Electron scattering by electron holograms: The physical interpretation of the Coulomb anomaly in disordered electron systems

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Two-dimensional disordered conductors show a resistance anomaly at low temperatures. In addition to weak localization there is also a contribution due to the Coulomb interaction which was discovered by Altshuler *et al.* In this paper the Hartree contribution is translated into transparent physics. The underlying physics is rather interesting and has some similarity with optical holography. If one considers the propagation of an electron on a closed loop (after many scattering events) then it interferes with itself and forms a charge pattern which contains all information about the impurities (and corresponds to a hologram). A conduction electron which is scattered by the same impurities generally experiences the same phase shifts. However, if it is scattered by the "charge hologram" the dephasing is readjusted, and one obtains a constructive interference.

I. INTRODUCTION

At low temperature the resistance of two-dimensional disordered electronic systems such as thin films show interesting deviations from the Boltzmann theory, in particular a logarithmic temperature dependence. These anomalies are due to electron interference. For normal metals one distinguishes essentially two contributions: quantum interference—generally called weak localization—and the Coulomb anomaly. The first part was discovered theoretically by Abrahams *et al.*,[1] and Gorkov *et al.*[2] Altshuler *et al.*[3] interpreted weak localization as the interference of two partial electron waves propagating on a closed loop in opposite directions. Independently the author[4] derived from the corresponding Kubo graph that a plane wave (not necessarily an electron wave), scattered by impurities builds up an echo in the backwards direction and that the application of a magnetic field allows time-of-flight experiments with the conduction electrons.

The Coulomb anomaly was discovered by Altshuler and Aronov[5] in three-dimensional disordered conductors and extended to two dimensions by Altshuler *et al.*[6] and Fukuyama.[7] Since then, the effect of the Coulomb interaction in disordered electron systems has been intensively studied and reviewed by several authors.[8–10] Although the evaluation of the Kubo diagrams is straightforward within the formalism of the Green's functions, and some papers give additional insight into its physics,[11–13] it is never discussed in terms of graphical physics. (By graphical, I mean "physically transparent" corresponding to the German word "anschaulich" which, when translated into English, means descriptive or graphic.) In this paper we want to close the gap between formalism and physical understanding. Of course, we have to start with some basic properties of the electrons, i.e., their wave character, the Pauli principle, and simple results of perturbation theory in particular that a potential U causes a transition between states. For the interpretation of the physics it is absolutely essential not to oversimplify the physical picture, but to perform a one-to-one translation of the diagrams into propagating electron waves. Within this process many questions arise which are either automatically solved by the theoretical formalism or are completely ignored because they lie beyond the formalism. The solution to these questions can be quite demanding and the physical interpretation which is presented here requires more space and discussion than the original theoretical evaluation. Therefore one might ask whether it is worth the effort. I would like to meet this challenge with the following arguments: (i) It is always desirable to understand the underlying physics of what one is doing. An interesting point is that one discovers behind the diagrams rather stimulating physics which gives considerable insight into the dynamics of the processes which is generally completely hidden in the diagrams. (ii) Because of the new insight into the problem one may be capable of making new predictions. (iii) One may transfer the physics from one field to another when one understands the inner connections.

As an example of the third point I would like to refer to the interpretation of weak localization as an echo of a plane wave scattered by impurities. As a consequence the scattering of laser light by random scatterers has been investigated during the last year[14,15] showing this echo in the backwards direction and promising to become a new field of research. I expect that the physical interpretation of the Hartree diagram describes nonlinear optical effects in disordered optical media.

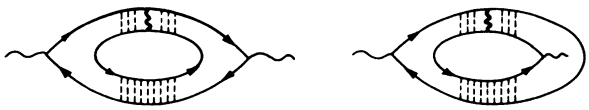

FIG. 1. Kubo diagrams which yield the conductance correction for the Hartree term in the presence of the Coulomb interaction.

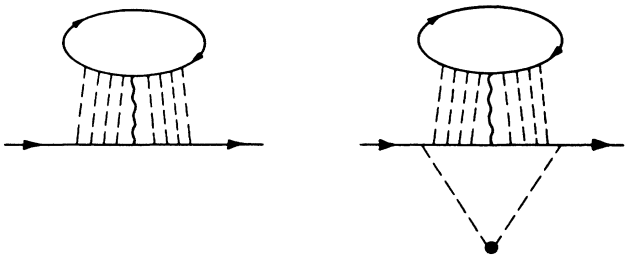

FIG. 2. The Hartree diagrams of self-energy.

Concerning insight into the physical problem the following description shows that the Coulomb anomaly corresponds to an interference experiment and that the inelastic lifetime should destroy this interference and therefore the anomaly. In the past it was thought that the inelastic lifetime of the conduction electrons had no effect on the Coulomb anomaly. Only recently has its destructive effect been recognized and calculated.[16]

The diagrams in Fig. 1 describe the Hartree correction to the conductance. In Sec. IV we will discuss its physical meaning. For its physical understanding it is quite helpful to consider first the so-called self-energy of a conduction electron, i.e., the effect the Coulomb interaction has on the frequency and damping of a conduction electron. The corresponding diagrams are shown in Fig. 2. The physics behind the different diagrams is quite similar although they partially compensate each other. We will discuss only the first one. Essentially we have to unfold this diagram and for our understanding the description in real time is most helpful. Such a diagram contains the recipe for the electron propagation and the information about the development of the electron phase. This will be considered later in detail.

The propagation of electrons may be evaluated in k space as well as in r space. Both pictures give (different) physical insight into the problem. In Sec. II we discuss electron propagation in real space and in Sec. III in momentum space. In real space it turns out that we have to consider the motion of an electron on a closed loop. By interference with the original (self-reconstructing) wave function it builds up a charge pattern. At finite temperature such a charge pattern is only possible if the thermal coherence length $l_T=\hbar v_F/(2\pi k_B T)$ is larger than the diffusion path on the closed loop. This brings the temperature into play. The propagation in real space yields an understanding of the influence of a magnetic field and the role of inelastic scattering processes. Finally in Sec. IV we consider the influence on the conductance.

In this article we concentrate on the Hartree term which is shown in Fig. 1. The Fock term has a similar meaning because it follows the same physics. On the other hand it has an additional inner dynamics because the effective interaction depends on the energy difference of the two electrons involved.

II. DIFFUSION IN REAL SPACE

The propagation of electrons in real space along trajectories is in many ways particularly graphical, although there is some complex physics behind it. To offer a description of sufficient accuracy a rather extended discussion is required. In this section we would prefer to extract the important physics of the Hartree diagram. Let us consider a free-electron gas in a (large) box with all states up to the Fermi frequency occupied. If we are far from the surface of the box, and if there are no impurities present, then the density of the electron gas is constant. The electronic wave function can be given in any basis: plane waves, spherical waves, etc. A single impurity produces the scattering of the electron waves and introduces interference. A free-electron wave which propagates from point C towards the impurity is scattered by the impurity, and the scattered wave interferes with the original wave at position C and modulates the charge density. This yields the Friedel oscillations (after summation over all occupied electron states). This concept can be directly generalized to many impurities.

In the following we use Huygens principle for the electronic wave function. We start with the assumption that the electron wave functions in the space between the impurities are free-electron wave functions ψ''. Then, according to Huygens principle, every point r' acts as a continuous source of an electron wave which propagates radially in all directions. In the presence of impurities the amplitude is damped since every impurity generates a new spherical wave (we assume s scattering in the following consideration). In $d=2$ or 3 dimensions this damped spherical electron wave has an amplitude at the position r

$$G(r',r)\propto\psi''(r')/|r-r'|^{(d-1)/2}\times e^{i|k||r-r'|}e^{-|r-r'|/2l}. \tag{2.1}$$

Its density decays exponentially after the mean free path l and reduces inverse proportionally to $1/r^{d-1}$, i.e., the surface of the d-dimensional sphere. The phase is modulated according to the wave vector k. For a single electron state $\psi''(r)$ the source has an infinite coherence time. In an electron system at finite temperature the electron source has a coherence time $\tau_T=\hbar/(2\pi k_B T)$ because of the superposition of different frequencies in the range of $\pi k_B T/\hbar$ at the Fermi frequency. In an electron system with impurities one is generally interested in a mean free path l which is much smaller than the thermal coherence length.

Let us consider an electron source at the position C in the lower part of Fig. 3. We follow the trajectory along the closed loop $C\rightarrow 1''\rightarrow 2''\rightarrow\cdots\rightarrow 9''\rightarrow 10''\rightarrow C$. The electron wave returns to C with amplitude A. The frequency of the electron source may be the Fermi frequency ϵ_F. On the closed loop the electron wave experiences a phase shift δ which is included in the amplitude A. If L is the total length of the closed loop ($L=v_F t$, where t is the diffusion time) then the main phase shift between the source and the returning amplitude A is given by Lk_F. The scattering by the impurities causes additional phase shifts. Electrons which differ in their frequency by ϵ'' from the Fermi frequency experience an additional phase shift $\epsilon'' t$ along the closed loop. The interference between the electron source and the returning wave modifies the (homogeneous) local electron charge by a factor

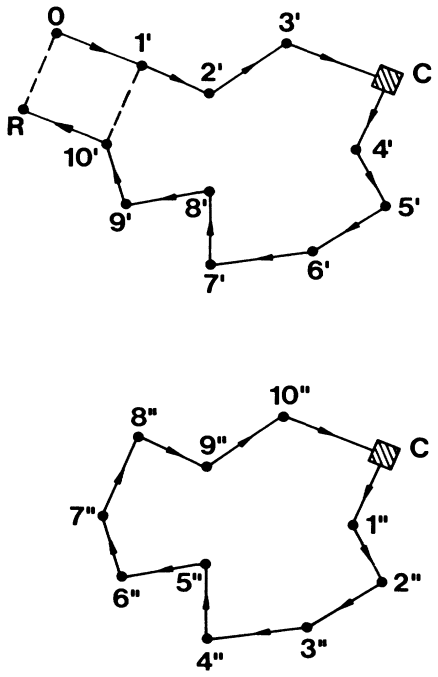

FIG. 3. The scattering sequence of an electron ('') (lower part) which propagates on a closed loop from $C \rightarrow 1'' \rightarrow 2'' \rightarrow \cdots \rightarrow C$. At C it interferes with itself and forms a charge hologram by which another electron (') can be scattered due to the Coulomb interaction. The upper part shows the interference of the electron (') going from O to R. From O it propagates on one hand directly to R and on the other hand it propagates along the closed loop where it is Coulomb scattered by the charge hologram at C. At R both partial waves interfere coherently because (i) the electron wave has a coherence length of $v_F\hbar/(2\pi k_B T)$ and (ii) the scattering by the charge hologram readjusts all the phase shifts by the impurities. This propagation is a direct translation of the (left) diagram in Fig. 2.

$$[1+A\exp(+\epsilon'' t)+A^*\exp(-i\epsilon'' t)] .$$

The same phase shift is obtained for all points C which lie on an ellipsoid around the points 1'' and 10''. Therefore the maxima and minima of the charge pattern lie on such ellipsoids. This charge pattern carries the information about the phase changes of the electron wave on the closed loop. It is an electron hologram. (Any impurity averaging eliminates this charge pattern mathematically.)

The amplitude of the charge pattern can be estimated. In a two-dimensional system (of unit area) the amplitude of the wave function $\psi''(r)$ at the position C is one. We concentrate on the charge pattern $A^*\exp(-i\epsilon'' t)$ resulting from the interference after encircling the closed loop. All occupied states $\psi''(r)$ contribute to the charge pattern according to the thermal occupation number $f(\epsilon'')$. The summation over all occupied states yields for the charge density

$$\begin{aligned}\rho(C) &= A^* N_0 \hbar \int_{-\infty}^{\infty} d\epsilon'' f(\epsilon'') \exp(-i\epsilon'' t) \\ &= i\epsilon_T A^* N_0 \hbar \sum_{l(\geq 0)} \exp(-\epsilon_l t) ,\end{aligned} \tag{2.2}$$

where N_0 [$= m/(\pi\hbar^2)$ in two dimensions] is the density of states ($N_0\hbar$ is the density of frequencies), ϵ_T is the characteristic frequency for the temperature T, and τ_T the corresponding time

$$\epsilon_T = 1/\tau_T = 2\pi k_B T/\hbar . \tag{2.3}$$

The Matsubara frequencies are defined as

$$\epsilon_l = (l+1/2)\epsilon_T . \tag{2.4}$$

The sum at the right side of (2.2) is evaluated in Eq. (3.8) below. For short times it can be replaced by an integral and yields

$$\rho(C) = iA^* N_0 \hbar/t \quad \text{for } t < \tau_T . \tag{2.2'}$$

This means that the magnitude of the charge pattern is inversely proportional to the length of the closed loop. For $t > \tau_T$ the charge pattern reduces proportionally to $\exp(-\epsilon_T t/2)$, which describes the fact that at finite temperature the electron wave packets have a coherence time of τ_T. For simplicity we replace the time dependence by a $1/t$ law with a cutoff at $\tau_t = 1/\epsilon_T$. Therefore one obtains the important result that the charge pattern is temperature dependent. The point C can lie between any two neighboring impurities on the closed loop. The resulting charge pattern is a generalization of the Friedel oscillations. Of course, this is not the only contribution to the inhomogeneous charge density. Every other closed loop yields its own pattern which characterizes the phase shift on this loop. We will see below that we have to consider each loop and its hologram individually.

In the next step we consider the propagation of a probe electron (') from the position point 0 (at the origin) to R as is shown in the upper part of Fig. 3. Its amplitude decays as given by Eq. (2.1). On the other hand the electron wave may also propagate on the same closed loop from 0 to point R. The electron propagates from point 0 via $1' \rightarrow 2' \rightarrow 3'$ to point C. Here it is scattered by the charge distribution $\rho(C)$ of the other electrons which we discussed above and continues its closed loop via 4'. . . to R, where it interferes with its original wave function. For simplicity we assume that the distance 0-1' and 10'-1' are the same as are the distances 0-R and 10'-R. Therefore the amplitudes at point R for the Coulomb scattered wave (along the closed loop) and for the direct path differ only by the propagator along the closed loop. Along the closed loop the probe electron experiences the same phase shift as if it had started at position C, propagated on the same closed loop to C, and performed the Coulomb scattering at C. The amplitude after the propagation along the closed loop from C to C is the same as for the electron (''), i.e., $Ae^{i\epsilon' t}$. The Coulomb potential for unscreened charges is $eU(r) = \int e\rho(r')/|r-r'|\,dr'$. (We work here in three dimensions because thin metal films are always three dimensional with respect to the screening length.) In momentum space this corresponds to the Coulomb potential $4\pi e^2/q^2$. In the case of Thomas-Fermi screening the potential is

$$4\pi e^2/(\kappa_s^2+q^2) = (1/N_0)1/[1+(q/\kappa_s)^2]$$

and for very strong screening it is just $1/N_0$

($\kappa_s^2 = 4\pi e^2 N_0$). In this case one obtains a δ function in real space, and the Coulomb potential is essentially $\rho(r)/N_0$. For finite screening the potential averages the charge distribution in real space over a finite distance of about $1/\kappa_s$. Since the charge pattern changes over a distance of half the Fermi wave length, its contributions cancel partially, and a less screened potential yields a smaller Coulomb effect. This imperfect screening is expressed by a screening factor F which means that the effective Coulomb potential in real space is about $F\rho(r)/N_0$.

The Coulomb scattering lasts for τ_0 (because the electron wave experiences impurity scattering) and contributes to the final amplitude of the returning electron wave (') with the factor

$$-i\langle U\rangle\tau_0/\hbar = -(i/\hbar)\tau_0\rho(r)F/N_0 = F\tau_0 A^*/t \ . \tag{2.5}$$

Since the Coulomb scattering can take place between every two scattering processes we obtain another factor of $n = t/\tau_0$ and the Coulomb scattering yields a total factor of FA^*. The total amplitude of the returning electron wave (') on the closed loop including one Coulomb scattering is just $F\,|A|^2$ as long as the diffusion time is less than τ_T. For times larger than τ_T there is an exponential cutoff.

The value of $|A|^2$ depends on the impurity distribution. However, if we sum over all loops and average over all impurity distributions, then we expect the probability for back scattering on a closed loop to be described by the diffusion probability. In two dimensions the total probability to return to the origin is $\Sigma\,|A|^2 \propto 1/(Dt)$, where Σ represents the sum over all closed loops.

The relative "amplitude" of the Coulomb scattered electron wave at the point R for times $t < \tau_T$ is proportional to $F/(Dt)$. In other words, besides the direct electron wave amplitude (for propagation from 0 to R) we also find a coherent amplitude at the position R which comes from the partial waves which started before the time t and propagated along all possible closed loops and reached finally the point R. Since all partial waves which started in the time interval $-\tau_T < t' < 0$ contribute coherently at the time $t = 0$, their amplitudes add up to $(F/D\tau_0)\ln(\tau_0/\tau_T)$.

Now we consider the four points 0, R, 1', and 10' more closely. The phase shift between the direct and the closed-loop partial waves is zero when the distances fulfill the following condition: distance (0-1')+(10'-1') is equal to (0-R)+(10'-R). Only those constellations contribute to the amplitude at R. For a given position of 0, R, and 10' the impurity 1' may lie on a hyperbola which goes through R and has its poles at 0 and 10'. Generally an area of length l and width $1/k_F$ contributes to the amplitude yielding a contribution proportional to

$$(l/k_F)(F/D\tau_0)\ln(\tau_0/\tau_T) \propto [F/(\epsilon_F\tau_0)]\ln(\tau_T/\tau_0) \ .$$

If the distance between 0 and R is small compared with l then the contribution is smaller because it reduces the most probable distance either between 0-1' or R-10'. It shows the tendency for an effective area of r/k_F where r is the distance between 0 and R, although I do not want to stress this point.

The effect of the hologram scattering on the electron wave which passes through the closed loop is a partial coherent repair of the electron wave function propagating from 0 to R. The "direct" propagator fades with increasing distance from 0 because of the impurities which scatter this wave and cause an overall decay as $\exp(-r/2l)$. The partial waves scattered by the hologram replace this loss of "coherent" amplitude partially and reduce the decay or increase the effective decaying length to $2l\{1+[F/(2\pi k_F l)]\ln(\tau_T/\tau_0)\}$. The relative difference between this and the old amplitude as a function of the distance r between 0 and R is $(r/2l)(F/2\pi k_F l)\ln(\tau_T/\tau_0)$. The effect of the hologram scattering is just that the metal appears to have a larger mean free path

$$l' = l\{1+[F/(2\pi k_F l)]\ln(\tau_T/\tau_0)\} \ .$$

III. COULOMB SCATTERING IN MOMENTUM SPACE

A. Closed-loop scattering of a probe electron

We consider a probe electron in an eigenstate ψ' of the disordered electron system. This eigenstate can be expressed in terms of plane waves,

$$\psi' = \sum_{k'} a'_{k'}\,|k'\rangle \ . \tag{3.1a}$$

The wave function is, of course, stationary, and oscillates with the frequency ϵ'.

The occupation of the $|k'\rangle$ component is, according to Thouless,[17] given by

$$\overline{|a'_{k'}|^2} = \frac{2}{\pi\hbar N_0}\frac{1/2\tau_0}{(\epsilon'-\epsilon_{k'})^2+(1/2\tau_0)^2} \ , \tag{3.2a}$$

where $N_0/2$ is the density of states per spin. This means that the eigenstate is roughly smeared over Z wave numbers $|k'\rangle$, where

$$Z = N_0\pi\hbar/\tau_0 \ . \tag{3.2b}$$

On the other hand, we may concentrate on the plane-wave component $|k'\rangle$. We denote its amplitude by $a' = a'_{k'}$. It will be scattered after the time τ_0, for example, into the state $|k'_1\rangle$. The number of possible scattering states is Z.

There is, of course, also a scattering into the plane-wave component $|k'\rangle$ from all other components and their scattering amplitudes add up and yield the amplitude a' [with the phase shift $\exp(-i\epsilon'\tau_0)$]. This scattering into the $|k'\rangle$ component can be expressed as a continuous source of momentum $|k'\rangle$.

The scattering sequence of $|k'\rangle$ may continue via $k'_2, k'_3, \ldots, k'_c$. The electron is not only scattered by the impurities, it also is affected by the inhomogeneous charge distribution of the other electrons. If we consider

for example the electron ('') in the state ψ'' in the Fermi sea, its wave function may be written as

$$\psi''=\sum_{k''} a''_{k''} \mid k'' \rangle = \sum_{k''} a''_{k''} e^{ik''r} . \tag{3.1b}$$

Such a wave function has an inhomogeneous charge distribution and contains a charge wave with the wave number Q. Its amplitude is simply obtained from the density of ψ'',

$$\begin{aligned} \mid \psi'' \mid^2 &= \sum_p \sum_{k''} a''_{k''+p} a''^{*}_{k''} e^{ipr} \\ &= \sum_p \rho_p e^{ipr} , \\ \rho_p &= \sum_{k''} a''_{k''+p} a''^{*}_{k''} . \end{aligned} \tag{3.3}$$

In the following we will see that we must analyze the charge waves much more closely, but at the moment we realize that our electron ('), i.e., its scattered plane wave component $\mid k'_c \rangle$, is affected by the inhomogeneous charge wave of any other electron, and there is a finite chance of being scattered, for example from $\mid k'_c \rangle$ to $\mid k'_{c+1} \rangle$ with $Q=k'_{c+1}-k'_c$. The screened Coulomb potential is replaced by its averaged value $\langle U_c \rangle = F/N_0$ (see Sec. II). For large screening $k_s > Q$, F is equal to one and for small screening it approaches zero.

Generally one considers the charge wave ρ_Q as sufficiently small that only one Coulomb scattering must be taken into account. Therefore we regard only further impurity scattering processes following the series

$$k'_{c+1} \rightarrow k'_{c+2} \rightarrow \cdots \rightarrow k'_n \rightarrow k'_{n+1} = k' .$$

There is always a finite chance that the scattering state is the original $\mid k' \rangle$. We assume that this happens after n impurity scattering events. Therefore the total scattering sequence is given by

$$k' \rightarrow k'_1 \rightarrow \cdots \rightarrow k'_c \Longrightarrow k'_{c+1} \rightarrow \cdots \rightarrow k'_n \rightarrow k'_{n+1} = k' . \tag{3.4}$$

This scattering series is shown in the upper part of Fig. 4 and corresponds to an indirect transition from the plane wave component $\mid k' \rangle$ into $\mid k' \rangle$. During this transition the amplitude and phase are changed.

After the time $n\tau_0$ we have the following situation. Since ψ' is an eigenstate all sequences of impurity scattering of all components $\mid k' \rangle$ of ψ' reproduce the eigenfunction and therefore also its $\mid k' \rangle$ component. But the phase has been oscillating and is shifted by $-\epsilon'(n\tau_0)$. [Although the actual scattering series takes $(n+1)$ times τ_0 we generally replace it by $n\tau_0$ because this difference does not matter.] Therefore one finds an additional phase factor $\exp[+i\epsilon'(n\tau_0)]$ between the amplitude of the $\mid k' \rangle$ components of the original state ψ' and the Coulomb satellite. On the other hand, we find an additional amplitude in (each) component $\mid k' \rangle$ from the scattering series (3.4). Since it contains the Coulomb scattering it is not included in the single-particle eigenstate.

The amplitude of the satellite in the final plane wave state $\mid k' \rangle$ has experienced a number of phase shifts

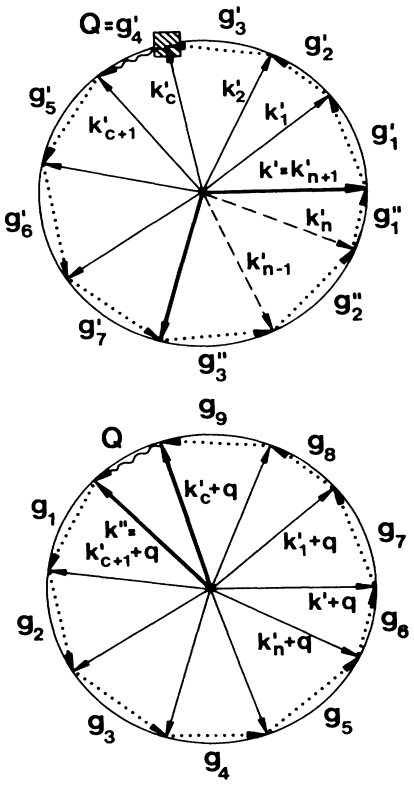

FIG. 4. The propagation of the electrons ('') and (') in momentum space. The lower part shows the scattering series of ('') which generates a charge pattern with momentum Q. The upper part shows the electron (') which is scattered by the charge pattern. Both scattering sequences contain almost the same momenta which differ only by a small q. This assures the conservation of coherence in the two sequences.

caused by the Coulomb scattering, the impurity scattering, and the phase oscillation according to the kinetic energy. The Coulomb scattering during the time interval τ_0 introduces a factor of $-i(F/N_0)\rho_Q\tau_0/\hbar$. The other phase shifts are discussed in the Appendix but their explicit knowledge is not required.

Finally, the amplitude of the Coulomb satellite is

$$-i\alpha' a'(F/N_0)\rho_Q\tau_0/\hbar , \tag{3.5}$$

where α' accumulated the effect of the impurity scattering and contains the phase shift due to the kinetic energy. This amplitude interferes with the stationary amplitude of the $\mid k' \rangle$ component of the eigenstate ψ'. The amplitude of this Coulomb satellite appears, however, to be rather chaotic. It contains many impurity matrix elements V_g. One might expect that this amplitude averages to zero because the phase shifts δ_i due to the impurity scattering have random values for statistical impurity distributions. However, the charge wave ρ_Q has a component which exactly compensates these phase shifts.

B. The coherent charge wave

We introduced in the scattering sequence of the plane wave component $|k'\rangle$ the Coulomb scattering by the charge pattern of the (occupied) state ψ'' of the electron ('). Now we want to select a particular component of this charge wave. For this purpose we consider in the eigenfunction ψ'' the plane wave component $|k''\rangle = |k'_{c+1}+q\rangle$. We denote its coefficient as $a''=a''_{k''}$. This component will be scattered by the impurities (as before the component $|k'\rangle$ of ψ'). We follow the scattering sequence,

$$k''=k'_{c+1}+q\rightarrow k'_{c+2}+q\rightarrow\cdots\rightarrow k'_{n+1}+q=k'+q\rightarrow k'_1+q\rightarrow k'_2+q\rightarrow\cdots\rightarrow k'_c+q\ . \tag{3.6}$$

This scattering series is shown in the lower part of Fig. 4 and is cyclically exchanged in comparison with the series (3.4) and its wave numbers are shifted by a small q.

The important properties of this scattering sequence are as follows:

(i) It contains all the wave numbers of our first scattering sequence besides a shift of q.

(ii) The order is cyclically changed.

(iii) The impurity matrix elements are the same.

The series due to the impurity scattering corresponds to a reduction α'' of the original amplitude a''.

As before, for the eigenfunction ψ' we now conclude that the scattered plane wave components reproduce ψ''. After n scattering processes we recover the plane wave component $|k''\rangle = |k'_{c+1}+q\rangle$ with a phase shift of $\exp[-i\epsilon''n\tau_0]$. On the other hand we see the coefficient of $|k'_c+q\rangle$ contains a term which comes originally from the plane wave component $|k'_{c+1}+q\rangle$. It has rather interesting properties. (a) Its magnitude is proportional to the coefficient of a''. (b) Its phase shift accumulates all the information about the scattering potentials which are the same as for our probe electron (').

Now we will consider the charge wave which is formed during the time interval from $n\tau_0$ to $(n+1)\tau_0$ by the stationary component $|k''\rangle = |k'_{c+1}+q\rangle$ and that part of $|k'_c+q\rangle$ which is obtained by the scattering sequence (3.6). The sum of the two plane wave components is

$$a''\{\exp[i(k'_{c+1}+q)r-i\epsilon''n\tau_0]+\alpha''\exp[i(k'_c+q)r]\}\ .$$

The resulting charge wave density with momentum Q contains the complex-conjugated amplitude of the scattering series, (3.6):

$$\rho_Q=\alpha''^{*}\,|a''|^2\exp(iQr-i\epsilon''n\tau_0)\ , \tag{3.7}$$

where $Q=k'_{c+1}-k'_c$. The charge wave with the wave number Q (and $-Q$) form an electron hologram by which the probe electron (') is scattered.

C. Phase readjustment

In Sec. III A we considered a probe electron (') which experienced the scattering sequence (3.4). This scattering sequence contains the same impurity matrix elements as the scattering sequence (3.6) and even the corresponding electron momenta are almost identical. Since the Coulomb potential contains the complex conjugate of the scattering amplitude of (3.6) and since it enters in the amplitude of the scattered probe electron as a factor we find complete cancellation of the impurity phase shifts.

We introduce the electron charge from Eq. (3.7) into Eq. (3.5) and obtain for the relative amplitude of the Coulomb satellite with respect to the k' component of the eigenstate

$$-i(F/N_0)(\tau_0/\hbar)\alpha'\alpha''^{*}\,|a''|^2 \times\exp\{[-i(\epsilon''-\epsilon')](n\tau_0)\}f(\epsilon'')\ .$$

The Fermi factor $f(\epsilon'')$ arises because the state ψ'' is thermally occupied.

The two impurity scattering amplitudes α' and α'' are almost identical. They arise from two parallel impurity scattering series of the electrons (') and (''). (i) The first corresponds to the scattering of the pair from $(k',k'+q)$ into (k'_c,k'_c+q). (ii) The second corresponds to the scattering of the pair from $(k'_{c+1},k'_{c+1}+q)$ into $(k',k'+q)$.

In the Appendix it is demonstrated that for each of the scattering series the two amplitudes of the electron (') and ('') lose their phase coherence very slowly as $\exp[-Dq^2t]$ because they experience the same impurity scattering. Each scattering series yields the factor

$$(1/Z)\exp(-iDq^2t_i)\ ,$$

where $t_i=n_i\tau_0$ is the duration of the (parallel) scattering series.

For the final evaluation one has to perform a number of summations and integrations.

(i) A summation over the intermediate states k'_c and k'_{c+1}. The summation over $k''=k'_{c+1}+q$ (which is equivalent to the summation over k'_{c+1}) sums all $|a''|^2=|a''_{k''}|^2$ and yields just one. The summation over k'_c yields just Z (and averages the Coulomb matrix element).

(ii) A summation over all occupied states which correspond to an integration over the frequency ϵ'' times the Fermi function. According to Eq. (3.2a) the two eigenfunctions ψ' and ψ'' contain only neighboring wave numbers k' and $k'+q$ when their frequency difference is less then $1/\tau_0$. Since ϵ' lies close to ϵ_F the frequency of the electron ('') is restricted to the frequency range $|\epsilon''|<1/\tau_0$. The sum over the frequency ϵ'' is equal to the integral over the frequency times the factor $\hbar N_0$. The integral yields

$$\int_{-1/\tau_0}^{1/\tau_0}d\epsilon''f(\epsilon'')\exp(-i\epsilon''t)=2\pi i(k_BT/\hbar)\sum_{l=0}^{l_c}\exp(-\epsilon_l t)\ , \tag{3.8}$$

with the Matsubara frequencies $\epsilon_l=(2l+1)\pi k_B T/\hbar$. Since the integration is limited to the frequency range of $1/\tau_0$, one restricts the ϵ_l summation to $|\epsilon_l|<1/\tau_0$ or $l<l_c$ with

$$l_c \approx 2\pi k_B T\tau_0/\hbar \ . \tag{3.9}$$

(iii) The integration over time. We want to determine the amplitude of the Coulomb satellite at a given time for example at the time $t=0$. Let us take n_2 impurity scatterings after the Coulomb scattering and n_1 before. Then we have to sum over n_1 and n_2 or integrate over dt_1/τ_0 and dt_2/τ_0 from zero to infinity where $t=t_1+t_2$.

(iv) The integration over $d^2q/(2\pi)^2$. When we perform the two time integrations then we obtain for the relative amplitude (with respect to the $|k'\rangle$ component of ψ' at the corresponding time $t=0$) the following expression:

$$\frac{F\tau_0}{Z}\frac{2\pi k_B T}{\hbar}\int\frac{d^2q}{(2\pi)^2}\sum_{l=0}^{l_c}\frac{1}{\tau_0^2(Dq+\epsilon_l-i\epsilon')^2} \ . \tag{3.10}$$

The integration and summation yield

$$\frac{F}{2\pi\epsilon_F\tau_0/\hbar}\frac{1}{2\tau_0}\left[\ln(2\pi k_B T\tau_0/\hbar)+\psi\left(\frac{1}{2}+\frac{i\epsilon'}{2\pi k_B T}\right)-\psi\left(\tfrac{1}{2}\right)\right] \ . \tag{3.11}$$

From the real part [the first term in Eq. (3.11)], we realize that the coherent Coulomb scattering increases the amplitude of the $|k'\rangle$ component. If we turn off the source the plane wave component k' decays normally with the rate $a'/2\tau_0$. Because of the Coulomb interference the elastic lifetime increases by a factor

$$[1+F/(2\pi\epsilon_F\tau_0)\ln(2\pi k_B T/\hbar)] \ .$$

This means that the decay of the plane wave component k' (and all other plane wave components) is partially compensated. The Hartree-Coulomb scattering partially restores the plane wave components and gives them a longer elastic lifetime. As a consequence the effective scattering is reduced. This corresponds to the result in real space where the coherently back scattered Coulomb satellite partially restored the exponentially decaying electron wave ('), yielding a slower decay with the same factor. The imaginary part changes the phase of the $|k'\rangle$ component and results in a frequency shift.

The second diagram in Fig. 2 yields just half the contribution of the first one with the opposite sign. According to this diagram the electron (') experiences two additional impurity scatterings (in real space by the same impurity and in momentum space with opposite momentum transfer) at the beginning and at the end of the scattering sequence. The evaluation is very similar to the one we just performed.

The important message of this section (and of Sec. II) is the physical process which underlies the Hartree diagram. Although we have not yet included the effect of an external electric field we have developed the dynamics of the electrons as a function of time. Within this picture it is obvious that the Hartree term is an "interference effect" which requires coherence of two electronic wave functions over a time period of τ_T.

IV. THE CONDUCTANCE CORRECTION

We want to discuss the contribution of the Hartree diagram only qualitatively. We treated the essential scattering processes quantitatively for the "self-energy" to root out the underlying physics. Now there is no point in going into the quantitative details again because the resulting equations and integrals are already known from the formal theory.

The highest threshold in the graphical understanding of the conductance is the Kubo formalism itself. The original derivation by Kubo[18] and Greenwood[19] is rather formal. There are now several review articles and textbooks available[20–22] but the physical transparency is still limited.

We are used to the fact that an electric field E accelerates the conduction electrons and each plane wave component k changes with time according to

$$\hbar dk/dt=eE \ . \tag{4.1}$$

Of course, the collisions scatter the conduction electrons to the back side of the Fermi sphere. However, for the moment we assume that we turn off the impurity scattering. As a consequence the whole Fermi sphere is shifted by $\kappa(t)$,

$$\hbar\kappa(t)=\int_{-\infty}^{t}dt'eE(t')=-eA(t) \ . \tag{4.2}$$

The Kubo formalism simulates the shift of the Fermi sphere in a rather sophisticated manner. If a_k is the coefficient of the k component of an electron in the eigenstate ψ then the electric field E causes a transition into the k component of an unoccupied state. With the above definition of $\kappa(t)$ the transition rate da_k/dt for the amplitude is

$$da_k/dt=-i\mathbf{v}_k\kappa a_k \ . \tag{4.3}$$

[One important difference to perturbation theory, which cannot be discussed here in detail, is the fact that da_k/dt and a_k are not perpendicular to each other, although the rate has the formal factor $-i$. As we see below, the occupation of the k component changes linear with time in the Kubo formalism. The use of integrated variables like $\kappa(t)$ or $A(t)$ hides this fact, because they introduce the factor $1/(-i\omega)$ for finite frequencies.]

The transition rate da_k/dt into the plane wave component k mixes states with different energy into the state ψ. Later one has to neglect all those contributions which belong to occupied states.

The Kubo formalism treats generally only finite frequencies $F(t)=F_0\exp(-i\omega t)$. For finite frequency ω (at $T=0$) the transitions occur from occupied states in the frequency range $-\omega<\epsilon<0$ into empty states with $0<\epsilon<\omega$. The final amplitude of the "field satellites" is $\mathbf{v}_k\kappa a_k/\omega$. If one calculates the change in the total occupation of the plane wave component, the finite frequency drops out. One obtains

$$\Delta n_k(t)=\sum_{(-\omega<\epsilon_n<0)}|a_k(n)|^2\frac{\mathbf{v}_k\boldsymbol{\kappa}(t)}{\omega}$$

$$=\frac{dn_k}{d\epsilon}\mathbf{v}_k\boldsymbol{\kappa}(t)\ ,$$

$$\Delta n_k(t)=\nabla_k n_k\boldsymbol{\kappa}(t)\ ,$$

or (4.4)

$$n_k(t)=n^0_{k-\kappa(t)}\ .$$

This means that one may simulate the displacement $\kappa(t)$ of the whole Fermi sphere by electronic transitions according to Eq. (4.3).

In the presence of impurities the plane wave component k decays with the elastic scattering time τ_0. This yields a stationary satellite amplitude

$$\Delta a_k=-i\mathbf{v}_k\boldsymbol{\kappa}\tau_0 a_k=e\mathbf{v}_k\mathbf{E}\tau_0 a_k/(\hbar\omega) \tag{4.5}$$

and corresponds to a shift of the Fermi sphere of $eE\tau_0/\hbar$ according to the change in occupation n_k

$$\Delta n_k(t)=\nabla_k n_k e\mathbf{E}\tau_0/\hbar$$

or (4.6)

$$n_k(t)=n^0_{k-eE\tau_0/\hbar}\ .$$

We realize that the Kubo formalism yields just the shift of the Fermi sphere by an electric field in the presence of impurities as the Boltzmann treatment. However, the use of the finite frequency ω and the final integration over the frequency range ω disturbs the transparency of this treatment on first sight. To clarify this point let us assume that at the time $t=0$ we apply a constant electric field E. We know that this field will shift the Fermi sphere by $eE\tau_0/\hbar$ roughly after the time τ_0. In the framework of the Kubo formalism this field causes transitions. In the first instant the transitions range over the whole Fermi sphere, i.e., the transition amplitudes Δa_k contain all frequencies. With progressing time the transition amplitudes Δa_k change their energy distribution. At the time $t\gg\tau_0$, the amplitude Δa_k, for states with $0<\epsilon<1/t$, has grown linear with t while Δa_k approaches zero for the other states with $\epsilon>1/t$. This means, that the amplitude for small-energy transfer grows with the reduction of high-energy transitions so the overall change in the occupation of the plane wave component k remains the same, as does the resulting shift of the Fermi sphere, i.e., $eE\tau_0/\hbar$. For a dc field this means that the final states lie within a very narrow (δ-like) shell at the Fermi frequency. Therefore the Kubo formalism yields the well-known shift of the Fermi sphere but in addition it yields inner dynamics by introducing transitions between occupied and empty states.

In the presence of impurities one obtains, as we discussed above, a stationary amplitude $\Delta a_k=e\mathbf{v}_k\mathbf{E}\tau_0 a_k/(\hbar\omega)$, because the impurities scatter this plane wave component. However, it is this scattered amplitude which may cause quantum interference after many scattering events, and we are going to observe its fate as a function of time.

Next we are going to discuss the time development of the Kubo diagram in Fig. 1 during the conductance process. We consider again the plane wave component k' in the eigenstate ψ'. k' will be scattered according to the sequence given by Eq. (3.4). In Fig. 5 we have sketched the time development. The scattering sequence starts at $-t_a<0$. At the time $t=0$ the electric field may act (either as a pulse or (even better) as a section of a slowly varying field). As a consequence the amplitude (and occupation) of the plane wave component k is changed without losing the coherence. This change is positive on the right side of the Fermi sphere and negative on the left side (if the electric field points to the right). In the following we trace only this differential partial wave. It performs further impurity scatterings until it experiences the Coulomb scattering at the time t_c by the hologram of the electron ('') which started already at the time $t_c-t_a-t_b$. [The electron ('') propagates opposite to the direction of the arrows. The reversed direction of the arrows is due to the fact that only its complex conjugated amplitude enters in the charge hologram which causes the Coulomb scattering.] After the Coulomb scattering the electron (') continues the scattering sequence until it is scattered into the original state k' at the time t_b. In the state k' the "satellite" (with respect to the electric field and the Coulomb interaction) interferes with the k' component of the original state ψ'. This interference yields finally the current contribution of this process.

In this process the electric field acts in the middle of the scattering sequence. Normally the left vertex in a Kubo diagram, (which represents the action of the electric field) is the starting point for the time evolution (see, for example, weak localization). In the Kubo diagrams for the Coulomb interaction we observe instead the fate of the conduction electron from the time $-t_a<0$ before the action of the field until the time t_b after the action of the field. This is the reason why the vertex is neighbored by two equivalent Green's functions (either advanced or retarded).

One easily realizes that in zeroth approximation this contribution of the interference in k' to the current is zero because of two reasons: (i) The electric field causes satellites which have opposite sign of the amplitude (respectively occupation) on opposite sides of the Fermi sphere. Since their contribution to the amplitude in the state k' just add, they compensate each other. (ii) The relative amplitude of the k' component of the satellite with respect to the original amplitude of ψ' is independent of

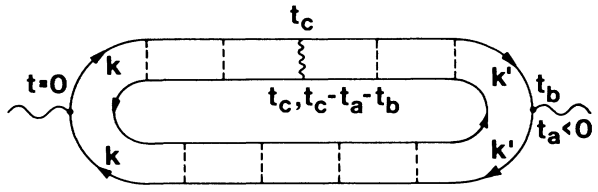

FIG. 5. The time evolution of the propagation of the two electrons (') and ('') in the presence of an electric field pulse at $t=0$.

k' and does not yield a current because of this spherical symmetry.

One has to be beyond the zeroth-order effects and look for asymmetries with respect to k and k'. Such asymmetries exist indeed. In the Appendix we see that the two electrons (') and (") in the plane wave components k and $k+q$ develop a phase difference during the time τ_0 which is

$$\exp(-i\mathbf{v}_k\mathbf{q}\tau_0)\approx(1-i\mathbf{v}_k\mathbf{q}\tau_0+\cdots).$$

The linear term generally averages to zero if one sums over all states k. However, after the action of the electric field the occupation of the states k and $-k$ have opposite signs, and the main contribution (i.e., the 1 in the parentheses) cancels, and the linear term in the bracket together with the amplitude of the field satellite $e\mathbf{v}_k\mathbf{E}\tau_0/(\hbar\omega)$ (after the transition time τ_0) yields a contribution proportional to $eD\mathbf{q}\mathbf{E}\tau_0/(\hbar\omega)$. (The average of $\langle v_k^2\rangle\tau_0$ yields the diffusion constant D).

The same applies for the final (i.e., initial) plane wave component k'. Between k' and $k'+q$ the corresponding phase shift develops and since the current is proportional to $v_{k'}$, one obtains only a contribution from the linear part in the phase coefficient, which yields the factor Dq.

The product of both factors is proportional to the square of q and is finite. As a consequence the whole Fermi sphere becomes asymmetric which corresponds to a shift of the Fermi sphere and causes a current. We realize that in momentum space the current and therefore the conductance correction depend very delicately on small phase shifts during their lifetime in the momentum states k and k'.

In the detailed calculation one has three sections in which the two electrons (') and (") scatter in parallel; from k' to k, from k to k_c and from k_{c+1} to k'. This yields three diffusion poles. Furthermore the electric field introduces for dc field the thermal transition factor $f(\epsilon')[1-f(\epsilon'+\omega)]$. There is, however, no point in replacing the formal calculation by a heuristic one. Instead we add a physical interpretation to this formal calculation.

The electric field can act before or after the Coulomb scattering. Furthermore one has also to include the effect of the electric field on the charge hologram. Since the electrons in the shifted Fermi sphere generate also the charge hologram one obtains also a change in this hologram which is also determined by Eq. (4.3). This yields a second contribution to the current which has opposite sign (which one can derive from the fact that the scattering states lie symmetrically around the momentum zero) and twice the magnitude of the first part. The conductance correction is not just due to the change in the effective mean free path or lifetime that we discussed at the end of Secs. II and III. (In the corresponding conductance process the electrical field and the interference would act on the same wave number k, i.e., the left diagram in Fig. 2 would be closed to a Kubo diagram which lacks the lower impurity scattering lines. This diagram is compensated by the Kubo diagram which is constructed from the right diagram in Fig. 2.)

It is valuable to say a few words about the propagation in real space. In Fig. 6 we have drawn two possible diffusion processes. In Fig. 6(a) the electron (") starts as usual at the position C, performs a closed diffusion loop and interferes with itself at C generating the charge hologram. The probe electron (') may diffuse from the position R along the closed loop back to R being Coulomb scattered by the charge hologram at C. At the time $t=0$ it may be located at the position O. Therefore this propagation yields an additional diffusion channel from O to R. It has the interesting aspect that it depends on the history of the electrons.

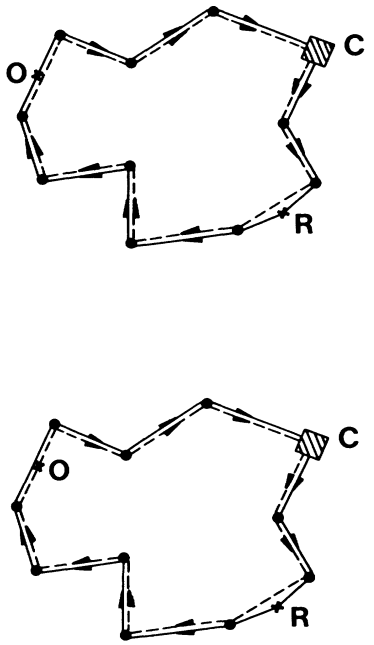

FIG. 6. The two contributions of the Coulomb satellite to the diffusion in real space.

The second contribution is shown in Fig. 6(b). Here the electron (") propagates from C to C along the closed loop and forms the charge hologram at C. At the time $t=0$ we find it at the position O. The other electron (') starts at R, propagates along the closed loop including the Coulomb scattering at C and returns to R. This is an indirect diffusion from O to R because the electron (') reaches only the point R when the electron (") is at the time $t=0$ at the position O. If we would remove the electron (") at $t=0$ from the position O the electron (') would not reach the point R (by this process). Therefore both processes modify the diffusion in real space.

V. CONCLUSIONS

In a disordered metal the electronic wave function has many possibilities to interfere with itself. As a consequence one finds an inhomogeneous electron density which stores the distributions of the impurity positions and which represents an electron hologram. This hologram contains the charge pattern for every closed scattering loop and acts as a (Coulomb) scattering potential itself. A second conduction electron which propagates on the same closed loop can readjust its phase, interfere

coherently at its starting point with its original wave functions and change the mean free path.

The coherence of the electronic wave functions is an essential requirement for the mechanism of the Hartree (and Fock) diagram. This coherence is limited by the thermal coherence time τ_T. This thermal coherence time is well known from the theory of superconductivity. Recently it has been suggested[23] that conductance fluctuation in small samples are also limited by the thermal coherence time.

In addition the Coulomb anomaly must be sensitive to inelastic scattering which destroys the coherence of the wave function. However, the effect of inelastic scattering is rather sophisticated. If we consider for example the electron paths in Fig. 3 there the electrons (') and (") propagate from $1'=8''$ at the same time along the same impurities. Any inelastic scattering on this part of the closed loop has little effect on the interference because the inelastic process occurs for both electrons with essentially the same probability and phase. However, along the path from C to $10'=7''$ the two electrons travel at quite different times (see Fig. 5) and inelastic processes are destructive for the phase coherence. This demonstrates that the understanding of the physical processes in real time are important for the evaluation of the phase coherence.

It is important that an electric field does not influence the coherence between $\psi'(r)$ and the Coulomb satellite. Of course, the electric field changes (in real space) the phase of the amplitude A'' and therefore the charge pattern. However, the phase shift is the same for both electrons on the same path and therefore cancels in the amplitude of the Coulomb satellite.

The same applies for a magnetic field. For the paths in Fig. 6 as well as for Fig. 3 we recognize easily that a magnetic field changes the electron hologram because the flux which is encircled by the closed loop changes the phases of the two partial waves and therefore the interference. As a consequence the phase shift caused by the magnetic field is also stored in the electron hologram. This (magnetic) phase shift compensates the phase shift which the conduction electron experiences on the same closed loop because the coherent Coulomb scattering selects just the complex conjugated component of the charge wave.

It is, however, remarkable that the magnetic field changes the charge pattern completely. In small systems where we do not have a perfect averaging of the impurity distribution,[23,24] the incoherent Coulomb scattering should enter into the resistance of the electrons.

The physical interpretation of the Hartree diagrams gives a simple understanding of the magnetoresistance in a magnetic field derived by Lee and Ramakrishnan.[25] In a magnetic field the spin-up and spin-down electrons have different Fermi wavelengths and therefore the hologram of spin up electrons (at the Fermi frequency) does not perfectly compensate the phase shifts of a spin down conduction electron and vice versa. For $\mu_B H \approx k_B T$ half of the electron hologram becomes ineffective in the phase adjustment and no longer contributes to the Coulomb anomaly. This yields the magneto-resistance with the weight $F/2$.

In the presence of spin-orbit scattering the spin of the electron will be rotated after the propagation on a closed loop. Therefore we expect not only a charge pattern (with half the amplitude) but also a spin pattern. This applies, of course, for nonmagnetic metals. For nearly magnetic electron systems these spin waves might have interesting consequences.

The whole discussion above depended essentially on the wave character of the electrons and the interaction between two different electrons. Therefore it suggests some generalization to laser experiments with high intensity. In a nonlinear medium with scattering centers the laser light forms a hologram as our electron (''). When the amplitude of this hologram is sufficiently large then the local dielectric constant is modulated according to the hologram. This solidified hologram will now scatter the scattered laser light as a phase correcting mirror and increase (or decrease depending on the sign of the nonlinearity) the amplitude of the original laser light. This has two consequences: (i) For a plane wave entering the scattering medium the extinction coefficient changes with laser intensity since the mean free path is changed in exactly the same manner as for the electron system. (ii) Under favorable conditions the laser light could select certain closed loops and stabilize its propagation along such loops.

Besides the Hartree diagram in the particle-hole channel the theory yields also a Hartree diagram in the particle-particle channel. For a single Coulomb scattering, as has first been discussed by Fukuyama,[7] the physical interpretation is essentially given by Fig. 3. The only difference is that the electron (") propagates on the closed loop in opposite direction. This does not change the charge hologram. However, the effect of a magnetic field is no longer compensated on the two paths but adds up and yields a magnetic phase shift which is $2e/\hbar$ times the enclosed magnetic flux. This contribution will therefore be destroyed by a magnetic field as in weak localization.

In Fig. 3 the two electrons (') and (") propagate together from O to C where the electron (') is scattered by the charge hologram of electron (''). This contradicts the Pauli principle. However, we do not have to worry about this fact because the Pauli principle is taken care of by subtracting the Fock diagram. Therefore we want to comment shortly on the Fock diagram. Since the Fock diagram is caused by the same Coulomb interaction as is the Hartree diagram, it describes essentially the same physics. One has to treat the Coulomb interaction in the correct two-particle wave function. In our description this means that we have to include in particular the "Pauli interference," i.e., the fact that the two different electron states can interfere because the two electrons are in both states at the same time. In addition the charge hologram oscillates with time and requires that the screening is calculated dynamically. In the Fock diagram the two electrons generally diffuse on a string, not on a closed loop and the Coulomb interaction acts over large distances.

APPENDIX: COHERENT PROPAGATION OF TWO PLANE WAVES

We consider the propagation of two plane waves $|k\rangle$ and $|k+q\rangle$. Both plane waves are scattered by the impurities according to the scattering sequences

$$k \to k_1 \to k_2 \to k_3 \to \cdots \to k_n \cdots = |k(t)\rangle$$

and (A1)

$$k+q \to k_1+q \to k_2+q \to \cdots \to k_n+q \cdots = |(k+q)(t)\rangle .$$

At first sight the scattering of the state $|k\rangle$ and $|k+q\rangle$ are independent and the first possible scattering processes might be $k \to k'$ and $k+q \to k''+q$. However, the two matrix elements $V_{k,k'}$ and $V_{k,k''}$ are only the same if $k'=k''$. Otherwise they are uncorrelated and destroy the phase coherence between the two scattered states. In the theory this is obtained by impurity averaging. Therefore the statistical distribution of impurities introduces a strict "conservation of momentum difference." (In the case of strong localization this conservation is no longer fulfilled. Here the difference $|k'-k''|$ is not zero but is smaller than the inverse localization length.)

The fact that the two states $\psi_1 = |k\rangle$ and $\psi_2 = |k+q\rangle$ preserve their phase coherence can be expressed by the product of $\psi_1^* \psi_2 = \Gamma(r,t)$. At $t=0$ this product yields $\exp(iqr)$.

The second plane wave $|k+q\rangle$ is often obtained as a satellite of the state $|k\rangle$ by means of a transition from the state $|k\rangle$ into the state $|k+q\rangle$ in a potential with wave number q. We assume that the amplitudes of the two states are phase coherent at time $t=0$. At time τ_0 the partial electron waves in the state k and the satellite $k+q$ are scattered. In this time interval the two states essentially remain coherent. There is, however, a small reduction due to the development of a phase shift between the state $|k\rangle$ and $|k+q\rangle$. They have a frequency difference of about $\mathbf{v}_k \mathbf{q}$ and therefore the phase of the satellite is shifted during the time interval τ_0 by

$$(\epsilon_{k+q} - \epsilon_k)\tau_0 \approx \mathbf{v}_k \mathbf{q} \tau_0 .$$

The amplitudes A' and A'' in the new states $|k_1\rangle$ and $|k_1+q\rangle$ are roughly given by $Z^{-1/2}\exp(i\delta_1)$ where $\exp(i\delta_1) \approx iV_g/|V_g|$, $g = k - k_1$ and Z is given by Eq. (3.2b). A' and A'' contain the same scattering matrix element and remain coherent. The square of the amplitude $|A|^2$ in the state k_1 is equal to $1/Z$ where Z is the number of possible scattering states. In the states k_1 and k_1+q the electron propagates again essentially for the time τ_0 without losing phase coherence. The small phase shift due to the different kinetic energy is this time $\varphi_1 = \mathbf{v}_1 \mathbf{q} \tau_0 (v_1 = \hbar k_1/m)$. Since we have to sum over k_1 the phase differences essentially average to zero because the angle between the Fermi velocity v_1 and the wave vector q takes all possible values and averaging yields only a broadening of the phase by $[(\mathbf{v}_1 \mathbf{q})\tau_0]^2 = Dq^2\tau_0$, where $D = \tau_0 v^2/d$ is the diffusion constant. If we form the product $\psi_1^* \psi_2$ at the time $t = 2\tau_0$, then the summation over the Z states $|k_1\rangle$ with occupation $1/Z$ yields just one (often called conservation of particles), and we obtain for the spatial part $\exp(iqr)$. As a function of time the amplitude is reduced by $\exp(-Dq^2\tau_0)$ in the time interval τ_0. The following scattering events preserve the momentum difference in the same manner so that the spatial part is always $\exp(iqr)$. After n scattering processes the phase coherence between ψ_1 and ψ_2, i.e., $\psi_1^* \psi_2$ is reduced by $\exp(-Dq^2 n\tau_0) = \exp(-Dq^2 t)$. The Fourier transformation in momentum and frequency yields

$$\Gamma(q,\omega) = \frac{1}{Dq^2 - i\omega} . \quad \text{(A2)}$$

This is the so-called diffusion pole in disordered electron systems and describes, as we showed, the coherent propagation of two plane waves with a small momentum difference.

In pure metals the phase difference of the two plane waves reaches the value one after the time $(\mathbf{v}_k \mathbf{q})^{-1}$. In some respects this means that the phase coherence in a disordered metal is better maintained than in a pure metal. This conservation of coherence can be considered as motional narrowing.

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