sinh(\gamma\beta\hbar\omega/2)}\right]\mathrm{sgn}(\tau)\ , \quad (34)$$

Eq. (31) should be compared to Eq. (2), being the limiting value of (31) for $\gamma\to 0$. The coefficients $Z_n(j_1,\ldots,j_n,\beta,B)$ are pretty complicated, but clearly analytic in β and B in some complex surrounding of the positive β and B axis. Since Z is a partition function, it is clearly positive. Therefore we find again that $F(\boldsymbol{\alpha},\beta,B)-F(\mathbf{0},\beta,B)$ is real analytic in α_j,β,B for $0\le\alpha_j$, $0<\beta<\infty$, and $0<B$.

Finally, we comment on the previous literature. To the best of our knowledge, the analytical properties of the ground-state energy $E_0(\boldsymbol{\alpha},B)$ are not clear at the moment. (See Note added in proof.) In a series of papers, Devreese and Peeters (see, e.g., Ref. 21) have discussed variational bounds on $F(\boldsymbol{\alpha},\beta,B)$. For optical polarons, they do find nonanalytical behavior of these bounds and conclude that, e.g., the mass stripping is discontinuous. They indicate themselves that this might be an artifact of the approximation.

VI. HOMOGENEOUS ELECTRIC FIELD

We proceed along similar lines as in Sec. V. It is straightforward to generalize the Fröhlich Hamiltonian and the function-integral formulation of the formal free energy. The electric field $\mathbf{E}$ is taken to be parallel to the 1 axis, $\mathbf{E}=(E,0,0)$. To avoid technical difficulties—the generalized Fröhlich Hamiltonian is unbounded, if the volume becomes infinite—we consider again the difference of the formal free energies $F(\boldsymbol{\alpha},\beta,E)-F(\mathbf{0},\beta,E)$, firstly for finite volume V. However, this difference is well defined even for $V\to\infty$. It can be derived from a partition function $Z(\boldsymbol{\alpha},\beta,E)$, where now

$$Z(\boldsymbol{\alpha},\beta,E):=\frac{\langle\exp(-S_I-S_e)\rangle_{S_0}}{\langle\exp(-S_e)\rangle_{S_0}}. \quad (35)$$

In (35),

$$S_e[\mathbf{R}]:=\overline{\gamma}\int_0^{\beta\hbar\omega}d\tau R^1(\tau)\ , \quad (36)$$

$$\overline{\gamma}:=\frac{|e|E}{\hbar\omega}\sqrt{\hbar/m\omega}\ . \quad (37)$$

Note that—in contrast to (27)—S_e is manifestly real. Inserting the explicit expression for S_I, we find

$$Z(\boldsymbol{\alpha},\beta,E)=1+\sum_{n=1}^{\infty}\sum_{j_1,\ldots,j_n=1}^{r}\alpha_{j_1},\ldots,\alpha_{j_n}Z_n(j_1,\ldots,j_n,\beta,E)\ , \quad (38)$$

where

$$Z_n(j_1,\ldots,j_n,\beta,E)=\frac{1}{n!}\frac{1}{(2\pi)^{3n}}\times\int_0^{\beta\hbar\omega}\cdots\int_0^{\beta\hbar\omega}d\tau_1\cdots d\tau_{2n}\int\cdots\int d^3k_1\cdots d^3k_n$$
$$\times\prod_{s=1}^{n}|\tilde{g}_{\mathbf{k}_s j_s}|^2\overline{G}(\hbar\omega_{\mathbf{k}_s j_s},\tau_{2s-1}-\tau_{2s})$$
$$\times\exp\left[-\beta\hbar\omega\sum_{j,l=1}^{n}A_{jl}\left[\frac{\tau_1}{\beta\hbar\omega},\ldots,\frac{\tau_{2n}}{\beta\hbar\omega}\right]\mathbf{k}_j\cdot\mathbf{k}_l\right]$$
$$\times\cos\left[\frac{\gamma}{2}\beta\hbar\omega\sum_{j=1}^{n}k_j^1\left[\frac{\tau_{2j-1}^2-\tau_{2j}^2}{\beta\hbar\omega}-\tau_{2j-1}+\tau_{2j}\right]\right]. \quad (39)$$

The only influences of the electric field are oscillating terms in (39). Hence we obtain

$$Z(\boldsymbol{\alpha},\beta,E)\le Z(\boldsymbol{\alpha},\beta,0)\ . \quad (40)$$

Recalling our results from Sec. II for $Z(\boldsymbol{\alpha},\beta,0)$, we conclude from (40) that the power series (38) converges for all complex α_j (provided we insert the familiar couplings $\tilde{g}_{\mathbf{k}j}$). Moreover, Z_n $(j_1,\ldots,j_n,\beta,E)$ is clearly analytic in β and E, if $0<\mathrm{Re}\beta<\infty$. Because of the positivity of the partition function $Z(\boldsymbol{\alpha},\beta,E)$, we arrive at the familiar result that $F(\boldsymbol{\alpha},\beta,E)-F(\mathbf{0},\beta,E)$ is real analytic for all α_j,β,E with $0\le\alpha_j$, $0<\beta<\infty$.

We mention that it was proposed to define a polaron mass by the response to an external electric field (see Arisawa and Saitoh[23]). As this mass is expressible as a second derivative of the free energy with respect to E, it cannot be discontinuous as a function of α_j or E. This question was also studied numerically by Jackson and Platzman.[24]

VII. RELATED PROBLEMS

Firstly, we mention that all our discussions are formally transferable to the exciton-phonon coupling, i.e., the formal free energy of an exciton in a phonon field is real analytic in all parameters like mass ratios, screening parameters, and coupling constants, but we shall discuss this assertion extensively elsewhere.

In the second place, we remark that a number of other physical problems can be cast into a similar path integral form (see Khandekar and Lawande[25] for a review). These are the propagation of waves in random media, the excluded volume problem in a polymer chain, and the density of states in disordered solids. To apply our methods to these problems we take the last subject as example.

The density of states of an electron moving in a random potential, subject to Gaussian statistics, may be found as the inverse Laplace transform of a partition function Z, which depends on an inverse temperature β and is given by

$$Z(\beta)=L^{\alpha}\left\langle\exp\left[\int_0^{\beta}d\tau\int_0^{\beta}d\tau'\ W[\mathbf{R}(\tau)-\mathbf{R}(\tau')]\right]\right\rangle_{S_0}, \quad (41)$$

where L is the system length, d the spatial dimension, and

$$W(\mathbf{R}):=\frac{1}{2}\int d^dx\ v(\mathbf{R}+\mathbf{x})v(\mathbf{x})=:\int d^dk\ f(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{R}}\ , \quad (42)$$

v being the scattering potential. Because of (42) and (41) this problem can be treated in the same manner as the polaron functional integral. Let W be positive. Then we have proved that $Z(\beta)$ is analytic in β $(0<\beta<\infty)$, if there exists a $Q\le\infty$ such that

$$\int d^dk\ |f(\mathbf{k})|\,\Theta(Q-k)<\infty \quad (43)$$

and

$$k^{d-1}|f(\mathbf{k})|\le C<\infty \quad \text{for all } k>Q\ . \quad (44)$$

The conditions (43) and (44) are fulfilled by familiar potentials v like Gaussian scattering and screened Coulomb potentials in three dimensions (see Samathiyakanit[26]) or Gaussian white noise in one dimension $[W(x)\sim\delta(x)]$. Gross[27] discusses the question, whether $Z(\beta)$ is an analytic function or not, but this is now clarified by the statement above.

Note added in proof. One of us (H.L.) has recently shown that also $E_0(\boldsymbol{\alpha},\beta)$ is real analytic for $0\le\alpha_j$, $0\le\beta$. A corresponding paper has been submitted for publication in J. Math. Phys.

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