Resistance fluctuations in randomly diluted networks

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The resistance $R(\mathbf{x},\mathbf{x}')$ between two connected terminals in a randomly diluted resistor network is studied on a d-dimensional hypercubic lattice at the percolation threshold p_c. When each individual resistor has a small random component of resistance, $R(\mathbf{x},\mathbf{x}')$ becomes a random variable with an associated probability distribution, which contains information on the distribution of currents in the individual resistors. The noise measured between the terminals may be characterized by the cumulants $M_q(\mathbf{x},\mathbf{x}')$ of $R(\mathbf{x},\mathbf{x}')$. When averaged over configurations of clusters, $\overline{M}_q(\mathbf{x},\mathbf{x}')\sim|\mathbf{x}-\mathbf{x}'|^{\tilde{\psi}(q)}$. We construct low-concentration series for the generalized resistive susceptibility, $\chi^{(q)}$, associated with $\overline{M}_q$, from which the critical exponents $\tilde{\psi}(q)$ are obtained. We prove that $\tilde{\psi}(q)$ is a convex monotonically decreasing function of q, which has the special values $\tilde{\psi}(0)=D_B$, $\tilde{\psi}(1)=\tilde{\zeta}_R$, and $\tilde{\psi}(\infty)=1/\nu$. ($D_B$ is the fractal dimension of the backbone, $\tilde{\zeta}_R$ is the usual scaling exponent for the average resistance, and ν is the correlation-length exponent.) Using the convexity property and the accepted values of these three exponents, we construct two approximant functions for $\psi(q)=\tilde{\psi}(q)\nu$, both of which agree with the series results for all $q>1$ and with existing numerical simulations. These approximants enabled us to obtain explicit approximate forms for the multifractal functions $\alpha(q)$ and $f(q)$ which, for a given q, characterize the scaling with size of the dominant value of the current and the number of bonds having this current. This scaling description fails for sufficiently large negative q, when the dominant (small) current decreases exponentially with size. In this case $\chi^{(q)}$ diverges at a lower threshold $p^*(q)$, which vanishes as $q\to-\infty$.

I. INTRODUCTION

The simplest geometrical properties of percolation clusters formed when each bond can randomly be either occupied with probability p or vacant with probability $1-p$ are well known. The existence of a divergent correlation length ξ as p approaches the threshold value for percolation, p_c, occurs in close analogy with thermodynamic phase transitions.[1-3] If occupied bonds are assigned unit resistance and unoccupied bonds infinite resistance, the bulk conductivity of such a random network can be formulated in terms of the percolation correlation length and a crossover exponent $\tilde{\zeta}_R$. The average resistance between two connected points $\mathbf{x}$ and $\mathbf{x}'$ on the random network scales as[4,5] $\overline{R}(\mathbf{x},\mathbf{x}')\sim|\mathbf{x}-\mathbf{x}'|^{\tilde{\zeta}_R}$, and the exponent $\tilde{\zeta}_R$ has been intensively studied for several years.[4-10] Recently Rammal *et al.*[11] have considered fluctuations in the resistance $R(\mathbf{x},\mathbf{x}')$ in a network due to fluctuations in the resistances, r_b, of single occupied bonds. Subsequently, de Arcangelis *et al.*[12] have focused attention on the distribution of currents that occurs in a two-terminal measurement. The fluctuations in resistance are directly related to the distribution of currents. An infinite set of exponents was introduced to describe either of these phenomena. The principal aim of this paper is to present a comprehensive study of the properties of this set of exponents.

To define these exponents consider a two-terminal measurement on an arbitrary fixed cluster in which a current I is injected at a source site $\mathbf{x}$ and removed at a sink site $\mathbf{x}'$. This imposed current will give rise to a distribution of currents in bonds covering a region associated with $\mathbf{x}$ and $\mathbf{x}'$. A simple way of characterizing this distribution involves its moments,[12] $\{M_q\}$:

$$M_q(\mathbf{x},\mathbf{x}')=\sum_b (I_b/I)^{2q}\equiv\sum_b i_b^{2q}\ , \tag{1.1}$$

where $\sum_b$ indicates a sum over all bonds b which carry a nonzero current I_b which depends implicitly on $\mathbf{x}$ and $\mathbf{x}'$. An equivalent formulation may be given in terms of the distribution of $R(\mathbf{x},\mathbf{x}')$ in the case when the resistance of each bond b independently has a small random component δr_b. Averages over the distribution of r_b will be denoted $\langle\ \rangle$ and cumulant averages, $\langle\ \rangle_c$. To lowest order in δr_b we write the cumulant average of the qth power of the resistance as[11]

$$\langle\delta R(\mathbf{x},\mathbf{x}')^q\rangle_c=\left\langle\left[\sum_b[\partial R(\mathbf{x},\mathbf{x}')/\partial r_b]\delta r_b\right]^q\right\rangle_c \tag{1.2a}$$

$$=\sum_b\{[\partial R(\mathbf{x},\mathbf{x}')/\partial r_b]^q\langle\delta r_b^q\rangle_c\}\ , \tag{1.2b}$$

where $\delta R(\mathbf{x},\mathbf{x}')=R(\mathbf{x},\mathbf{x}')-\langle R(\mathbf{x},\mathbf{x}')\rangle$. With the use of[13] $\partial R(\mathbf{x},\mathbf{x}')/\partial r_b=i_b^2$ one sees that Eqs. (1.1) and (1.2) are

essentially the same:

$$\langle \delta R(\mathbf{x},\mathbf{x}')^q \rangle_c = M_q(\mathbf{x},\mathbf{x}') \langle \delta r_b^q \rangle_c \equiv \delta_q(\mathbf{x},\mathbf{x}') \ . \tag{1.2c}$$

Rammal *et al.*[11] have suggested that such resistance fluctuations may provide a model from which the spectrum of noise in a two-terminal measurement can be calculated. They relate the noise to the variance of the resistance, summed over $\mathbf{x}$ and $\mathbf{x}'$ and then averaged over percolation clusters. For this purpose they were led to generalize the usual resistive susceptibility[6] to a qth-order resistance-fluctuation susceptibility (per site), as

$$\chi^{(q)}(p) = \sum_{\mathbf{x}'} [\nu(\mathbf{x},\mathbf{x}') M_q(\mathbf{x},\mathbf{x}')]_{\rm av} \ , \tag{1.3}$$

where $\nu(\mathbf{x},\mathbf{x}')$ is the indicator function for percolation: i.e., it is unity if sites $\mathbf{x}$ and $\mathbf{x}'$ are in the same cluster and is zero otherwise, and $[\ ]_{\rm av}$ indicates an average over all configurations of occupied and unoccupied bonds. Critical exponents can be defined in association with each moment via

$$\overline{M}_q \equiv [\nu(\mathbf{x},\mathbf{x}') M_q(\mathbf{x},\mathbf{x}')]_{\rm av} / [\nu(\mathbf{x},\mathbf{x}')]_{\rm av} \sim |\mathbf{x}-\mathbf{x}'|^{\tilde{\psi}(q)}, \quad 1 \ll |\mathbf{x}-\mathbf{x}'| \ll \xi \ , \tag{1.4}$$

where ξ is the percolation correlation length: $\xi \sim |p_c - p|^{-\nu}$. From Eq. (1.4) it follows that

$$\chi^{(q)}(p) \sim |p_c - p|^{-\gamma - \psi(q)} \ , \tag{1.5}$$

where γ is the susceptibility exponent for percolation and $\psi(q) = \tilde{\psi}(q)\nu$. Throughout this paper we shall follow the convention that exponents with tildes are equal to those without tildes divided by ν.

Some special values of q have readily identifiable physical significance.[12] For example, $\overline{M}_1(\mathbf{x},\mathbf{x}')$ gives the average resistance between sites $\mathbf{x}$ and $\mathbf{x}'$. Since $\overline{M}_0(\mathbf{x},\mathbf{x}')$ counts the average number of sites in the backbone, $\tilde{\psi}(0)$ gives the fractal dimensionality of the backbone D_B. Also, $M_\infty(\mathbf{x},\mathbf{x}')$ counts the number of singly connected bonds, so that $\tilde{\psi}(\infty)$ can be identified as the fractal dimension of the set of such singly connected bonds (through which the full current I flows). In summary, then, we have the following limits:

$$\lim_{q\to 0} \tilde{\psi}(q) = D_B \ , \tag{1.6a}$$

$$\lim_{q\to 1} \tilde{\psi}(q) = \tilde{\zeta}_R \ , \tag{1.6b}$$

where $\tilde{\zeta}_R$ is the usual ohmic resistivity crossover exponent;[6-10] and[14,15]

$$\lim_{q\to\infty} \tilde{\psi}(q) = \frac{\ln N_{\rm SC}}{\ln |\mathbf{x}-\mathbf{x}'|} = 1/\nu \ , \tag{1.6c}$$

where $N_{\rm SC}$ is the number of singly connected bonds.

Some analytic and numerical results concerning these moments or the $\psi(q)$'s have already appeared. For instance, upper and lower bounds for $\psi(2)$ have been given by Wright *et al.*[16] and by Tremblay and Feng.[17] These exponents have been studied using numerical simulation[12,18,19] in two and three spatial dimensions d. For d near six, Park *et al.*[20] have given results to first order in $\epsilon = 6-d$ for $\psi(q)$, for arbitrary q.

In this paper we present a number of new results concerning $\psi(q)$. In particular, in Sec. II we derive a convexity relation which these exponents must satisfy. We also show that all the central moments can be given in terms of $\psi(2)$ and $\psi(3)$ only. In Sec. III we give the results of the determination of $\psi(q)$ from series expansions in spatial dimensions d for $2 \le d \le 6$. These results represent the first study of $\psi(q)$ for $d > 3$ and agree within the limit of their accuracy with the results of the ϵ expansion as well as with available results for $d=2$ and $d=3$. In view of our increasing knowledge concerning these exponents, in Sec. IV we construct approximants which are constrained to reproduce the accepted values for the limiting cases in Eq. (1.6). These approximants reproduce our series values for all $q > 1$, and provide convenient representations for $\psi(q)$. We use these approximants to facilitate a discussion, given in Sec. V, of the application of the multifractal formalism to the $\psi(q)$. In Sec. VI this formalism is shown to break down for negative q, and in particular the threshold concentration p^* at which $\chi^{(q)}$ diverges is shown to depend on q and to vanish as $q \to -\infty$. Our conclusions are briefly summarized in Sec. VII.

II. ANALYTIC PROPERTIES OF CRITICAL EXPONENTS

In this section we discuss various relations which the critical exponents $\psi(q)$ must satisfy. Perhaps the most important of these is the convexity relation,

$$\psi(k) + \psi(m) \ge 2\psi\left[\frac{k+m}{2}\right] \ . \tag{2.1}$$

We also show that the central moments of the resistance fluctuations, when each individual bond has a small random component, can be characterized by $\psi(2)$ and $\psi(3)$ only.

We start by discussing some simple properties of the $\psi(q)$'s. We may write the definition of $\tilde{\psi}(q)$ as

$$\tilde{\psi}(q) = \lim_{|\mathbf{x}-\mathbf{x}'|\to\infty} \ln \left[\sum_b i_b^{2q}\right]_{\rm av} \Big/ \ln |\mathbf{x}-\mathbf{x}'| \ , \quad (p = p_c) \ , \tag{2.2}$$

in the notation of Eq. (1.1). Since $i_b \le 1$, it is clear that $\tilde{\psi}(q)$ is a nonincreasing function of q. Thus, as was shown by Park *et al.*,[20]

$$D_B \ge \tilde{\zeta}_R \ge \tilde{\psi}(2) \ge \tilde{\psi}(3) \ge \cdots \ge 1/\nu \ . \tag{2.3}$$

Alternatively, the monotonicity of $\psi(q)$ follows from the expression for $d\psi(q)/dq$:

$$\frac{d\tilde{\psi}(q)}{dq} = \lim_{|\mathbf{x}-\mathbf{x}'|\to\infty} 2 \left[\frac{\sum_b i_b^{2q} \ln i_b}{\sum_b i_b^{2q}}\right] \Big/ \ln |\mathbf{x}-\mathbf{x}'| \ . \tag{2.4a}$$

The second derivative is

$$\frac{d\tilde{\psi}^2(q)}{dq^2}=\lim_{|\mathbf{x}-\mathbf{x}'|\to\infty}4\left[\frac{\sum_b i_b^{2q}(\ln i_b)^2}{\sum_b i_b^{2q}}-\left[\frac{\sum_b i_b^{2q}\ln i_b}{\sum_b i_b^{2q}}\right]^2\right]\Big/\ln|\mathbf{x}-\mathbf{x}'| \ . \tag{2.4b}$$

Since this quantity can be interpreted as the variance of $\ln i$ within the distribution defined by i_b^2, it is clear that $\psi(q)$ has a positive second derivative, i.e., it is a convex function of q. From Eq. (2.3) it is immediately deducible that

$$\tfrac{1}{2}\tilde{\zeta}_R\le\tilde{P}\le\tilde{\zeta}_R-1/(2\nu) \ , \tag{2.5}$$

where $\tilde{P}=\tilde{\psi}(1)-\frac{1}{2}\tilde{\psi}(2)$ was introduced by Rammal *et al.*[11] to characterize the relative noise. This exponent is essentially equivalent to the $n=4$ member of a generalized set of exponents $p(n)$ introduced by de Arcangelis *et al.*,[19] which in our notation is given by

$$p(n)=n\psi(1)-\psi(n/2) \ . \tag{2.6}$$

In terms of the $p(n)$'s Eq. (2.3) is

$$p(2n-2)+2\psi(1)\le p(2n)\le p(2n+2)-2\psi(1) \ . \tag{2.7a}$$

The bound $\psi(n)\ge 1$ becomes

$$p(2n)\le 2n\psi(1)-1 \quad \text{or} \quad p(2n)/\nu\le 2n\tilde{\psi}(1)-1/\nu \ . \tag{2.7b}$$

Since $\tilde{\psi}(1)=p(2)/\nu$ and $\nu=4/3$ for[21] $d=2$, one sees that the bound is satisfied for $n=2$ but violated for $n=3$ by the estimates of Ref. 19: $p(2)/\nu=0.976$, $p(4)/\nu=3.12$, and $p(6)/\nu=5.15$. [If their quoted uncertainties are taken into account, their values just barely satisfy Eq. (2.7b) for $n=3$].

Next we prove Eq. (2.1). For this purpose recall the Schwartz inequality:

$$\left[\sum_{j=1}^n a_j^2\right]\left[\sum_{j=1}^n b_j^2\right]\ge\left[\sum_{j=1}^n a_j b_j\right]^2 , \tag{2.8}$$

where equality holds only if the vectors **a** and **b** are parallel. Assigning the values $(i_j)^k$ to a_j and $(i_j)^m$ to b_j (where j here labels the bonds), we have

$$\ln\left[\sum_j i_j^{2k}\right]+\ln\left[\sum_j i_j^{2m}\right]\ge 2\ln\left[\sum_j i_j^{k+m}\right] , \tag{2.9}$$

which in conjunction with Eq. (2.2) implies Eq. (2.1). The equality holds in Eq. (2.1) only if $i_b=1$ for all bonds which contribute to the sums in Eq. (2.9). For $d>6$ this happens because the blobs are irrelevant.[22,7] Thus the sums are dominated by the contributions from singly connected bonds, for which $i_b=1$. For $d<6$ the equality only holds in the limit when k and $m\to\infty$. By generalizing Eq. (2.4) we see that all the derivatives $d^m\psi(q)/dq^m$ vanish in the limit $q\to\infty$. This situation is equivalent to having equality in Eq. (2.1). In view of Eq. (2.6), the convexity of $\psi(q)$ implies that $p(q)$ is concave:

$$p(k)+p(m)\le 2p\left[\frac{k+m}{2}\right] . \tag{2.10}$$

Next we consider the average central moments in the case where each bond has independently a small random component of resistance δr_b. We define the "central moments" $\bar{\mu}_n$ by

$$\bar{\mu}_n(\mathbf{x},\mathbf{x}')=[\nu(\mathbf{x},\mathbf{x}')\langle\delta R(\mathbf{x},\mathbf{x}')^N\rangle]_{\rm av}/[\nu(\mathbf{x},\mathbf{x}')]_{\rm av} \ , \tag{2.11}$$

where $\langle\ \rangle$ denotes an average over δr_b. In terms of the cumulants δ_n, similarly defined in Eq. (1.2c) we have

$$\bar{\mu}_2(\mathbf{x},\mathbf{x}')=\bar{\delta}_2(\mathbf{x},\mathbf{x}') \ , \tag{2.12a}$$

$$\bar{\mu}_3(\mathbf{x},\mathbf{x}')=\bar{\delta}_3(\mathbf{x},\mathbf{x}') \ , \tag{2.12b}$$

$$\bar{\mu}_4(\mathbf{x},\mathbf{x}')=\bar{\delta}_4(\mathbf{x},\mathbf{x}')+3\frac{[\nu(\mathbf{x},\mathbf{x}')\langle\delta R(\mathbf{x},\mathbf{x}')^2\rangle_c^2]_{\rm av}}{[\nu(\mathbf{x},\mathbf{x}')]_{\rm av}} \ . \tag{2.12c}$$

Apart from corrections to scaling, the distribution of $R(\mathbf{x},\mathbf{x}')$ associated with the average, $[\ ]_{\rm av}$, over cluster configurations is a scaling function described by a single exponent.[9,20,23] Thus, in counting powers of $L\equiv|\mathbf{x}-\mathbf{x}'|$ it is permissible to decouple the averages in Eq. (2.12c):

$$\frac{[\nu(\mathbf{x},\mathbf{x}')\langle\delta R(\mathbf{x},\mathbf{x}')^l\rangle_c^k]_{\rm av}}{[\nu(\mathbf{x},\mathbf{x}')]_{\rm av}}=\left[\frac{[\nu(\mathbf{x},\mathbf{x}')\langle\delta R(\mathbf{x},\mathbf{x}')^l\rangle_c]_{\rm av}}{[\nu(\mathbf{x},\mathbf{x}')]_{\rm av}}\right]^k=\bar{\delta}_l^{\,k} \ . \tag{2.13}$$

More generally for percolation we expect to be able to write

$$[\nu(\mathbf{x},\mathbf{x}')AB]_{\rm av}=[\nu(\mathbf{x},\mathbf{x}')A]_{\rm av}[\nu(\mathbf{x},\mathbf{x}')B]_{\rm av}/[\nu(\mathbf{x},\mathbf{x}')]_{\rm av} \ ,$$

where A and B are functions of $R(\mathbf{x},\mathbf{x}')$. Thus we have

$$\bar{\mu}_4(\mathbf{x},\mathbf{x}')=\bar{\delta}_4(\mathbf{x},\mathbf{x}')+\frac{3[\nu(\mathbf{x},\mathbf{x}')\langle R(\mathbf{x},\mathbf{x}')^2\rangle_c]_{\rm av}^2}{[\nu(\mathbf{x},\mathbf{x}')]_{\rm av}^2} \tag{2.12c'}$$

$$=\bar{\delta}_4(\mathbf{x},\mathbf{x}')+3\bar{\delta}_2(\mathbf{x},\mathbf{x}')^2 \ . \tag{2.12c''}$$

Now consider the scaling behavior of this result. The nth cumulant scales[20] as $\bar{\delta}_n\sim L^{\tilde{\psi}(n)}$. Thus the two terms in Eq. (2.12c'') scale as $L^{\tilde{y}_i}$, with

$$y_1=\psi(4), \quad y_2=2\psi(2) \ , \tag{2.14}$$

so that the second term dominates. The decoupling of Eq. (2.13) can be invoked for general n to recover an expansion of the usual form for $\bar{\mu}_n$ in terms of $\bar{\delta}_n$:

$$\bar{\mu}_n(\mathbf{x},\mathbf{x}')=\sum_j c_j\prod_l[\bar{\delta}_{m(l)}]^{h(l)} \ , \tag{2.15}$$

where the index j labels the partitions of n of the form $n=\sum_l m(l)h(l)$. As in Eq. (2.12c''), each term in Eq. (2.15) scales with a different power of L. In particular, the jth term in Eq. (2.15) has a scaling exponent given by

$$y_j=\sum_l \psi[m(l)]h(l) . \tag{2.16}$$

First consider the even central moments, i.e., $n=2k$. Note that $m(l)$ in Eq. (2.16) is greater than one and also that $\psi(l)$, and therefore also $\psi(l)/l$, decrease with increasing l. Thus we may write

$$\sum_l \psi[m(l)]h(l) \le \tfrac{1}{2}\psi(2)\sum_l m(l)h(l)=k\psi(2) . \tag{2.17}$$

All the even moments contain the term $\bar{\delta}_2^k$, which by Eq. (2.17) is the dominant one at large L, so that

$$\bar{\mu}_{2k}(\mathbf{x},\mathbf{x}')\sim L^{k\tilde{\psi}(2)} . \tag{2.18}$$

In a similar manner one can show that the dominant term in the $(2k+1)$th central moment is that proportional to $\bar{\delta}_3\bar{\delta}_2^{k-1}$, so that

$$\bar{\mu}_{2k+1}(\mathbf{x},\mathbf{x}')\sim L^{(k-1)\tilde{\psi}(2)+\tilde{\psi}(3)} . \tag{2.19}$$

Thus the central moments depend only on $\psi(2)$ and $\psi(3)$.

III. SERIES RESULTS

We constructed low density series for the qth susceptibility defined in Eq. (1.3), above. To do this we literally followed the prescription implied by Eq. (1.3): For a given cluster of bonds and a given location on that cluster of the source and sink at which unit current was put in and taken out, respectively, we solved Kirchhoff's circuit equations to find the current in every bond of the cluster. We could then calculate arbitrary moments of the current distribution for that arrangement of source, sink, and cluster. We then summed the $2q$th moment of the bond currents over positions of the source and sink, as indicated in Eqs. (1.1) and (1.3). To perform the configurational average we could, in principle, weight each cluster with the appropriate factors of p and $1-p$ for occupied bonds and vacant perimeter bonds, respectively. However, to avoid counting perimeter bonds, we had recourse to the usual cumulant subtraction, namely we define the cumulant susceptibilities via

$$\chi_c(\Gamma)=\chi_0(\Gamma)-\sum_{\gamma\in\Gamma}\chi_c(\gamma) , \tag{3.1}$$

where $\gamma\in\Gamma$ means that γ is a subdiagram of Γ (but not equal to Γ), and $\chi_0(\Gamma)$ is the "bare" susceptibility for the

TABLE I. Series coefficients $c_{k,l}^{(2)}$ defined in Eq. (3.3b). For this tabulation we write the coefficients of $F_k(d)=\sum_{l=1,k} c_{k,l}^{(2)}d^l$ in decreasing powers of d starting from d^k. Thus $F_4(d)=32d^4-48d^3+4.90625d^2+15.09375d$.

k	$F_k(d)$		
1	$0.100\,000\,000\,000\,00\times10^{1}$		
2	$0.400\,000\,000\,000\,00\times10^{1}$	$-0.200\,000\,000\,000\,00\times10^{1}$	
3	$0.120\,000\,000\,000\,00\times10^{2}$	$-0.120\,000\,000\,000\,00\times10^{2}$	$0.300\,000\,000\,000\,00\times10^{1}$
4	$0.320\,000\,000\,000\,00\times10^{2}$ $0.150\,937\,500\,000\,00\times10^{2}$	$-0.480\,000\,000\,000\,00\times10^{2}$	$0.490\,625\,000\,000\,00\times10^{1}$
5	$0.800\,000\,000\,000\,00\times10^{2}$ $0.142\,750\,000\,000\,00\times10^{3}$	$-0.160\,000\,000\,000\,00\times10^{3}$ $-0.813\,750\,000\,000\,00\times10^{2}$	$0.236\,250\,000\,000\,00\times10^{2}$
6	$0.192\,000\,000\,000\,00\times10^{3}$ $0.324\,888\,888\,888\,89\times10^{3}$	$-0.480\,000\,000\,000\,00\times10^{3}$ $0.164\,416\,666\,666\,67\times10^{3}$	$0.130\,875\,000\,000\,00\times10^{3}$ $-0.326\,180\,555\,555\,55\times10^{3}$
7	$0.448\,000\,000\,000\,00\times10^{3}$ $0.761\,305\,555\,555\,55\times10^{3}$ $0.291\,875\,810\,555\,55\times10^{4}$	$-0.134\,400\,000\,000\,00\times10^{4}$ $0.256\,442\,670\,000\,00\times10^{4}$	$0.589\,000\,000\,000\,00\times10^{3}$ $-0.593\,049\,036\,111\,11\times10^{4}$
8	$0.102\,400\,000\,000\,00\times10^{4}$ $0.137\,116\,666\,666\,67\times10^{4}$ $-0.313\,180\,381\,972\,23\times10^{5}$	$-0.358\,400\,000\,000\,00\times10^{4}$ $0.380\,695\,332\,777\,78\times10^{4}$ $0.225\,498\,378\,750\,00\times10^{5}$	$0.224\,850\,000\,000\,00\times10^{4}$ $0.354\,958\,032\,777\,78\times10^{4}$
9	$0.230\,400\,000\,000\,00\times10^{4}$ $0.356\,244\,444\,444\,44\times10^{4}$ $-0.365\,754\,840\,208\,56\times10^{6}$	$-0.921\,600\,000\,000\,00\times10^{4}$ $0.400\,374\,540\,000\,00\times10^{4}$ $0.460\,790\,969\,836\,53\times10^{6}$	$0.766\,200\,000\,000\,00\times10^{4}$ $0.834\,104\,501\,619\,64\times10^{5}$ $-0.186\,753\,769\,634\,37\times10^{6}$
10	$0.512\,000\,000\,000\,00\times10^{4}$ $0.585\,911\,111\,111\,11\times10^{4}$ $-0.960\,128\,571\,086\,67\times10^{5}$ $-0.217\,591\,530\,173\,33\times10^{7}$	$-0.230\,400\,000\,000\,00\times10^{5}$ $-0.146\,205\,393\,333\,33\times10^{4}$ $-0.194\,854\,961\,876\,93\times10^{7}$	$0.240\,860\,000\,000\,00\times10^{5}$ $0.158\,715\,424\,736\,80\times10^{6}$ $0.405\,120\,929\,569\,70\times10^{7}$
11	$0.112\,640\,000\,000\,00\times10^{5}$ $0.407\,866\,666\,666\,67\times10^{4}$ $0.240\,872\,214\,229\,67\times10^{7}$ $-0.447\,190\,165\,753\,46\times10^{8}$	$-0.563\,200\,000\,000\,00\times10^{5}$ $-0.266\,746\,417\,777\,78\times10^{5}$ $-0.200\,210\,407\,707\,34\times10^{8}$ $0.155\,033\,995\,771\,63\times10^{8}$	$0.713\,280\,000\,000\,00\times10^{5}$ $0.307\,350\,654\,432\,55\times10^{6}$ $0.465\,169\,199\,473\,00\times10^{8}$

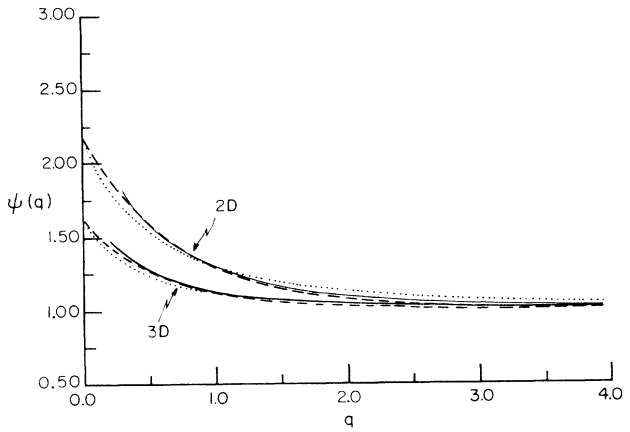

FIG. 1. Series results for $\psi(q)$ vs q for $d=2$ and $d=3$ spatial dimensions. The solid curve gives our results, the dashed curve is from the approximant of Eq. (4.1a), and the dotted curve is that from Eq. (4.1b).

cluster Γ:

$$\chi_0^{(q)}(\Gamma)=\sum_{\mathbf{x},\mathbf{x}'\in\Gamma}\sum_b i_b^{2q}=\sum_{\mathbf{x},\mathbf{x}'\in\Gamma} M_q(\mathbf{x},\mathbf{x}')\ , \tag{3.2}$$

where $M_q(\mathbf{x},\mathbf{x}')$ is calculated according to Eq. (1.1) and i_b is the current in bond b when a unit current is inserted at $\mathbf{x}$ and removed at $\mathbf{x}'$. Since the solution to Kirchhoff's equations only depends on the topology of the cluster, we only need to calculate $\chi_0^{(q)}(\Gamma)$ for diagrams Γ which are topologically inequivalent. This calculation is done once, and then used for all dimensions. Let $w(\Gamma;d)$ be the number of diagrams per site which are topologically equivalent to Γ in dimension d. $w(\Gamma;d)$ is usually referred to as the weak embedding constant. Summing only over inequivalent diagrams, one has

$$\chi^{(q)}(p)=\sum_\Gamma w(\Gamma;d)\chi_c^{(q)}(\Gamma)p^{n_b(\Gamma)} \tag{3.3a}$$

$$=\sum_{k,l} c_{k,l}^{(q)}p^k d^l\ , \tag{3.3b}$$

where $n_b(\Gamma)$ is the number of bonds in the diagram Γ. This series was calculated up to and including all diagrams with 11 bonds. For $q=2$ the coefficients $c_{k,l}^{(2)}$ are given in Table I.

The results were analyzed using generalized Padé approximants in the manner described previously.[24] We obtained series for arbitrary values of q in the range $0<q<10$ from which we obtained the values of $\psi(q)+\gamma$. We then subtracted the value of γ from the analysis of the percolation series of the same order, so as to exclude any systematic errors. The value of γ we used were $\gamma=2.41,1.81,1.40,1.16$ for spatial dimension $d=2,3,4,5$, respectively. For $d=2$ and $d=3$ our results are shown in Fig. 1. For $q=2$, 3, and 4 the results in dimensions 2 and 3 are given in Table II. For $d>3$ these exponents are hardly distinguishable from unity within the accuracy of our determination. Our results for $q\to 0$ in $d=2$ are consistent with series (of nine terms) results for the backbone fractal dimension obtained by Hong and Stanley.[25] These series become less useful as $q\to 0$ because the series are not long enough to contain large blobs which determine the properties of the backbone. It appears that more terms in the series will be required to obtain results consistent with recent numerical simulations for the backbone.[26–28] Nevertheless, our results are qualitatively reasonable and are consistent both with the convexity property of Eq. (2.1) as well as with the monotonic behavior expressed by Eq. (2.3). Since our results for $q=1$ agree with previous determinations of $\tilde{\zeta}_R$ using various techniques, and since the blobs become less important for increasing q, we conclude that our series estimates are quite reliable for $q\geq 1$. Our results can be compared with the ϵ expansion results of Park *et al.*,[20] who give, for $d=6-\epsilon$,

$$\psi(q)=1+\frac{\epsilon}{7(q+1)(2q+1)}\ . \tag{3.4}$$

Although a direct comparison of our data for $q>1$ with Eq. (3.4) is inconclusive, we will see indirect support for the form of the result, Eq. (3.4), in the next section.

IV. APPROXIMANTS FOR $\psi(q)$

The simple analytic properties of the function $\psi(q)$, i.e., monotonicity and convexity, the identification of the three values $\psi(0)$, $\psi(1)$, and $\psi(\infty)$ with exponents which have been determined previously, and the fact that all the derivatives of $\psi(q)$ approach zero as $q\to\infty$, all restrict the possible functional dependence of ψ on q. It is therefore tempting to construct simple approximate functional forms for $\psi(q)$. Such analytic approximants are very useful, both for getting rough easy estimates of exponents and for algebraic manipulation (see below). Similar approximants have been used widely in critical phenomena. They were recently proposed for percolation, in Refs. 12 and 29. We have constructed two such approximants, i.e.,

$$\psi(q)=1+(\nu D_B-1)^{1-q}(\zeta_R-1)^q \tag{4.1a}$$

and

$$\psi(q)=1+\frac{a}{(q+1)(q+b)}\ . \tag{4.1b}$$

In each case the parameters were adjusted to reproduce the best available data for the cases, $q=0$ (backbone), $q=1$ (resistance crossover), and $q=\infty$ (singly connected bonds). The form in Eq. (4.1a) obviously reproduces the values νD_B, ζ_R, and 1 for $q=0$, 1, and ∞, respectively. This form was chosen because it imitates the expression for $\psi(q)$ on the simple Mandelbrot-Given fractal model,[30] which has been used successfully[29,31] for the nonlinear[32] resistor network. The second form was chosen to imitate the form found in the ϵ expansion, Eq. (3.4). The parameters a and b were adjusted to fit the values of νD_B and ζ_R. The values used as input, and the resulting values of a and b are listed in Table III (Refs. 32 and 33) for dimensions $2\leq d\leq 6$.

The resulting values of $\psi(q)$, from both Eqs. (4.1a) and (4.1b), are compared with our series results in Fig. 1 for $d=2$ and $d=3$. The agreement for $q\geq 1$ is satisfactory.

TABLE II. Comparison of values of $\psi(n)$ from series, approximants, numerical simulations, and the ϵ expansion.

d	Method	$\psi(2)$	$\psi(3)$	$\psi(4)$	$-\psi(1)'$
2	Eq. (4.1a)	1.076±0.007	1.020 ±0.000 03	1.0005±0.0001	0.40 ±0.01
	Eq. (4.1b)	1.13 ±0.01	1.05 ±0.02	1.02 ±0.02	0.29 ±0.02
	Series[a]	1.10 ±0.08	1.04 ±0.08	1.02 ±−0.08	0.36 ±0.08
	Simul[b]	1.04 ±0.09	0.95 ±0.09		
	Simul[c]	1.08 ±0.03	1.03 ±0.04	0.99 ±0.03	
	(4.2)[d]				0.44 ±0.02
3	Eq. (4.1a)	1.024±0.002	1.0046±0.0004	1.000 9±0.000 07	0.20 ±0.04
	Eq. (4.1b)	1.05 ±0.02	1.03 ±0.01	1.017 ±0.007	0.13 ±0.03
	Series[a]	1.05 ±0.03	1.02 ±0.03	1.01 ±0.03	0.16 ±0.03
	(4.2)[d]				0.18 ±0.06
4	Eq. (4.1a)	1.008±0.002	1.0012±0.0002	1.000 19±0.000 03	0.09 ±0.05
	Eq. (4.1b)	1.02 ±0.02	1.01 ±0.01	1.007 ±0.005	0.06 ±0.03
	(4.2)[d]				0.10 ±0.04
5	Eq. (4.1a)	1.0025±0.0007	1.0003±0.0001	1.000 04±0.000 01	0.041±0.04
	Eq. (4.1b)	1.01 ±0.01	1.01 ±0.01	1.003 ±0.007	0.02 ±0.03
	(4.2)[d]				0.04 ±0.04
$6-\epsilon$	Eq. (4.1a)	$1+\frac{\epsilon}{252}$	$1+\frac{\epsilon}{1512}$	$+\frac{\epsilon}{9072}$	$\frac{\ln 6}{42}=0.04\epsilon$
	Eq. (4.1b)	$1+\frac{\epsilon}{105}$	$1+\frac{\epsilon}{196}$	$1+\frac{\epsilon}{315}$	$\frac{\epsilon}{36}$
	(4.2)[d]				0.04ϵ
	ϵ expt[e]	$1+\frac{\epsilon}{105}$	$1+\frac{\epsilon}{196}$	$1+\frac{\epsilon}{315}$	$\frac{\epsilon}{36}$

[a]This work.
[b]Numerical simulation, see Ref. 19.
[c]Numerical simulation, see Ref. 18.
[d]Using Eq. (4.2) with the right-hand side taken from Ref. 31.
[e]ϵ expansion, see Ref. 20.

The discrepancy for $q<1$ is due to the fact that we used values for $q=0$ for our fit which were not very close to the series values. This difficulty was apparent when earlier shorter series[25] were compared to numerical simulation data.[26–28] Table II contains comparisons between specific approximant values of $\psi(q)$ and other available estimates, and the agreement is again quite good. Note, however, that even at $d=2$ and 3 Eq. (4.1b) fits the data better than Eq. (4.1a). Since Eq. (4.1b) goes smoothly into the ϵ-expansion result of Eq. (3.4), this fact provides indirect support for the ϵ expansion, and reveals some of the drawbacks of models like that used in Ref. 12.

Table II also contains values for $-d\psi(q)/dq\,|_{q=1}$. This latter quantity is of some interest in that it is related to the critical exponent $\zeta(\alpha)$ associated with the *nonlinear*[29,31,32,35] resistor network. Harris[36] has derived the relation

$$\left.\frac{\partial\psi(q)}{\partial q}\right|_{q=1}=2\left.\frac{\partial\zeta(\alpha)}{\partial\alpha}\right|_{\alpha=1}. \tag{4.2}$$

We have tested this relation by comparing the left-hand side and the right-hand side in Table II. The left-hand side is well represented by the derivative of the approximants in Eq. (4.1). Also, as was noted in Ref. 31, a similar approximant for $\zeta(\alpha)$ can be used to obtain the right-hand side of Eq. (4.2) without significant error and the resulting values are listed in Table II. In summary then, we conclude that the approximants in Eq. (4.1) provide useful representations of the $\psi(q)$'s. Equation (4.1b) is more satisfactory for $q>1$.

V. MULTIFRACTALS

Recently there has been increasing interest in the multifractal description of the probability distributions on fractal structures.[37–39] These formulations provide a mathematical framework within which it is possible to discuss systematically families of fractal measures which may be used to characterize a fractal set. Originally an

TABLE III. The values of the input and output parameters for the approximants given in Eq. (4.1).

d	νD_B	ζ_R	a	b
2	2.16±0.03[a]	1.297±0.007[b]	1.22±0.01	1.05±0.1
3	1.61±0.07[c]	1.12 ±0.02[d]	0.40±0.06	0.65±0.08
4	1.32±0.1[c]	1.05 ±0.02[d]	0.15±0.04	0.45±0.1
5	1.16±0.1[c]	1.02 ±0.02[d]	0.05±0.05	0.33±0.3
$6-\epsilon$	$1+\frac{\epsilon}{7}$[e]	$1+\frac{\epsilon}{42}$[f]	$\frac{\epsilon}{14}$	$\frac{1}{2}$

[a]Reference 26.
[b]Reference 10.
[c]Reference 25.
[d]Reference 34.
[e]Reference 33.
[f]Reference 9.

idea of this type was proposed by Mandelbrot[40] to treat nonuniform turbulence. The more recent interest in this phenomenon has been motivated by the nonuniform measures generated by diffusion-limited aggregates,[39] chaotic behavior,[38] and the distribution of currents on percolating clusters.[19] Although the multifractal description contains exactly the same information as does the function $\psi(q)$, it is sometimes useful to discuss the fractal behavior in terms of the multifractal functions f and α defined below. Our analytic approximants allow explicit construction of these functions.

For convenience, in the discussion which follows we consider a single large cluster of linear size L. A typical resistance, $R(L)$, between two points in the cluster whose separation is of order L is given by

$$R(L)=\sum_b i_b^2 \equiv M_1(L) , \tag{5.1}$$

we can define a bond variable $p_b=i_b^2/R(L)$, so that $\sum_b p_b=1$. We can now rewrite Eq. (1.4) in the form

$$\sum_b p_b^q = M_q(L)/R(L)^q \sim L^{\tilde{\psi}(q)-q\tilde{\psi}(1)} . \tag{5.2}$$

We can now follow Halsey *et al.*,[38] and define a set of fractal dimensionalities $D(q)$ via

$$\left[\sum_b p_b^q\right]^{1/(1-q)} \sim L^{D(q)} , \tag{5.3}$$

i.e.,

$$D(q)=[q\tilde{\psi}(1)-\tilde{\psi}(q)]/(q-1) . \tag{5.4}$$

The $D(q)$ are obviously similar to the $p(q)$ of Eq. (2.6) introduced in Ref. 19. It is easy to see that $D(0)=\tilde{\psi}(0)=D_B$, $D(\infty)=\tilde{\psi}(1)=\tilde{\zeta}_R$, and $D(1)=\tilde{\psi}(1)-d\tilde{\psi}/dq\,|_{q=1}$. As shown by Hentschel and Procaccia[37] $D(q)$ is also a monotonically decreasing function.

Following Halsey *et al.*,[38] we can now rewrite $\sum_b p_b^q$ as

$$\sum_b p_b^q = \sum_p n(p) p^q , \tag{5.5}$$

where $n(p)$ is the number of bonds on which the fractional current i obeys $i^2/R=p$. Using steepest descent, the sum will be dominated by a value of p denoted p^*, for which $d\log n/d\log p=-q$. Assuming now that both p^* and $n(p^*)$ scale as powers of L, we write

$$p^* \sim L^{-\alpha(q)} , \tag{5.6a}$$

$$n(p^*) \sim L^{f(q)} , \tag{5.6b}$$

and we identify

$$(q-1)D(q)=q\alpha(q)-f(q) , \tag{5.7a}$$

$$\alpha(q)=\frac{d}{dq}[(q-1)D(q)] . \tag{5.7b}$$

Using Eqs. (5.4) and (5.7) we thus find

$$\alpha(q)=\tilde{\psi}(1)-\frac{d}{dq}\tilde{\psi}(q) , \tag{5.8a}$$

$$f(q)=\tilde{\psi}(q)-q\frac{d}{dq}\tilde{\psi}(q) . \tag{5.8b}$$

These two equations may now be used to construct $f(\alpha)$. Using Eq. (4.1b) we evaluated these functions and we show the results for $D(q)$, $f(q)$, and $\alpha(q)$ in Fig. 2 and for $f(\alpha)$ in Fig. 3 for $d=2$. Similar results for all these functions are obtained for $d>2$. Note that, unlike $D(q)$, $f(q)$ has the geometrical interpretation as a fractal dimensionality, of the subset of bonds which dominate M_q.

We have used the approximant of Eq. (4.1b) to illustrate the analysis because it correctly reflects the fact (discussed in detail in the next section) that $\psi(q)\to\infty$ for $q\to q_c$, where q_c is a critical value of q which is nonpositive and possibly small in magnitude. Quantitatively, Eq.

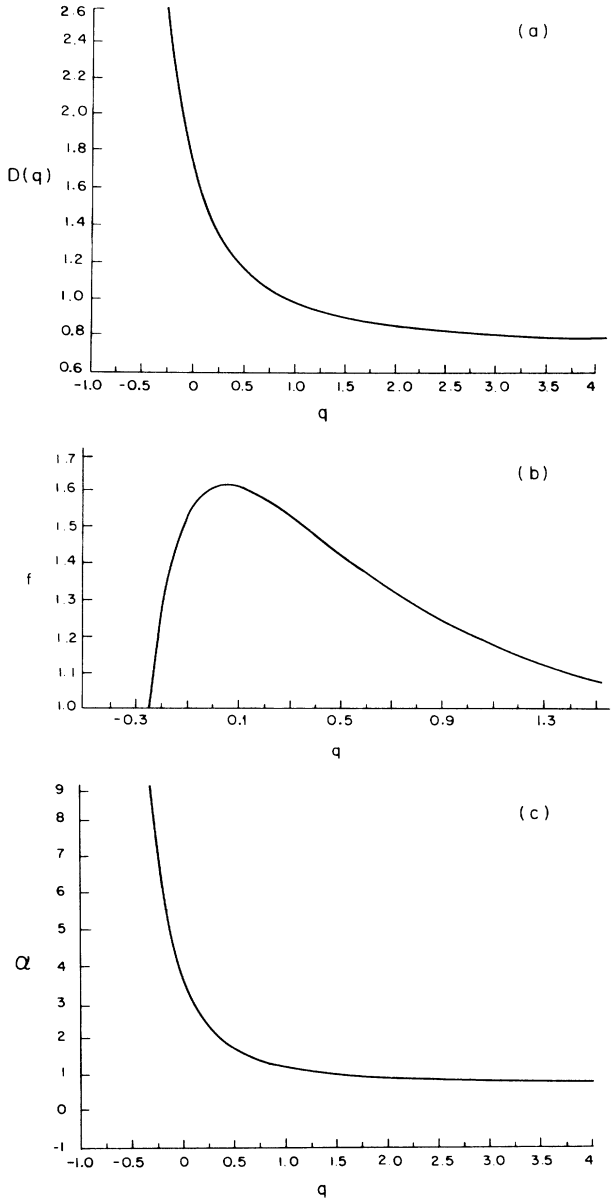

FIG. 2. $D(q)$, $f(q)$, and $\alpha(q)$ for $d=2$, as obtained from the approximant of Eq. (4.1b).

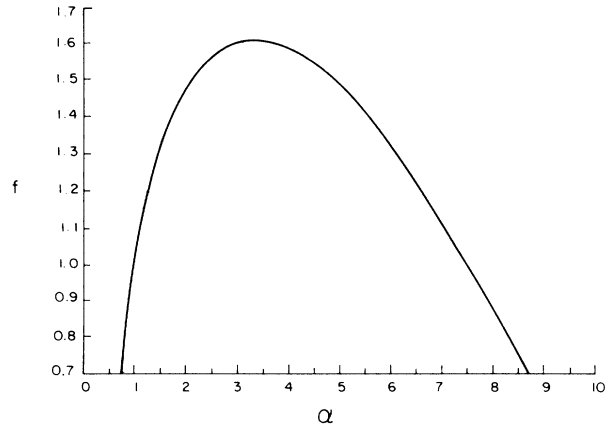

FIG. 3. $f(\alpha)$ for $d=2$ constructed from the functions $f(q)$ and $\alpha(q)$ shown in Fig. 2.

(4.1b), may not be very accurate for $q<0$. However, it does illustrate the type of behavior one can expect from the exact $\psi(q)$. In particular, $\psi(q)$, $\alpha(q)$, and $f(\alpha)$ possess the general properties discussed by Halsey *et al.*[38] For instance, from Eq. (5.8) we obtain

$$\frac{d^2f}{d\alpha^2}=\left[\frac{d^2f}{dq^2}\frac{d\alpha}{dq}-\frac{df}{dq}\frac{d^2\alpha}{dq^2}\right]\Big/\left[\frac{d\alpha}{dq}\right]^3 \tag{5.9a}$$

$$=-\left[\frac{d^2\psi}{dq^2}\right]^{-1}<0\ . \tag{5.9b}$$

We now discuss the results for $d=2$ in more detail. Note that for $q=\infty$, the values of $p_b=i_b^2/R$ correspond to R^{-1} which scales with L as $L^{-\alpha(\infty)}$, with $\alpha(\infty)=\tilde{\xi}_R=0.973$, and the set of bonds having $i_b=1$ has a fractal dimension $f(\infty)=D(\infty)=1/\nu=3/4$. Also, f assumes its maximum value, the fractal dimension of the backbone, $D_B=1.62$ for $q=0$. Finally, as $q\to q_c$ [for the approximant of Eq. (4.1b), $q_c=-1$], the functions f and α describe the set of bonds for which $p_b=i_b^2/R$ scales with the largest $\alpha(\alpha\to\infty)$ and whose fractal dimension f becomes very small ($f\to-\infty$). The curve of f versus α therefore characterizes all the fractal sets, i.e., those sets which exhibit power law scaling. However, this description is not a complete one, because it does not deal with the regime $q<q_c$. As we shall see in the next section, in that regime the ordinary power law scaling breaks down: $p_b=i_b^2/R$ then scales as an exponential function of L.

We now compare our results with those of Ref. 19. To facilitate this comparison we denote their quantities by the subscript ARC and ours by the subscript BMAH. Then we have

$$f_{\rm ARC}(2q)=f_{\rm BMAH}(q) \tag{5.10a}$$

and

$$\alpha_{\rm BMAH}=2\alpha_{\rm ARC}-\tilde{\psi}(1)\ . \tag{5.10b}$$

Otherwise, for $q>0$ our results do not differ substantively from theirs except that, as noted above, their central estimate for $p(6)$ does not satisfy the inequalities of Eq. (2.7b). On the other hand, for $q<0$ our results are quite different from theirs. The anomalous behavior we find for $q\to q_c$ is not well represented in their numerical simulation. As we discuss in the next section, the magnitude of their sharp increase in α for $q<0$ is probably dependent on their sampling procedure. With more extensive sampling, their "maximum" in $\alpha(q)$ would probably increase without limit. To see the non-power-law scaling we find would require extensive numerical work.

VI. BEHAVIOR OF $\psi(q)$ FOR $q<0$

Within the multifractal formulation it is of interest to consider the $\psi(q)$'s for negative q.[41] Recall that within this picture, for each value of q one has a fractal dimension associated with the set of currents assuming values in a corresponding narrow range. As q becomes negative and large, obviously the $\psi(q)$'s emphasize the smallest nonzero currents, and thereby would seem to give information on the fractal dimension of the set of bonds carrying some small value of the current. In the existing theories, this phenomenon is reflected by a continuous variation in $\psi(q)$ as $q\to-\infty$, with the understanding, of course, that the scaling result of Eq. (1.4) remains valid for all q. Here we show that this is *not* the case. In particular, there exists a critical value of q denoted q_c, such that for $q<q_c$ the sum over backbone bonds needed to construct $[M_q(\mathbf{x},\mathbf{x}')]_{\rm av}$ does not converge for $p=p_c$. Physically, this occurs because the rate at which the current falls off as the size of the system is increased, is fast enough so that when the current is raised to a large negative power, it can overcome the exponential decay associated with the percolation correlations for $p<p_c$. As a result, for $q<q_c$, there exists a critical curve $p^*(q)$ with $p^*(q)<p_c$ which gives the largest value of p for which $M_q(\mathbf{x},\mathbf{x}')$ is finite for a given q.

To investigate this possibility we analyzed our series for negative q. We show in Fig. 4 the results for the threshold value of p as a function of q for $d=2$ and $d=3$. For positive q our results are as expected, viz., the threshold is

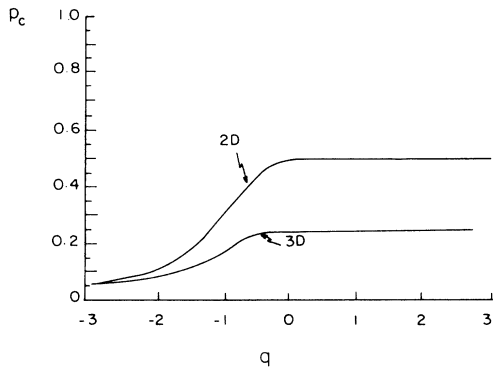

FIG. 4. Dependence of the apparent threshold, $p^*(q)$ on q. For $q\geq 0$, $p^*(q)$ coincides with p_c. The critical value of q at which $p^*(q)$ differs from p_c seems to be about -0.3.

independent of q and agrees with the accepted value. However, for q negative we see that the threshold begins to decrease. Whether this decrease begins strictly at $q=0$ or not is unclear from our data. However, it is undeniable that this data establishes that $p^*(q)<p_c$ for $q<-0.3$ in $d=2$ and for $q<-0.6$ in $d=3$.

We now develop a bound which proves the above assertions. To do this we wish to focus on structures which give very small currents. Although we cannot specify precisely which structures are the dominant ones at a given concentration p, it seems clear that they should contain loops within loops. To obtain a bound we consider some special configurations of this type which are susceptible to exact analysis. To start, consider a "ladder" of m plaquettes as shown in Fig. 5 with unit current imposed across the bottom rungs. Crudely, one expects that the current in each successively higher rung will be smaller by a factor of 3 than in that below it. An exact solution, illustrated in Fig. 5, gives the current in the topmost (i.e., mth) rung as

$$I_{\min}=\frac{2\sqrt{3}}{(2+\sqrt{3})^{m+1}-(2-\sqrt{3})^{m+1}} . \tag{6.1}$$

For large m this gives

$$I_{\min}=ai_0^{-m} , \tag{6.2}$$

where

$$i_0=2+\sqrt{3} , \tag{6.3}$$

rather than 3 as intuited above.

The probability that such a ladder occur starting at a given bond and having orientation in a given direction is

$$p^{3m+1}(1-p)^{2mz+2z-6m-2} . \tag{6.4}$$

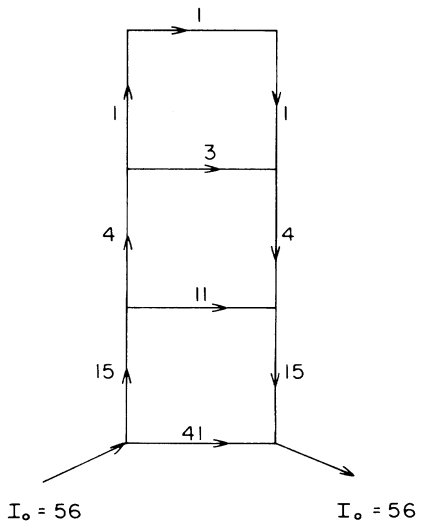

FIG. 5. A ladder configuration with $m=3$ plaquettes which gives rise to small currents. The relative current (normalized to unity in the topmost rung) is indicated. If the external current is normalized to unity, the current in the topmost rung is $I_{\min}=\frac{1}{56}$, as given by Eq. (6.1) for $m=3$.

Here the factors of $1-p$ are included to ensure that the perimeter bonds are indeed unoccupied. [The precise form of the terms in the exponent of $(1-p)$ which are independent of m is not important in what follows.] We might also take account of the many equivalent configurations which can be constructed by allowing the ladder to be distorted at each stage (as in Fig. 6). Since at each stage there are three choices of which bond to use for attachment, and there are $2d-3$ orientations for the successive plaquettes, one would estimate the total number of distorted configurations of an m rung ladder to be of order $[3(2d-3)]^m$. As in the theory of self-avoiding walks,[42] we expect that a more rigorous treatment would replace this result by μ^m, where μ is an asymptotic branching ratio not too different from $3(2d-3)$. Actually, the existence of a bound could even be obtained by setting $\mu=1$, i.e., considering only straight ladders. In summary, the average number of m-rung ladders attached to a given bond is of order

$$Q(m)=\mu^m p^{3m+1}(1-p)^{2mz+2z-6m-2} . \tag{6.5}$$

Now consider the situation when two terminals at separation $\mathbf{r}$ are the source and sink for a unit imposed current. If these terminals are l steps apart, the above m-rung ladder configuration surely occurs at least $[p(1-p)^{z-2}]^l Q(m)$ times. Thus, the m-rung ladder contributes to $M_q(\mathbf{x},\mathbf{x}')$ an amount at least as large as

$$\delta M_q^{(m)}\sim[p(1-p)^{z-2}]^l Q(m) a^{2q} i_0^{-2mq} \tag{6.6a}$$

$$\sim\exp[-mA+Bm(-q)-Cl] , \tag{6.6b}$$

where $e^{-A}=\mu p^3(1-p)^{2z-6}$, $e^B=i_0^2$, and $e^{-C}=p(1-p)^{z-2}$. We now consider the contribution of these ladders to $M_q(\mathbf{x},\mathbf{x}')$. This contribution, denoted δM_q, is obtained by summing the above result over m. Obviously this sum will diverge for $-Bq>A$. If q is not strongly negative, $-Bq<A$ for $p<p_c$ and the bound in Eq. (6.6) doesn't provide any useful information. On the other hand, when q is large and negative, then $-Bq>A$ for $p=p_c$. This fact indicates that the qth-order susceptibility in question will actually become infinite as p in-

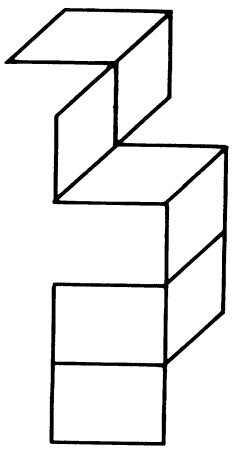

FIG. 6. A distorted ladder configuration.

creases toward a critical value of p (which is less than p_c) denoted $p^*(q)$. The lower bound on δM_q from Eq. (6.6) provides the following upper bound on $p^*(q)$: $p^*(q) < p_>(q)$, where

$$\mu p_>(q)^3[1-p_>(q)]^{2z-6} = i_0^{-2q} . \tag{6.7}$$

As $q \to -\infty$, $p^*(q) \to 0$ exponentially fast, at least. The above argument is rather crude but is given to show definitively that there is a critical value of q below which $p^*(q)$ becomes less than p_c.

That there can be two thresholds, the conventional one for percolation at $p = p_c$, and an anomalous one for q sufficiently negative where M_q becomes infinite, is not a totally unheard of phenomenon. In essence, it results from taking the average of a correlation function which grows exponentially with a correlation length ξ_0 which is not equal to the percolation correlation length ξ_p. The threshold for such a correlation function occurs when p becomes large enough so that ξ_p is no longer less than ξ_0. Since ξ_0 depends on q, this threshold clearly depends on q also. The converse effect, where the critical point for a correlation function occurs within the ordered phase (as defined by the thermodynamic functions), is predicted in the nematic to smectic-A transition.[43] There one has

$$G(\mathbf{x}) = G_{\mathrm{SC}}(\mathbf{x}) e^{-R(\mathbf{x})} , \tag{6.8}$$

where $G_{\mathrm{SC}}(\mathbf{x})$ is a normal correlation function which becomes long ranged in the low-temperature phase. Thermodynamic singularities are regulated by G_{SC}. The observable correlation function, $G(\mathbf{x})$, however, does not become long ranged until, within the low-temperature phase, $e^{-R(\mathbf{x})}$ becomes long ranged. Since this occurs at some temperature below the transition temperature, one sees the existence of two critical points. Which one is relevant depends on what physical properties are being monitored.

To make closer contact with the multifractal analysis, we apply finite size scaling to the above argument, i.e., we consider the application of the above argument to a finite-sized sample when $p = p_c$. In that case we should not sum over all values of the number of rungs m, but rather only over values of m which correspond to ladders which might actually fit into a finite size box of length L. Thus we should modify Eq. (6.6) above by cutting off the sum when the self-avoiding walk of plaquettes has a characteristic size of order L. This can be done heuristically by replacing $Q(m)$ by $\tilde{Q}(m)$, where

$$\tilde{Q}(m) = Q(m) \exp[-a(m^x/L)^y] , \tag{6.9}$$

where $x = \nu_{\mathrm{SAW}}$ is the correlation length exponent for self-avoiding walks.[44] The appearance of the variable m^x/L expresses the fact that the size of the walk is limited to be L. The values of the constant a and the exponent y are uncertain, but y should be greater than unity and a large enough to cause the distribution to decrease rapidly as m^x approaches the size L. For simplicity we set $y = 2$.

We can now perform the sum

$$\delta M_q(\mathbf{x},\mathbf{x}') = \sum_m \delta M_q^{(m)} > \sum_m \exp\left[-mA + Bm(-q) - Cl - a\left[\frac{m^x}{L}\right]^2\right] \tag{6.10}$$

by steepest descent. The result is then that the ladder contribution to $M_q(\mathbf{x},\mathbf{x}')$ when $|\mathbf{x}-\mathbf{x}'| \sim L$ obeys

$$\delta M_q(L) > \exp\{\mathrm{const}[L(-A-Bq)^x]^{2/(2x-1)}\} . \tag{6.11}$$

The meaning of this result is as follows. The present analysis is restricted to $p = p_c$ and q large and negative so that $-Bq > A$. Then one sees that the qth moment of the current grows with L, not as $L^{p(q)/\nu}$, as given by de Arcangelis *et al.*,[19] but rather it grows much more strongly, in fact, $\ln M_q \sim L^{\phi}$ with $\phi = 2/(2x-1)$. We do not claim have a rigorous identification of ϕ because of the assumptions implicit in the above argument. First of all, the ansatz that $y = 2$ is unlikely to be exact. Also, the exponent $x = \nu_{\mathrm{SAW}}$ may not be exactly the one to characterize the more general structure with hierarchical loops which gives the minimum current. It may not be easy to numerically verify our results because, as with all Lifshitz phenomena,[45] it would require a truly definitive sampling in order to see the very special configurations which give rise to the unusually small currents needed for the above argument. The ladders used here are only the simplest candidates for small current configurations. However, it is significant that numerical work does show a large increase for negative q in the scaling exponents $-p(q)$ and $\alpha(q)$ used by de Arcangelis *et al.*[19] to describe the growth of V^{-1}. Although our series results do not directly probe the size dependence of the moments of the current, our results nevertheless do indicate quite clearly that the associated effect, namely, that $p^*(q)$ should become less than p_c, does occur.

It should also be noted that the exponential size dependence in $M_q(L)$ only occurs for $p > p^*(q)$. For $p \approx p^*(q)$ we expect that $A - Bq \sim p^*(q) - p$ and thus that $\chi^{(q)}$ will exhibit a power law divergence for $p \to p^*(q)$. The associated critical exponent γ_q defined by

$$\chi^{(q)}(p) \sim |p^*(q) - p|^{-\gamma_q}, \quad (q < q_c) , \tag{6.12}$$

remains to be clarified, although one might speculate that it is related to properties of random loop structures. In view of the fact that cluster statistics for very large clusters away from the percolation threshold are governed by the fixed point for lattice animals,[46] it seems very unlikely that γ_q is expressible in terms of percolation exponents.

VII. CONCLUSION

In this paper we have presented a comprehensive study of the exponents $\psi(q)$ which describe how the qth moment of the current distributions scales with the distance between the source (of external current) and the sink. Our main results are as follows.

(1) We show both analytically and by series expansions

that $\psi(q)$ is a monotonically decreasing and convex function of q: $d\psi/dq < 0$, $d^2\psi/dq^2 > 0$.

(2) The values of $\psi(q)$ we obtain from series expansions agree with previous results in the cases where these are available.

(3) We construct two approximants for $\psi(q)$ which are consistent with the known properties of $\psi(q)$ and which provide useful representations of our numerical results for $\psi(q)$.

(4) Using these approximants, we carry out a multifractal analysis of the type recently suggested by Halsey *et al.*[38] For $q > 0$ our results agree substantially with those of de Arcangelis *et al.*[19]

(5) We show that the multifractal scaling behavior which describes $\psi(q)$ for $q > 0$ breaks down for $q < 0$. The power law dependence of M_q on size becomes, for q sufficiently negative, an exponential dependence on size. Thus the threshold concentration, $p^*(q)$, where the resistive susceptibility associated with $\psi(q)$ diverges, becomes a function of q, for q sufficiently large and negative.

(6) The result (5) suggests that it would be of interest to establish whether the critical value of q below which this behavior occurs is or is not equal to zero. Also it remains to clarify the status of the exponents associated with the divergence of $\chi^{(q)}(p)$ for $p \to p^*(q)$.

Note added. After the completion of this paper, we received a copy of work by Bhatti and Essam[47] prior to publication in which they constructed and analyzed series for the moments of the current distribution in a random diode-insulator network in two and three dimensions.[47] Their estimates of the exponents are consistent with ours. They also constructed an approximant very similar to the ones we introduced in Eq. (4.1a).

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