Frustrated XY model with screening

D. Stroud

Department of Physics, Ohio State University, Columbus, Ohio 43210

S. Kivelson

Department of Physics, State University of New York, Stony Brook, New York 11794

(Received 14 August 1986)

We consider a Josephson- or proximity-coupled superconducting array with finite geometrical inductance. The finite self-inductance is shown to screen the vortex-antivortex interaction in the arrays, converting the logarithmic interaction into a $1/r$ interaction at large separations in analogy with the effects of screening found by Pearl for a homogeneous superconducting film. We explicitly calculate the screening length in terms of the geometrical inductance and the intergrain Josephson-coupling strength. For typical arrays, the screening length is found to be of order 10 cm, independent of the plaquette dimensions of the array, although the length can be made smaller for strong couplings. An analogous type of screening is shown to occur in the cooperative-ring-exchange model of the fractional quantum Hall effect. In this latter case, the screening arises from the vibrations of the two-dimensional Wigner lattice in the presence of a strong magnetic field, producing fluctuations in the flux per lattice plaquette. The screening in this case also produces $1/r$ interactions between the elementary excitations of the system. However, the screening is very slow, and hence only effective at very long times.

I. INTRODUCTION

The collective properties of many systems can be described in terms of a local phase variable ϕ_j defined on a lattice whose sites are labeled by the index j. (We will consider several explicit physical examples below.) Implicit in the notion of a phase variable is the understanding that the configuration of the phases is periodic under the transformation $\phi_j \to \phi_j + 2\pi$. For some purposes, it is convenient to think of ϕ_j as specifying the orientation of a fixed-length planar spin; such systems can thus be described by a model of the XY class. A common feature of such models is that the spins couple to an additional field, which we will call the vector potential A_{ij}, which is defined on the bonds of the lattice. In the presence of a vector potential, the interaction between two nearest-neighbor lattice sites is minimized when $\phi_i - \phi_j = A_{ij}$. When the A_{ij} are such that the various contributions to the interaction energy cannot all be minimized simultaneously, the model is said to be frustrated.

Examples of physical systems that can be well described by a frustrated XY model are abundant. We will consider three explicit two-dimensional examples here, each of which is illustrative of different limits of the general model we consider in this paper.

Example 1: The most obvious example is an array of coupled Josephson junctions, or Josephson-coupled grains in a granular superconductor, in a transverse magnetic field.[1] Here ϕ_j is the phase of the superconducting order parameter on grain j. (Fluctuations in the magnitude of the order parameter are assumed to be unimportant.) A_{ij} is partially determined by the flux Φ_α through plaquette α according to the relation

$$\sum A_{ij} = 2\pi f_\alpha , \tag{1}$$

where $f_\alpha = \Phi_0$, $\Phi_0 = hc/2e$ is the quantum of superconducting flux and the sum runs over the bonds that enclose plaquette α.[2] The fact that Eq. (1) does not fully specify A_{ij} is a reflection of the gauge invariance of the system; a transformation in which $A_{ij} \to A_{ij} + \omega_i - \omega_j$ and $\phi_i \to \phi_i + \omega_i$ for any ω_i leaves the system physically unchanged.

Example 2: The second example we consider arises from a recently developed semiclassical theory of the low-temperature properties of the two-dimensional electron gas in a high magnetic field, relevant to the fractionally quantized Hall effect. The classical ground state of this system is a triangular Wigner crystal. Kivelson, Kallin, Arovas, and Schrieffer[3] (KKAS) considered the corrections to the classical ground-state energy of this system due to cooperative ring exchanges, which they identified as being the most important class of quantum fluctuations for producing a commensurate lock-in between the electron density and the magnetic flux density. KKAS showed that the sum over cooperative ring exchanges is equivalent to a classical spin model. This equivalence is best expressed in terms of a discrete Gaussian model, but that in turn is related via a duality transformation (see below) to an XY-like model. The exact physical meaning of the phase variables is thus difficult to identify directly. Roughly, however, they derive from that fact that each ring exchange contributes to the energy with a well-defined phase which arises in part from the Aharonov-Bohm effect (and is proportional to the enclosed flux) and in part from Fermi statistics. The contribution of each plaquette to the overall phase can be expressed, as in Eq. (1), in terms of an effective vector potential. However, in this case, the effective flux which enters Eq. (1) is given by $f_\alpha = \Phi_\alpha/\Phi_0 - \frac{1}{2}$, where the factor of $-\frac{1}{2}$ results from Fermi statistics and $\Phi_0 = hc/e$

rather than $hc/2e$, as it is for the superconductor. (For a fractional filling of the lowest Landau level $v=\eta_e\Phi_0/B$, the average "flux" per plaquette is $f_\alpha=v^{-1}-\frac{1}{2}$.) As in the previous example, A_{ij} is defined only up to a gauge transformation.

Example 3: Perhaps the most widely studied example of a frustrated phase model is that relevant to a charge-density wave interacting with a periodic substrate potential, where now ϕ_i represents the relative phase difference between the substrate potential and the charge-density wave at site j. Here, A_{ij} represents the difference in the natural periodicity of the charge-density wave and the substrate. This system is quite different from the other two, however, in that it is not invariant under the gauge transformation described above. We will therefore have little to say about this system.

The frustrated XY model has been moderately well studied.[2] While many questions remain as to the exact nature of the interplay between the continuous Kosterlitz-Thouless transition and the discrete order-disorder transitions which occur in this model, some aspects of phase diagram are well understood, particularly at low-order rational uniform frustrations. The new ingredient introduced in the present work is that the flux variable f_α is a fluctuating quantity as well. In the Josephson arrays, this is equivalent to including the effect of the finite geometrical inductance of the array. In the quantum Hall example, it is the plaquette areas themselves which fluctuate. The effect of this will be seen, in both cases, to screen the vortex-vortex interaction and so to destroy the Kosterlitz-Thouless transition,[4] leaving only various order-disorder transitions. We will see that for the case of the Josephson-junction arrays, the screened interaction falls off as $1/r$ for large separations, and that the screening length is typically larger than the size of most currently fabricated arrays, so that the effect is of little practical importance at present in this case. In the quantum Hall effect, the interactions also fall off as $1/r$ at large separation, but the screening is much stronger, so that it plays an important role in determining the properties of the quantum Hall system.

We turn now to the body of the paper. Section II presents the frustrated XY model, including the effects of geometrical inductance as they apply to Josephson-junction arrays. The screening fields are eliminated in favor of the vortex variables, leaving a screened interaction between vortices which is shown to differ little from the original logarithmic interaction in the range of greatest current experimental interest. Section III describes the analogous model for the fractionally quantized Hall effect, as described in the KKAS model. Here the screening is strong, but dynamic (i.e., retardation) effects are important. Similar retardation effects also play a role in Josephson arrays with finite capacitance. Finally, Sec. IV gives a brief discussion of our results and presents our conclusions.

II. SCREENING IN SUPERCONDUCTING ARRAYS IN A MAGNETIC FIELD

We consider an array described (in the absence of screening) by the following model Hamiltonian:

$$H_{XY}=-\sum_{(ij)}K_{ij}\cos(\phi_i-\phi_j-A_{ij}) ,\qquad (2)$$

$$A_{ij}=(2\pi/\Phi_0)\int_{x_i}^{x_j}\mathbf{A}\cdot d\mathbf{l} .\qquad (3)$$

Here K_{ij} is the coupling energy due to the Josephson coupling between grains i and j (assumed to be nearest neighbors for simplicity), x_i is the position of the center of grain i, $\Phi_0=hc/2e$ is the superconducting flux quantum, ϕ_i is the phase of the superconducting order parameter on the ith grain, and $\mathbf{A}$ is the vector potential. The coupling K_{ij} is related to the critical current I_c flowing through link (ij) by $K_{ij}=I_c(h/2e)$. Model (1) omits the charging energy which may be significant in Josephson arrays with small capacitance; when $\mathbf{A}$ is taken as arising from an externally imposed magnetic field, it also omits screening currents. Finally, it also leaves out dissipative effects associated with the normal current carried in parallel with the supercurrent in the so-called resistively-shunted-junction (RSJ) model of Josephson junctions;[5] such dissipative effects may play an important role in the phase transition when the grains have small capacitance.[6,7]

To include the effects of screening in the model, we add to the Hamiltonian (1) the field energy

$$H_{\text{field}}=(1/2c)\int (\mathbf{J}_0+\mathbf{J}_1)\cdot(\mathbf{A}_0+\mathbf{A}_1)d^3x ,\qquad (4)$$

where $\mathbf{J}_0$ represents the external current source producing the external vector potential $\mathbf{A}_0$, $\mathbf{J}_1$ is the induced supercurrent flowing in the Josephson links, and $\mathbf{A}_1$ is the corresponding vector potential. The first term in (4) involves only the external fields and currents and plays no role in the properties of the array. The two cross terms in (4) are equal because of the linearity of Maxwell's equations. Each may be transformed into a summation of the form

$$H'=(1/c)\sum_\alpha I_\alpha\Phi_\alpha^0=(\Phi_0^2/c^2)\sum_{\alpha,\beta}(M^{-1})_{\alpha\beta}\Phi_\alpha^0\delta\Phi_\beta ,\qquad (5)$$

where I_α is the current through the αth closed loop in the array, Φ_α^0 is the corresponding external flux, $\delta\Phi_\alpha$ is the induced flux, and M is the mutual-inductance matrix discussed below.

We wish to express this Hamiltonian entirely in terms of the flux variables Φ_α. To do this, we make two approximations: (i) We assume that I_α is small so that it is related to the induced flux $\delta\Phi_\alpha$ by a linear relation

$$\delta\Phi_\alpha(t)=\Phi_0\sum_\beta\int_{-\infty}^t d\tau\, m_{\alpha\beta}(t-\tau)I_\beta(\tau) ,\qquad (6a)$$

where $\Phi_0=hc/2e$ is the superconducting flux quantum. (ii) We assume that the response function $m_{\alpha\beta}(t)$ has a characteristic response time which is short compared to the characteristic time over which the screening currents I_β can change. Thus,

$$\delta\Phi_\alpha(t)\approx\Phi_0\sum_\beta M_{\alpha\beta}I_\beta(t) ,\qquad (6b)$$

where M is the mutual inductance matrix,

$$M_{\alpha\beta}=\int_0^\infty dt\, m_{\alpha\beta}(t) .\qquad (6c)$$

We generally expect approximation (i) to be valid in Josephson-junction arrays, where the characteristic field

strengths for nonlinear response are expected to be of order the critical field B_c of the bulk superconductor. To consider the validity of approximation (ii), we note that the characteristic response time for $m_{\alpha\beta}(t)$ is the time required for a signal to propagate across the sample at the speed of light, whereas the time characterizing the rate of change of the screening currents is the shorter of the LC response frequency or the L/R damping time. For typical superconducting-normal-superconductor (S-N-S) arrays,[8] the capacitive energy term in the RSJ model can be neglected, and the important time is the L/R damping time. If we make the crude estimate $L = a/c^2$, where a is the grain spacing and c is the speed of light, and use for the signal time a/c, then approximation (ii) is valid if the junction resistance $R \lesssim R_0/137$, where $R_0 = \hbar/e^2$ is the Thouless resistance and 137 is the fine-structure constant $\hbar c/e^2$. Thus, we expect the nonretarded approximation to be valid only for very high-coupling S-N-S arrays. It may sometimes be necessary to consider retardation effects, although we will not do so here for the array problem. In the following section, we will return to this problem, and consider the case in which the time-retarded nature of the response function is included.

With these two approximations, the total Hamiltonian can be expressed in terms of dimensionless flux variables $f_\alpha = \Phi_\alpha/\Phi_0$ as

$$H = H_{XY} + (\Phi_0^2/2c^2) \sum \delta f_\alpha (M^{-1})_{\alpha\beta} \delta f_\beta , \quad (7)$$

where H_{XY} is given in Eq. (2). This expression is somewhat inconvenient since the phase variables which appear in H_{XY} are defined on the direct lattice while the flux variables are defined on the dual (plaquette) lattice. However, it is well known that the free energy of the frustrated XY model can be expressed approximately as the sum of a spin-wave piece, F_{SW}, which is relatively innocuous, and a vortex piece, in which the vortices are defined on the dual lattice. This decomposition is exact for a frustrated Villain[9] model, which has the same symmetries as the XY model, and so is believed to be in the same universality class as that model. Thus, many important properties of the system, in particular its behavior in the scaling regime, can be derived from an effective Hamiltonian defined in terms of plaquette variables alone:

$$H = (\pi\tilde{J}/2) \sum_{\alpha,\beta} (n_\alpha - f_\alpha) G_{\alpha\beta} (n_\beta - f_\beta)$$

$$+ (\Phi_0^2/c^2) \sum_{\alpha,\beta} \delta f_\alpha (M^{-1})_{\alpha\beta} f_\beta^0$$

$$+ (\Phi_0^2/2c^2) \sum_{\alpha,\beta} \delta f_\alpha (M^{-1})_{\alpha\beta} \delta f_\beta . \quad (8)$$

Here $\tilde{J}$ is the effective vortex charge (equals πJ for an ordered square lattice, $\sqrt{3}\pi J$ for a honeycomb lattice), f_α is the flux through the αth plaquette in units of a quantum

of flux $hc/2e$, and f_α^0 and δf_α are the applied and induced fluxes through the αth plaquette in the same units. The integers n_α represent the vortex charges on the plaquettes labeled by α and can take on any positive or negative integer values. $G_{\alpha\beta}$ is the lattice Green's function, which is defined by the relation

$$G_{\alpha\beta} = \sum_{(k\neq 0)} \frac{1}{\Delta(\mathbf{k})} e^{i\mathbf{k}\cdot(\mathbf{R}_\alpha - \mathbf{R}_\beta)} ,$$

$$\Delta(\mathbf{k}) = \sum_{\mathbf{R}_\alpha} \Delta_{0\alpha} e^{i\mathbf{k}\cdot\mathbf{R}_\alpha} ,$$

where the sum on $\mathbf{k}$ runs over the first Brillouin zone and, for an ordered lattice, the off-diagonal element $\Delta_{\alpha\beta} = -1$ if plaquettes α and β have a bond in common, and zero otherwise, while $\Delta_{\alpha\alpha} = -\sum_{\beta(\neq\alpha)} \Delta_{\alpha\beta}$. (The definition of G can be straightforwardly generalized to a disordered lattice.)

We now integrate over thermal fluctuations of the screening currents. Up to an unimportant constant Jacobian factor, this is equivalent to integrating over all flux configurations δf_α. Because the effective Hamiltonian is a quadratic form in δf_α, this integration can be done exactly. First, we bring the Hamiltonian into a diagonal form by Fourier transform:

$$H = (\pi\tilde{J}/2N) \sum_{\mathbf{q}(\neq 0)} |n_\mathbf{q} - f_\mathbf{q}|^2 G(\mathbf{q})$$

$$+ (\Phi_0^2/c^2) \sum_{\mathbf{q}(\neq 0)} \delta f_\mathbf{q} M^{-1}(\mathbf{q}) f_\mathbf{q}^0$$

$$+ (\Phi_0^2/2Nc^2) \sum_{\mathbf{q}(\neq 0)} |\delta f_\mathbf{q}|^2 M^{-1}(\mathbf{q}) , \quad (9)$$

where N is the number of plaquettes and the integrations involve all wave vectors in the first Brillouin zone. The function $G(\mathbf{q})$ is the Fourier transform of the Green's function $G_{\alpha\beta}$ and varies as $A/(q^2 a^2)$ for small q, where A is a constant of order unity (equals 1 for a square lattice) and a is the lattice constant. The Fourier transforms are defined by the relations

$$f_\mathbf{q}^0 = N^{-1} \sum_\alpha f_\alpha^0 \exp(i\mathbf{q}\cdot\mathbf{R}_\alpha) , \quad (10)$$

$$f_\mathbf{q} = N^{-1} \sum_\alpha f_\alpha \exp(i\mathbf{q}\cdot\mathbf{R}_\alpha) , \quad (11)$$

$$\delta f_\mathbf{q} = f_\mathbf{q} - f_\mathbf{q}^0 , \quad (12)$$

$$n_\mathbf{q} = N^{-1} \sum_\alpha (n_\alpha - \bar{f}) \exp(i\mathbf{q}\cdot\mathbf{R}_\alpha) , \quad (13)$$

where $\bar{f}$ is the average flux per plaquette. We now carry out an integration over δf_α and then invert the Fourier transform to obtain the following effective partition function, in terms of the vortex variables n_α:

$$Z = Z_0 \sum_{\{n_\alpha\}} \exp\left[\left[(\pi\tilde{J}/2) \sum_{\alpha,\beta} (n_\alpha - f_\alpha) G_{\alpha\beta}^{\text{scr}} (n_\beta - f_\beta)\right] \Big/ k_B T\right] , \quad (14)$$

where Z_0 is the contribution of the spin waves plus Gaussian fluctuations and f_α is the applied flux through plaquette α in units of Φ_0. The screened interaction $G_{\alpha\beta}^{\text{scr}}$ takes the form

$$\pi\tilde{J}G_{\alpha\beta}^{\text{scr}}=(1/N)\sum_{\mathbf{q}}\exp[i\mathbf{q}\cdot(\mathbf{R}_\alpha-\mathbf{R}_\beta)]\{[M(q)c^2/\Phi_0^2]G(q)/[\pi G(q)JM(q)c^2/\Phi_0^2+1]\}\ ,\tag{15}$$

where $G_{\alpha\beta}^{\text{scr}}$ represents the screened interaction between vortices.

To make further progress, we must determine the form of the mutual inductance matrix $M(\mathbf{q})$ at small q, which in turn controls the behavior of the screened interaction at large separation. This is determined by two conditions on $M_{\alpha\beta}$:

$$\sum_\beta M_{\alpha\beta}=0\ ,\tag{16}$$

$$M_{\alpha\beta}\rightarrow S^2/(\,|\,\mathbf{R}_\alpha-\mathbf{R}_\beta\,|^3c^2)\ \text{ as }\ |\,\mathbf{R}_\alpha-\mathbf{R}_\beta\,|\rightarrow\infty\ ,\tag{17}$$

where S is the area of one of the plaquettes. The first of these conditions follows from the continuity of magnetic flux lines: The total flux through all the plaquettes induced by current flowing around one plaquette (including the flux through that plaquette itself) is zero. The second condition arises from the fact that the field produced by a current loop behaves at large distances like a magnetic dipole field. Equation (16) implies that $M(\mathbf{q}=0)=0$. Equation (17) implies (in two dimensions) that $M(\mathbf{q})$ is linear in $|\mathbf{q}|$ at small values of $|\mathbf{q}|$:

$$M(\mathbf{q})\rightarrow\pi S\,|\,\mathbf{q}\,|\,/c^2\ .\tag{18}$$

Substituting (18) into (15) and using $G(q)\sim1/(q^2a^2)$ for a square lattice, we get the following result for the interaction between vortices, valid at large separation ($R\gg a$):

$$\pi\tilde{J}G_{\alpha\beta}^{\text{scr}}\sim\tilde{J}\ln(\,|\,\mathbf{R}_{\alpha\beta}\,|\,/\lambda),\quad R\ll\lambda\ ,\tag{19}$$

$$\sim\tilde{J}\lambda/|\,\mathbf{R}\,|,\quad R\gg\lambda\ ,\tag{20}$$

where λ, the screening length, takes the form

$$\lambda=\Phi_0^2/(\pi\tilde{J})\ .\tag{21}$$

It is interesting that this form is "universal," i.e., independent of the lattice constant. For typical couplings ($J\sim10$ K), λ is of order 10 cm, larger than the dimensions of many arrays. We may conclude that screening of this kind is unlikely to alter the Kosterlitz-Thouless transition in Josephson-coupled arrays at zero applied field, nor any of the many transitions at finite fields, unless either the arrays are made much larger or the coupling considerably stronger.

Form (21) has been obtained previously by Lobb, Abraham, and Tinkham[10] by a completely different argument. These authors translate the usual formula for the transverse penetration length of a homogeneous, but highly resistive, superconducting film, to the case of two-dimensional arrays. However, the present work is the first to take explicit account of the discrete structure of arrays.

Note that our basic result (logarithmic vortex-vortex interactions at short range, $1/r$ at long range) is the discrete (lattice) analog of the vortex-vortex interactions in thin superconducting films, as first obtained by Pearl.[11] The present work shows that this result is obtained also in the lattice case, at length scales large compared to the lattice constant a. More importantly, it also pertains at high magnetic fields where the density of vortices is not small.

III. DYNAMIC SCREENING IN A FRUSTRATED *XY* MODEL

We now consider the case of a frustrated *XY* model in which the screening fields have nontrivial dynamics. One example of this is the Josephson array with finite capacitance. However, the example we will treat explicitly here is a model of the two-dimensional electron gas (2DEG) in a high magnetic field, recently proposed by Kivelson, Kallin, Arovas, and Schrieffer[3] to account for the fractionally quantized Hall effect.[12] Their model is based on a semiclassical analysis. Thus, they start from the classical ground state of the 2DEG, which at the relevant densities is a triangular Wigner crystal. They then consider quantum fluctuations about this classical ground state. KKAS argue that the quantum fluctuations which determine the important zero-temperature properties of the system are cooperative-ring-exchange processes, that is, processes in which the electrons around closed polygons of various sizes exchange positions by a coherent translation around the circumference. The phase with which each ring contributes to the free energy is determined by the enclosed area via the Bohm-Aharonov effect. There are thus two natural units of area in this model: (i) an area which is inversely proportional to the electron density, which in the model of KKAS is the area of a plaquette of the triangular Wigner crystal, and (ii) the area per quantum of magnetic flux,

$$2\pi l_0^2=\Phi_0/B\ ,\tag{22}$$

where B is the applied magnetic field and $\Phi_0=hc/e$ is the flux quantum for the normal electrons. The quantity ν, which is the number of electrons per flux quantum, is therefore the ratio of these areas. If the plaquette area is a half-integer multiple of $2\pi l_0^2$, then one can show that all ring exchanges add in phase. (In this case, the Bohm-Aharonov phase factor exactly cancels the $+1$ or -1 factors which arise from the Fermi statistics for exchange processes.) When the plaquette area is an irrational multiple of $2\pi l_0^2$, the model is frustrated in the sense that different ring exchanges interfere in a complicated fashion. KKAS showed that the effects of the collections of all ring exchanges could in fact be approximately represented as a uniformly frustrated *XY* model.

The screening which occurs in this model is formally similar to that in the superconducting arrays, but its physical origin is quite different; in the case of the 2DEG, the screening is due to the vibrational modes (magnetophonons) of the electron lattice in a strong magnetic field. These modes alter the flux through a plaquette by varying

the *area* of the plaquettes, and hence changing the frustration. The screening is intrinsically dynamic because the modes have characteristic frequencies associated with them and cannot be taken as infinitely fast, as was done for the response of screening currents in the superconducting array. In particular, the long-wavelength magnetophonons, which are responsible for the screening at long distances, have vanishingly small frequencies (see the Appendix).

Because of the nontrivial dynamics, it is most convenient to compute the partition function Z in terms of a path integral of $\exp(-S/\hbar)$ where S is the Euclidean action, which has dynamics in imaginary time, $\tau = \beta \hbar$. Thus we write the partition function in the form

$$Z = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{\alpha}{}' \mathscr{D} h_{\alpha}$$

$$\times \sum_{\{q_{\alpha}\}} \exp\{-S[h_{\alpha}(\tau), q_{\alpha}(\tau)]/\hbar\} , \tag{22'}$$

where S is an action which in the adiabatic approximation is

$$S/\hbar = \int_0^{\beta\hbar} d\tau K \sum_{\alpha,\beta} [q_{\alpha}(\tau) - h_{\alpha}(\tau)] G_{\alpha\beta}[q_{\beta}(\tau) - h_{\beta}(\tau)]$$

$$+ \int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau'(1/\beta^2) \sum_{\alpha,\beta} h_{\alpha}(\tau) g_{\alpha\beta}^{-1}(\tau - \tau') h_{\beta}(\tau') . \tag{23}$$

In formula (23), $q_{\alpha}(\tau)$ is the vortex charge at time τ (a positive or negative integer); $h_{\alpha}(\tau) = \frac{1}{2}[(\Phi_{\alpha}(\tau)/\Phi_0 - 1)]$, where $\Phi_{\alpha}(\tau)$ is the flux through plaquette α at time τ. The integral $\prod_{\alpha}{}' \mathscr{D} h_{\alpha}$ is a functional integral over all lattice configurations; the prime denotes that the functional integral is to be carried out subject to the constraint that charge neutrality is preserved at all times, i.e., that $\sum_{\alpha} [q_{\alpha}(\tau) - h_{\alpha}(\tau)] = 0$ at all τ. Finally, $G_{\alpha\beta}$ is the same

lattice Green's function defined in Sec. II [e.g., Eq. (8)], and $g_{\alpha\beta}$ is related to the longitudinal part of the magnetophonon propagator (it is discussed below and in the Appendix). This latter propagator describes the vibrations of the electron lattice in a high magnetic field, and is responsible for the density fluctuations which screen the interactions between the vortices. The second term in (23) thus refers to the retarded interaction between the extra flux in the various plaquettes produced by the lattice vibrations. The notation $\sum_{\{q_{\alpha}\}}$ indicates the sum over all integer values of the vortex charges, $q_{\alpha}(\tau)$, associated with each plaquette.

There is one additional subtlety associated with the expression in Eq. (23). The cooperative ring exchanges, whose effects are expressed in terms of the vortex charges $q_{\alpha}(\tau)$, do not occur instantaneously, but rather over a characteristic imaginary time scale τ_0. Therefore, $q_{\alpha}(\tau)$ is undefined at time scales less than τ_0. To get an estimate of τ_0, we note that $q_{\alpha}(\tau)$ is approximately related to the curl of the tunneling current at plaquette α. The frequency $1/\tau_0$ is of order a tunneling prefactor (in imaginary time) for the motion of a charge from one plaquette to another.[13] Since the relevant excitations of the lattice are magnetophonons, this frequency is of order a zone center magnetophonon frequency, and may be estimated as $\hbar/\tau_0 \sim e^2/\epsilon l$, where l is the lattice constant of the Wigner crystal and ϵ is the background static dielectric constant. Strictly speaking, therefore, Eq. (23) is correct only if the integral is interpreted as a double summation over time slices of width τ_0, as discussed by KKAS.

Except for the complication of dynamic screening, the action (23) is essentially the same as that treated in Sec. II. Just as in Sec. II, therefore, the action can be Fourier transformed, and the integrals over the Fourer components of the fields h_{α} can be carried out, since these integrals are Gaussian. The result is

$$S^{\text{eff}}/\hbar = K \int_0^{\beta\hbar} \int_0^{\beta\hbar} \frac{d\tau \, d\tau'}{(\beta\hbar)} \sum_{\alpha,\beta} [q_{\alpha}(\tau) - \bar{h}] \widetilde{G}_{\alpha\beta}(\tau - \tau')[q_{\beta}(\tau) - \bar{h}] , \tag{24}$$

where $\widetilde{G}_{\alpha\beta}$ is the screened interaction, whose Fourier transform is

$$\widetilde{G}(\mathbf{k}, \omega) = \{G^{-1}(\mathbf{k}) + Kg(\mathbf{k}, \omega)\}^{-1} . \tag{25}$$

Here $G(\mathbf{k})$ is the spatial Fourier transform of the bare vortex-vortex interaction, $G(\mathbf{R}_{\alpha} - \mathbf{R}_{\beta})$, while $g(\mathbf{k}, \omega)$ is the Fourier transform of the longitudinal propagator for the Wigner lattice in the limit of strong magnetic fields. The nature of the screened interactions at large separations is now determined by the behavior of G and g at small $\mathbf{k}$ and an appropriate range of frequencies. Note that the form of Eq. (25) is the same as that of Eq. (15), but $G(k)$ and $g(k, \omega)$ differ from the corresponding quantities for the superconducting case. This causes certain differences in the screened interactions in the two cases.

Since $G(k)$ is the Fourier transform of an interaction

which has an asymptotic logarithmic dependence on separation, it varies as

$$G(\mathbf{k}) \sim 1/(Dk^2) \tag{26}$$

at small values of k, where D is a constant of order unity. The magnetophonon propagator $g(k, \omega)$ has a more complicated dependence on its arguments, but can be calculated straightforwardly; details are given in the Appendix. The result takes the form

$$g^{-1}(\mathbf{k}, \omega) = X(k)[\omega^2 + \xi^2(k)] , \tag{27}$$

$$X(k) \sim (k_1/k)^4 \quad \text{as } k \to 0 , \tag{28}$$

$$\xi^2(k) \sim V(k)(k/k_2)^4 \quad \text{as } k \to 0 , \tag{29}$$

where k_1 and k_2 are defined in the Appendix. Note that there is a plus rather than a minus sign in Eq. (27); this

sign arises because we are dealing with a formalism in imaginary rather than real time. Note also that in the limit of small k and ω, Eqs. (27)—(29) lead to a resonant frequency $\omega_{\text{res}} \sim k^{3/2}$, the well-known long-wavelength behavior of magnetophonons in two dimensions.[14]

Equations (25)—(29) can easily be combined to obtain the asymptotic form for the screened interaction at zero frequencies and small k;

$$\widetilde{G}(\mathbf{k}, \omega=0) = 1/(Ak^2 + Bk) , \tag{30}$$

where A and B are related to the other constant of Eqs. (25)—(29). This asymptotic form is the same as that obtained in the preceding section for the screened interaction between vortices in a superconducting array, and leads to the same static interaction between vortices: logarithmic, for separation less than a screening length λ and $(e^{*2})/\epsilon r$ for greater separations, where $e^* = ev$, the charge per quantum of flux, is the total charge in the screening cloud surrounding the vortex and ϵ is the background dielectric constant. Thus, we see that, near a "magic" density where $\bar{h} = m$ is an integer [that is, $v = 1/(2m+1)$], Eq. (24) can be interpreted simply to be the action of a dilute gas of quasiparticles with core size λ and charge e^*. The quasiparticle creation energy (that is, the energy to create one vortex) is

$$E_{\text{QP}} = K \int_{-\infty}^{\infty} \frac{d\tau}{\beta} \widetilde{G}_{\alpha\alpha}(\tau) . \tag{31}$$

When $K > \hbar/\tau_0$, which seems to be the case in the 2DEG at most relevant densities, λ is of order the lattice constant l, and hence E_{QP} can be seen to be roughly $E_{\text{QP}} \sim 0.05(e^*)^2/\epsilon l$. Because of the retarded nature of the screening in Eq. (24), the quasiparticles have complicated dynamics. While we expect that in the dilute limit, at least, the quasiparticles should have the same dynamics as point particles of charge e'' in a large magnetic field, we have not been able to demonstrate this from Eq. (24).

Finally, we note that if the bare interaction between the electrons in the Wigner lattice were not the Coulomb interaction, $V(r) = e^2/\epsilon r$, but rather some other shorter-ranged interaction, the results would not be fundamentally changed. The quasiparticle interaction still has a $\ln|r|$ core for $r > \lambda$, where λ is again the core radius, but for $r < \lambda$ the interactions vary as $V(r)$ so long as $V(r)$ is shorter range than $\ln|r|$.

IV. DISCUSSION

We have presented in this paper a model Hamiltonian consisting of a frustrated *XY* model with screening, and have applied the model to two physical examples: a superconducting array in an applied transverse magnetic field, and a two-dimensional Wigner lattice of electrons in a strong magnetic field, dominated by cooperative ring exchange among electrons. In the first example, the screening is produced by Josephson supercurrents flowing between the superconducting grains in response to the applied magnetic field. The magnetic field generated by these supercurrents screens the interactions between the fractionally charged vortices in the array, converting them from logarithmic to $1/r$ at sufficiently large separations,

but the screening length proves to be larger than the size of the sample in most cases of practical interest. In the case of a Wigner lattice in a strong magnetic field, the screening between the elementary excitations is due to the magnetophonons of the lattice, which couple to the applied magnetic field by causing the area, and hence the flux per plaquette, to vary dynamically. Once again, in the static limit, the logarithmic vortex-vortex interaction is converted to $1/r$ at large separations by the effects of screening.

The basic Hamiltonian discussed here, consisting of a frustrated *XY* model with an additional screening term, no doubt has other applications It is suggested, for example, whenever a tendency for a commensurate lock-in transition of a vortex liquid (embodied in the frustrated *XY* model) competes with a tendency for the liquid to remain at a fixed density (embodied in the screening term). Our results show that the screening can be treated in a straightforward way to result in a renormalized interaction between the excitations of the system.

ACKNOWLEDGMENTS

APPENDIX: DERIVATION OF MAGNETOPHONON PROPAGATOR

In this appendix, we derive Eq. (27) for the inverse magnetophonon propagator, $g(k,\omega)$, needed for calculating the screened interaction for the cooperative-ring-exchange model. This propagator is simply the phonon propagator for a two-dimensional Wigner crystal in the limit of a strong magnetic field. It is more easily derived from the Euclidean action

$$S_\beta = \int_0^\beta d\tau \left[-i \sum_{\mathbf{R}} \frac{d\boldsymbol{\phi}_{\mathbf{R}}}{d\tau} \times \boldsymbol{\phi}_{\mathbf{R}'}/(2l_0^2) \right.$$

$$\left. + \frac{1}{2} \sum_{\mathbf{R}} \sum_{\mathbf{R}'} \boldsymbol{\phi}_{\mathbf{R}} \cdot \overset{\leftrightarrow}{\mathbf{D}}(\mathbf{R} - \mathbf{R}') \cdot \boldsymbol{\phi}_{\mathbf{R}'} \right] . \tag{A1}$$

Here $\mathbf{R}$ labels the direct lattice to which the lattice of plaquettes is dual, and $\boldsymbol{\phi}_{\mathbf{R}}$ is the displacement of the electron at site $\mathbf{R}$ from its equilibrium position. $\overset{\leftrightarrow}{\mathbf{D}}(\mathbf{R} - \mathbf{R}')$ is the dynamical matrix for the Wigner lattice in the absence of a magnetic field, while the first term denotes the additional contribution to the action, $\int (1/c)\mathbf{J} \cdot \mathbf{A} d\tau$, arising from the presence of the magnetic field.

Next, we introduce a vector operator $\boldsymbol{\Delta}_\alpha$ which relates the field h_α through the αth plaquette to the displacements $\boldsymbol{\phi}_{\mathbf{R}}$:

$$h_\alpha = \sum_R \Delta_\alpha \cdot \phi_R \ . \tag{A2}$$

(Δ_α is the discrete gradient operator.) We Fourier-transform the action (A1) to obtain

$$S = \int_0^\beta d\tau \sum_k \left[-i \left| \frac{d\phi_k}{d\tau} \times \phi_k \right| \bigg/ (2l_0^2) + |k \cdot \phi_k|^2 [V(k)/2] \right.$$

$$\left. + |k \times \phi_k|^2 [K(k)/2] \right] , \tag{A3}$$

which can be written more conveniently with the decomposition

$$\phi_k = u\hat{k} + v\hat{k} \times \hat{z} \tag{A4}$$

as

$$S = \int_0^\beta \sum_k \{ [Kk^2/2][v - iu/(Kk^2l_0^2)]^2 + \dot{u}^2/(2Kk^2l_0^4)$$

$$+ 2k^2 Vu^2/2 \} d\tau \ . \tag{A5}$$

We now integrate out v and make the substitution $h(k) = \Delta(k)u_k$. The resulting expression for S is a quadratic form in h:

$$S = \sum_k \int_0^\beta d\tau \{\dot{h}^2/[2\Delta^2 Kk^4 l_0^4] + Vh^2/(2\Delta^2)\}$$

$$= \sum_k \sum_\omega |h(k,\omega)|^2 g^{-1}(k,\omega) , \tag{A6}$$

where the sum over ω runs over Matsubara frequencies $\omega = 2\pi n / \beta\hbar$ and

$$g^{-1}(k,\omega) = (\tau_0/2)[(\omega\tau_0^{-1})^2/(\Delta^2 Kk^4 l_0^4) + V(k)/\Delta^2] \tag{A7}$$

$$= X(k)[\omega^2 + \Omega^2(k)] , \tag{A8}$$

where g^{-1} is the Fourier transform of the propagator in Eq. (23). Here

$$X(k) = 1/\{2\tau_0\Delta^2(k)K[kl_0]^4\} \rightarrow (k_1/k)^4 \text{ as } k \rightarrow 0 ,$$

$$\Omega^2(k) = (\tau_0)^2 V(k)K(k)[kl_0]^4 \rightarrow (k/k_0)^3 \text{ as } k \rightarrow 0 .$$

The final expressions follow from the fact that $\Delta(k) \sim k$ and $V(k) \sim e^2/(\epsilon k)$ at small k, and $K(k) \sim \kappa_S$ at small k, where κ_S is the shear modulus. Note that the propagator involves a denominator of $\omega^2 + \Omega^2$ instead of $\omega^2 - \Omega^2$. This difference of sign is a result of the integration over imaginary time appearing in the action.

When Eqs. (A8)–(A10) are used in screening the interaction between quasiparticles in Sec. III, the remaining results follow in a straightforward way.

[1]For recent experiments on Josephson- or proximity-effect arrays in a magnetic field, see, e.g., B. Pannetier, J. Chaussy, D. Rammal, and J. C. Villegier, Phys. Rev. Lett. **53**, 1845 (1984); R. A. Webb, R. F. Voss, G. Grinstein, and P. M. Horn, *ibid.* **47**, 1542 (1983); D. J. Resnick, J. C. Garland, J. T. Boyd, S. Shoemaker, and R. S. Newrock, *ibid.* **47**, 1542 (1981); Ch. Leeman, Ph. Lerch, G.-A. Racine, and P. Martinoli, *ibid.* **56**, 1291 (1986); A. Behrooz, M. J. Burns, H. Deckman, D. Levine, B. Whitehead, and P. M. Chaikin, *ibid.* **57**, 368 (1986); C. J. Lobb, D. W. Abraham, and M. Tinkham, Phys. Rev. B **27**, 150 (1983).

[2]For recent studies of the frustrated XY model relevant to superconducting arrays, see, e.g., S. Teitel and C. Jayaprakash, Phys. Rev. B **27**, 598 (1983); Phys. Rev. Lett. **51**, 1999 (1983); W. Y. Shih and D. Stroud, Phys. Rev. B **28**, 6575 (1983); **30**, 6774 (1984); **32**, 158 (1985); M. Y. Choi and S. Doniach, *ibid.* **31**, 4516 (1985); T. C. Halsey, *ibid.* **31**, 4728 (1985); Phys. Rev. Lett. **55**, 1018 (1985); E. Granato and J. M. Kosterlitz, Phys. Rev. B **33**, 4767; **33**, 6533 (1986).

[3]S. Kivelson, C. Kallin, D. P. Arovas, and J. R. Schrieffer, Phys. Rev. Lett. **56**, 873 (1986).

[4]J. M. Kosterlitz and D. J. Thouless, J. Phys. C **6**, 1181 (1973).

[5]For a description of the resistively shunted junction model, see, e.g., M. Tinkham, *Introduction to Superconductivity*

(McGraw-Hill, New York, 1975).

[6]S. Chakavarty, G.-L. Ingold, S. Kivelson, and A. Luther, Phys. Rev. Lett. **56**, 2303 (1986).

[7]B. G. Orr, H. M. Jaeger, A. M. Goldman, and C. G. Kuper, Phys. Rev. Lett. **56**, 378 (1986).

[8]See, e.g., M. Tinkham, *Introduction to Superconductivity*, Ref. 5.

[9]J. Villain, J. Phys. C **10**, 1717 (1977); **10**, 4793 (1977).

[10]C. J. Lobb, D. Abraham, and M. Tinkham, Ref. 1.

[11]J. Pearl, in *Proceedings of the 9th Conference on Low Temperature Physics, Columbus, Ohio, 1964*, edited by J. G. Daunt, D. O. Edwards, F. J. Milford, and M. Yaqub (Plenum, New York, 1965), p. 506.

[12]See, for instance, D. C. Tsui, H. L. Stormer, and A. L. Gossard, Phys. Rev. Lett. **48**, 1559 (1982); H. L. Stormer, A. M. Chang, D. C. Tsui, J. C. M. Hwang, A. C. Gossard, and W. Wiegmann, *ibid.* **50**, 1953 (1983).

[13]For more discussion of the treatment of tunneling by the Euclidean action formalism, see, e.g., S. Coleman, in *The Whys of Subnuclear Physics*, edited by A. Zichichi (Plenum, New York, 1977).

[14]See, for instance, L. Bonsall and A. A. Maradudin, Phys. Rev. B **15**, 1959 (1977).