RAPID COMMUNICATIONS

PHYSICAL REVIEW B VOLUME 35, NUMBER 7 1 MARCH 1987

Magnetic properties of the two-dimensional $n=0$ vector model

H. Saleur

Service de Physique Théorique, Centre d'Etudes Nucléaires de Saclay, 91191 Gif-sur-Yvette Cedex, France

(Received 29 September 1986; revised manuscript received 1 December 1986)

We determine the magnetic properties of the two-dimensional $n=0$ vector model using transfer-matrix calculations and finite-size scaling. In the high-temperature region, we obtain results similar to the mean-field picture. In the low-temperature region, we find a very different behavior with a susceptibility which is always negative and an infinite spontaneous magnetization. This is shown to be in agreement with some works of Nienhuis. The physical interpretation is discussed.

The properties of self-avoiding walks (SAW's) can be obtained by considering the $n=0$ limit of the n-vector model[1,2] and this correspondence has been the basis of many calculations in polymer physics,[3,4] although more direct approaches have also been developed.[4]

Apart from being this useful intermediate, the $n=0$ vector model is of some interest in itself. Substantial work has been done to determine the nature of its transition[5-7] and to study the different thermodynamic instabilities which may appear.[8-14] The magnetic properties of the low-temperature phase have also been much discussed and several questions remain open. First, one may wonder whether there is a spontaneous magnetization and what is its behavior.[3,7,12] A related intriguing point concerns the longitudinal susceptibility χ. Several arguments (see Ref. 15, and references therein) suggest that—at least for n integer and $d>2$—the $(n-1)$ Goldstone modes transversing to the spontaneous magnetization make χ diverge in small magnetic fields h like

$$\chi \sim (n-1)h^{d/2-2},\ T<T^c,\ h\to 0\ . \tag{1}$$

If $n=0$ in formula (1), one finds that χ becomes *negative* when the coexistence curve is approached. Although a negative susceptibility has no consequence on the stability of the polymer problem,[11,13,14] several authors have suggested that formula (1) cannot be applied for $n<1$ and that χ must remain positive. Approximate calculations where this is indeed the case have been produced.[11,13]

In this paper we study these questions in the two-dimensional case. We shall first recall some general results which have been obtained by Nienhuis.[16] Using a series of model transformations, this author has shown how a particular $O(n)$ model on the hexagonal lattice can be mapped onto a Coulomb gas. The properties of the critical point can be deduced from this mapping and the exponents obtained in this way are in good agreement with numerical estimates, especially for the polymer problem (see Ref. 17, and references therein). In addition, the mapping is expected to also give the spin correlation function in the low-temperature phase when the magnetic field vanishes.[15] Parametrizing,

$$n=-2\cos\frac{2\pi}{t}\quad (t>2)\ . \tag{2a}$$

Nienhuis has suggested that—at least for $n>1$ (Ref. 16)—this correlation function behaves like $\langle S_i S_j\rangle \sim |i-j|^{-2x}$, where

$$-x+2=y=1+\frac{3}{4t}+\frac{t}{4}\ . \tag{2b}$$

As far as one knows, this result has never been tested numerically.

If we now apply these formulas to the case $n=0$, we find $t=4$, $y=\frac{35}{16}$, and $x=\frac{3}{16}$. This negative value of x corresponds to a correlation function *increasing* with the distance, a behavior which is observed in other cases, for instance, the Lee-Yang problem.[18] Then the scaling relations[18] give

$$\xi \sim h^{-16/35}\ , \tag{3a}$$

$$m\sim h^{-3/35},\ T<T^c,\ h\to 0\ , \tag{3b}$$

$$\chi=\frac{\partial m}{\partial h}\sim -h^{-38/35}\ . \tag{3c}$$

Within conjecture (2), the susceptibility (3c) is negative and diverges as was the case in (1). The magnetization (3b) is also expected to diverge, in opposition to the result of mean-field-type calculations[7] or perturbation expansions for the quantum version of the model.[13]

In order to test the validity of (3) we shall now present a numerical study based on transfer-matrix calculations and finite-size scaling. We consider the n-vector model on a lattice with the Hamiltonian ($\beta=1/T$)

$$\beta\mathcal{H}=-\beta\sum_{\langle i,j\rangle}\mathbf{s}_i\cdot\mathbf{s}_j-h\sum_i s_i^1\ , \tag{4}$$

where $\mathbf{s}$ is a n-component classical spin and $\sum_{a=1}^{n}(s^a)^2=n$. The high-temperature expansion for the partition function of (4) in a volume Λ can be analytically continued to $n=0$ (Refs. 9 and 11) giving

$$Z(\beta,h)=\left(1+\frac{h^2}{2}\right)^{\Lambda}\left[1+\sum_{\substack{p\geq 1\\ l\geq p}}\Omega_{pl}h^{2p}\beta^l\left(1+\frac{h^2}{2}\right)^{-p-l}\right]\ . \tag{5}$$

In this formula Ω_{pl} is the number of configurations of p nonintersecting SAW's of total length l on the lattice of Λ

spins. $Z(\beta,h)$ describes thus a grand canonical ensemble of nonintersecting polymers of variable length. The number of monomers is controlled by the fugacity β, and the number of chain extremities by h, a special weight $h^2/2$ being associated to polymers of zero length.[11] Very similar expressions are obtained in other versions of the $n=0$ limit.[4] In this limit, the critical temperature of (4) becomes $T^c=(\beta^c)^{-1}=\mu$, where μ is the connectivity constant of the lattice. I have shown in Refs. 17 and 19 how one can calculate (5) on strips of width L, generalizing to the case of several polymers the transfer matrix of Ref. 20. The configurations at one column are defined by the sites which are connected together and also by the sites which are extremities of a polymer, the other extremity of which is in the left part of the strip. The matrix elements depend now on β and h. If $h=0$ one cannot go from one column to the next by adding extremity sites. The transfer matrix M_L therefore has a "blockwise triangular" structure[19] and the complete spectrum can be obtained by diagonalizing submatrices $M_L^{(p)}$ acting on configurations with p extremity sites which correspond to p polymers present on the strip. In the following we shall denote the largest eigenvalue of $M_L^{(p)}$ by $\lambda_L^{(p)}$. An important property of (5) is the existence of a phase transition on strips of finite width.[21,22] This occurs because the connectivity constant of strips μ_L is nonzero so if $\beta=\beta_L^c=\mu_L^{-1}$, $\lambda_L^{(0)}=1$ and $\lambda_L^{(1)}$ become degenerate. At this point, there is a spontaneous magnetization[7,21] which, however, scales as[17] $L^{-\beta/\nu}=L^{-5/48}$ (where β and ν are the standard order parameter and correlation length exponents for the $n=0$ critical point[16] $\beta=\frac{5}{64}$, $\nu=\frac{3}{4}$) and becomes negligible for L large enough. If $\beta>\beta_L^c$, this spontaneous magnetization disappears while $\lambda_L^{(1)}$ becomes larger than 1, which corresponds to the presence of a finite density of monomer[5,7] $\rho_L=(1/L)\times\partial\ln\lambda_L^{(1)}/\partial\ln\beta$.

These properties are already well known and we shall now investigate what happens if we apply a magnetic field. In this case one must diagonalize the complete transfer matrix M_L since configurations with different numbers of polymers are now coupled. The magnetization is obtained by

$$m_L(\beta,h)=\frac{1}{L}\frac{\partial}{\partial h}\ln\lambda_L(\beta,h)\ , \tag{6}$$

where λ_L is the largest eigenvalue of M_L. I have calculated these magnetizations for the square lattice and strips with periodic boundary conditions. The critical point is known with a good precision[17] $\beta^c\simeq 0.37905$. In the high-temperature (low-β) regime, we have observed for m_L a behavior very similar to the single spin or mean-field results.[7] In Fig. 1(a) there is a typical example for $\beta=0.36$. One sees that the different $m_L(h)$ converge rapidly to a limiting curve which has a smooth maximum $m_{\max}(\beta)$ after which the susceptibility becomes negative.[7] This maximum depends now on β with $m_{\max}(\beta)\geq m_{\max}(\beta=0)=1/\sqrt{2}$ while it is constant and equal to $1/\sqrt{2}$ in the mean field. An example of the low-temperature (high β) regime is given in Fig. 1(b) where $\beta=0.40$. In this case the curves $m_L(h)$ show sharp maxima at values of h which go to zero while the height of these maxima do not stabilize when L increases. This is rather different from the standard picture[3,7,13] where one expects a finite spontaneous magnetization m_{sp} which tends to zero as $\beta-\beta^c$. If this were the case here, the curves $m_L(h)$ should asymptote to m_{sp} in the vicinity of $h=0$, as is observed in the Ising ($n=1$) case.[23] Our results are in much better agreement with (3b). To test this more precisely we have used several methods. We have verified that the $m_L(h)$ take the scaling form $m_L(h)\simeq L^{3/16}F(hL^{35/16})$ expected from (3), and so the maxima of Fig. 1(b) increase like $L^{3/16}$.

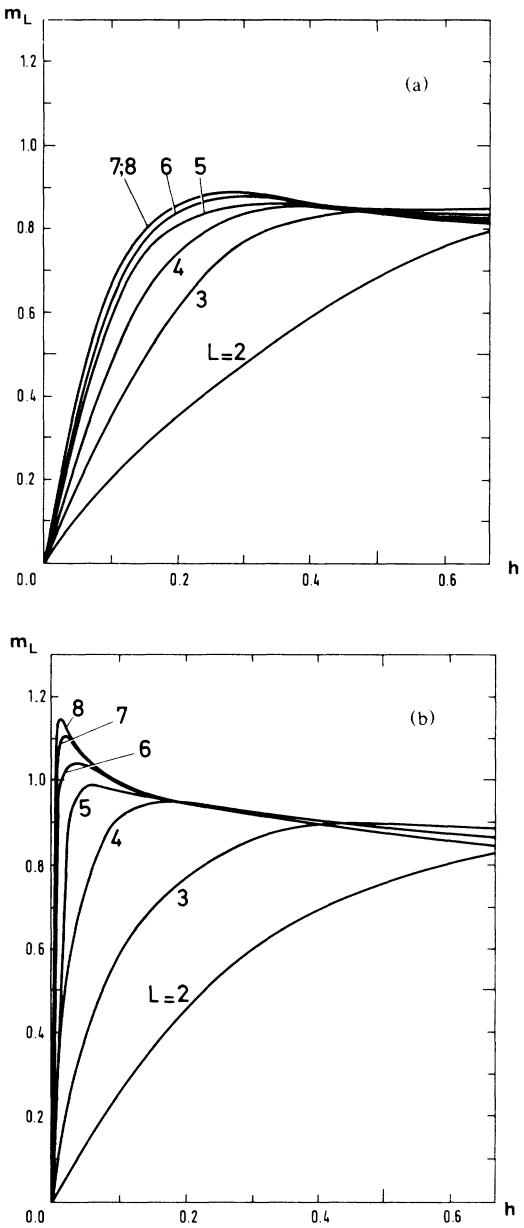

FIG. 1. (a) Magnetizations for a typical high-temperature value $\beta=0.36$. The $m_L(h)$ saturate to a limiting curve which has a smooth maxima after which χ becomes negative. This is similar to the mean-field picture. (b) Magnetizations for $\beta=0.40$. The $m_L(h)$ now have sharp maxima which do not seem to have a limit. This is very different from (a) or from what should be observed if there were a finite spontaneous magnetization.

We have also checked that ξ diverges as in (3a) by applying phenomenological normalization.[23] Our best results, however, have been obtained by studying the amplitudes of the correlation lengths. As Cardy[24] has shown, conformal invariance at the critical point gives the general relation

$$\xi_{L,\phi} \simeq \frac{L}{2\pi x_\phi}, \quad L \to \infty , \tag{7}$$

where x_ϕ is the dimension of any operator ϕ and $\xi_{L,\phi}$ the length of the ϕ correlations on a strip of width L and periodic boundary conditions. Relation (7) has already been used[17,19] to study various properties of (5) at $\beta=\beta^c$, $h=0$. In the low-temperature regime of (5), however (7) cannot hold directly since the dimension x of the spin is expected to be negative. It has been discussed in Ref. 25 how (7) must be modified in such cases. For the spin correlation length this gives the simple result

$$\xi_L \simeq \frac{L}{2\pi |x|}, \quad L \to \infty . \tag{8}$$

At any point $\beta > \beta^c$, $h=0$, ξ_L is given for L large enough by

$$\xi_L = -(\ln \lambda_L^{(2)}/\lambda_L^{(1)})^{-1} . \tag{9}$$

We have thus studied the quantities

$$A_L = \frac{L}{2\pi} \ln \frac{\lambda_L^{(1)}}{\lambda_L^{(2)}} ,$$

which are presented for different widths up to $L=10$ in Fig. 2. (One can work with larger sizes here since $h=0$ and it suffices to diagonalize submatrices of M_L.) For $\beta < \beta^c$, (5) is not critical, ξ_L remains finite, and the different A_L move apart. For $\beta=\beta^c$, we have the standard critical point of the polymer problem. The different A_L cross at a value already studied in Ref. 19, $A_L(\beta^c) = \frac{27}{48}$.

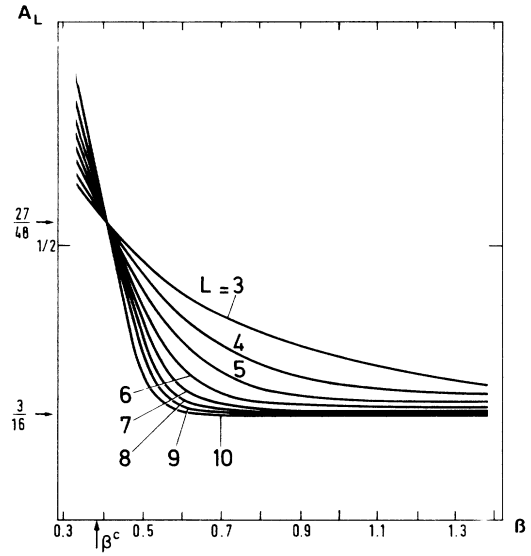

FIG. 2. Scaling behavior of the correlation length. The curves collapse to the expected value $|x| = \frac{3}{16}$ (see the text) in all the low-temperature region.

Finally, for $\beta > \beta^c$, the curves become more and more constant with increasing L, collapsing at the expected value $|x| = \frac{3}{16}$ (this can be checked more precisely with standard extrapolation methods[17]). It must be noted that the study of A_L confirms only the absolute value of x. The behavior of the other physical quantities, like the magnetization (6), shows that x is, in fact, negative, but this can also be seen more directly on the spectrum of M_L. For instance, following the arguments of Ref. 26 one can show that the central charge of the critical region $\beta > \beta^c$, $h=0$ is $C=-2$ (while the standard critical point $\beta=\beta^c$, $h=0$ has $C=0$[19]). As explained in Ref. 25, the presence of a negative dimension can then be detected by studying the scaling behavior of the free energy $f_L = (1/L)\ln\lambda_L^{(1)}$. Defining $\hat{C}$ by

$$f_L \simeq f_0 + \frac{\pi \hat{C}}{6L^2}, \quad L \to \infty , \tag{10}$$

we have checked that $\hat{C} = C - 12x = \frac{1}{4}$ instead of $\hat{C} = C = -2$ which should be observed if x were positive.

Our calculations thus confirm the conjectures (3). Moreover, we find that χ is, in fact, negative in all low-temperature regions. Let us recall that this has no consequence for the stability of the polymer problem. The correct stability criterion is not $\chi \geq 0$ but[14] $\langle p^2 \rangle - \langle p \rangle^2 \geq 0$. This takes a rather complicated form[11] for (5), but it simplifies in the limit $h \to 0$ to give[11,14]

$$\chi + \frac{m}{h} \geq 0, \quad h \to 0 , \tag{11}$$

which is satisfied by (3). Note that (11) requires that m diverges no faster than h^{-1}. A negative susceptibility corresponds to a repulsion between the extremities of polymers in the dense phase.[14,27] If m_{sp} is finite, as in the mean field case,[7] this effect of repulsion is rather weak: When $\langle p \rangle \ll \Lambda$, the extremities behave independently,[14] a defect free energy $\ln m_{sp}$ being associated to each of them,[7] and the distribution of p in the ensemble (5) is Poissonian. This is not what happens in two dimensions where $\langle S_i S_j \rangle$ diverges like $|i-j|^{3/8}$ so m_{sp} is infinite. If we consider one single polymer in Λ and $\beta > \beta^c$, its extremities are never independent but repel each other[27] with a potential $\frac{3}{8}$ ln (distance).

In addition, the distribution of p for $1 \ll \langle p \rangle \ll \Lambda$ is no longer Poissonian, in opposition to what was suggested in Ref. 14. Using (3b) and taking the inverse Laplace transform of (5) we find

$$\mathcal{P}(p) \sim \left[\frac{\langle 2p \rangle^{2p}}{(2p)!} \right]^{35/32} \exp -\tfrac{35}{16} \langle p \rangle, \quad 1 \ll p, \ \langle p \rangle \ll \Lambda . \tag{12}$$

It would be interesting to know whether such behavior extends to $d=3$ or is particular to $d=2$. In any case, the divergence of m does not modify the main result[1,2] of the semidilute regime. If there is a finite spontaneous magnetization m_{sp}, the free energy has the form for $\beta \to \beta^{c\dagger}$ and $h \to 0$ (Δ is the usual gap exponent $\Delta = \frac{91}{64}$ for $d=2$),

$$f \sim (\beta - \beta^c)^{d\nu} + (\beta - \beta^c)^{d\nu - \Delta} h . \tag{13a}$$

A magnetization diverging as $m \sim h^{\zeta-1}(\zeta = \frac{32}{35}$ for $d=2)$ gives

$$f \sim (\beta-\beta^c)^{d\nu} + (\beta-\beta^c)^{d\nu-\zeta\Delta} h^{\zeta} . \tag{13b}$$

However, only the first term is needed to give the osmotic pressure π:

$$\pi \sim \rho^{2\nu/2\nu-1} \tag{14}$$

in the limit where the monomer and polymer concentration are small but the chains are very long.

The results (3) being established, one could now use them for predicting some geometrical properties of dense polymers.[28,29]

[1]P. G. de Gennes, Phys. Lett. **38A**, 339 (1972).
[2]J. des Cloizeaux, J. Phys. (Paris) **36**, 281 (1975).
[3]P. G. de Gennes, *Scaling Concepts in Polymer Physics* (Cornell Univ. Press, Ithaca, NY, 1975).
[4]J. des Cloizeaus and G. Jannink, *Polymères en Solution. Leur Modélisation et Leur Structure* (Editions de Physique, Paris, in press).
[5]S. Redner and J. P. Reynolds, J. Phys. A **14**, L55 (1981).
[6]G. Parisi and N. Sourlas, J. Phys. (Paris) Lett. **17**, 403 (1980).
[7]P. D. Gujrati, Phys. Rev. Lett. **53**, 2453 (1984).
[8]R. B. Griffiths and P. D. Gujrati, J. Stat. Phys. **30**, 563 (1983).
[9]J. C. Wheeler and P. Pfeuty, Phys. Rev. A **23**, 1531 (1981).
[10]J. C. Wheeler and P. Pfeuty, Phys. Rev. A **24**, 1050 (1981).
[11]P. D. Gujrati, Phys. Rev. A **24**, 2096 (1981).
[12]P. D. Gujrati, Phys. Rev. B **25**, 3381 (1982).
[13]M. A. Moore and C. A. Wilson, J. Phys. A **13**, 3501 (1980).
[14]S. P. Obukhov, J. Phys. A **15**, L211 (1982).
[15]D. J. Wallace, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, London, 1976), Vol. 6.
[16]B. Nienhuis, Phys. Rev. Lett. **49**, 1062 (1982).
[17]H. Saleur and B. Derrida, J. Stat. Phys. **44**, 225 (1986).
[18]M. E. Fisher, Phys. Rev. Lett. **40**, 1610 (1978).
[19]H. Saleur, J. Phys. A (to be published).
[20]B. Derrida, J. Phys. A **14**, L5 (1981).
[21]R. Balian and G. Toulouse, Ann. Phys. (N.Y.) **83**, 28 (1974).
[22]P. D. Gujrati, Phys. Rev. B **31**, 4375 (1985).
[23]M. N. Barber, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and J. L. Lebowitz (Academic, London, 1983), Vol. 8.
[24]J. L. Cardy, J. Phys. A **16**, L385 (1983).
[25]C. Itzykson, H. Saleur, and J. B. Zuber, Europhys. Lett. **2**, 91 (1986).
[26]H. Blöte, M. P. Nightingale, and J. L. Cardy, Phys. Rev. Lett. **56**, 742 (1986).
[27]G. Gaspari and J. Rudnick, Phys. Rev. B **33**, 3295 (1986).
[28]H. Saleur (unpublished).
[29]B. Duplantier (unpublished).