PHYSICAL REVIEW B VOLUME 35, NUMBER 7 1 MARCH 1987

Phenomenology and neglect of irrelevant variables for ferromagnetic systems

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(Received 2 September 1986)

An earlier proposed phenomenology for the description of the singular behavior of a ferromagnet in an extended region about its critical point is reexamined in light of certain renormalization-group results. It is established that if irrelevant variables are neglected, then the free energy predicted by the renormalization group will be identical to that of the phenomenology, provided the analytic coefficient functions u and v associated with the latter are selected appropriately. Explicit formulas for u and v in terms of the underlying nonlinear scaling fields g_t and g_h are obtained and used to derive exact relations among the analytic corrections to scaling for the leading singular parts of certain thermodynamic quantities. The results are compared with experimental values for the zero-field susceptibility of nickel.

I. INTRODUCTION

Consider a ferromagnetic, or similar thermodynamic system that for sufficiently low temperatures is characterized by a first-order phase boundary ending in an ordinary critical point. In an earlier publication[1] we proposed a phenomenological description for such a system in an extended region around the critical point including the phase boundary. The phenomenology involves two functions u and v, each analytic in t and h, where[2] $t=T/T_c-1$ is the relative temperature and $h=H/k_BT$ with H the external field and determined so that the free energy F for the system is an appropriate solution of the first-order partial differential equation

$$u\frac{\partial F}{\partial t}+v\frac{\partial F}{\partial h}=F+A\ , \tag{1}$$

where $A\equiv A(t,h)$ is an analytic background term. The singular behavior of F in the asymptotic region about the critical point is assured by the requirement[1] that u and v vanish there simultaneously as

$$u\simeq t/(2-\alpha),\ \ v\simeq h\Delta/(2-\alpha),\ \ h,t\to 0\ , \tag{2}$$

where α is the specific-heat exponent and $\Delta=2-\alpha-\beta$ with β the exponent for the spontaneous magnetization. Since F is an even function of h, it follows that u and A must also be even in h and that v must be odd in h.

Consider now this same system but from the viewpoint of the renormalization group[3–5] (RG). According to RG studies, if we neglect the effects of irrelevant variables, then near the critical point the singular part of the free energy, F_s may by expressed in the form[6]

$$F_s(t,h)=|g_t|^{2-\alpha}Y_\pm(g_h/|g_t|^{\Delta})\ , \tag{3}$$

where α and Δ are the critical exponents and t and h are, respectively, the relative temperature and field strength as defined above. The $Y_\pm$ are two universal functions of the indicated variable and g_t and g_h are the nonlinear scaling fields of the RG which in the absence of irrelevant variables are[6] analytic functions of t and h. By symmetry, g_t and g_h are even and odd in h, respectively, and in the asymptotic region about the critical point they vary as

$$g_t\simeq t,\ \ g_h\simeq h,\ \ t,h\to 0 \tag{4}$$

relations which also serve to fix overall factors in g_t and g_h.

The purpose of this paper is to establish that if g_t and g_h are related to u and v in a certain way [Eq. (5), below], then the phenomenology given by Eq. (1) is essentially equivalent to the RG formula in Eq. (3). Specifically, we first show that given g_t and g_h, the singular part of the free energy, F_s, as given by the RG in the absence of irrelevant variables, will satisfy the phenomenological Eq. (1) (with $A\equiv 0$), provided the analytic functions u and v there are expressed appropriately in terms of the nonlinear scaling fields g_t and g_h. Secondly, we establish the converse; that is, that if the coefficient functions u and v are known—experimentally or otherwise—then g_t and g_h can be determined (up to constant factors) in such a way that F_s in Eq. (3) reproduces whatever experimental or theoretical data were used to obtain u and v in Eq. (1) in the first instance. In this sense, then, Eqs. (1) and (3) are equivalent. Thirdly, we make use of Eqs. (1) and (3) to develop formulas valid for small h and compare the results with experimental values for the zero-field susceptibility of nickel.

II. PROOF OF EQUIVALENCE

Suppose first that the nonlinear scaling fields g_t and g_h are known. It is straightforward to establish for this case that a sufficient condition that F_s, as given in Eq. (3), satisfy the singular part of Eq. (1) (with $A\equiv 0$) is that u and v be determined by the relations

$$u\frac{\partial g_t}{\partial t}+v\frac{\partial g_t}{\partial h}=\frac{1}{2-\alpha}g_t\ ,\qquad u\frac{\partial g_h}{\partial t}+v\frac{\partial g_h}{\partial h}=\frac{\Delta}{2-\alpha}g_h\ . \tag{5}$$

For if we substitute the form for F_s as given in Eq. (3) into the phenomenological Eq. (1) (with $A\equiv 0$), we find that F_s is indeed a solution for any (differentiable) choice of the universal functions $Y_\pm$, provided only that g_t and g_h satisfy Eqs. (5). Note that Eqs. (5) are consistent with (i) the fact that $u(v)$ and $g_t(g_h)$ are even (odd) in h, and (ii) the limiting forms for u, v, g_t, and g_h, as given in Eqs. (2) and (4). For substituting, say, Eqs. (4) into Eqs. (5), we regain Eqs. (2). Similarly, if we substitute Eqs. (2) into Eqs. (5), we regain Eqs. (4) but only up to constant factors in this case.

More generally, if we solve Eqs. (5) for u and v, there results

$$u=\frac{1}{(2-\alpha)J}(g_t g_{hh}-\Delta g_h g_{th}) ,$$
$$v=\frac{1}{(2-\alpha)J}(\Delta g_h g_{tt}-g_{ht}) , \tag{6}$$

where $g_{tt}=\partial g_t/\partial t$, $g_{th}=\partial g_t/\partial h$, etc., and $J\equiv J(t,h)$ is the Jacobian

$$J=g_{tt}g_{hh}-g_{th}g_{ht} . \tag{7}$$

Since g_t and g_h are each analytic in t and h, it follows that, as required, so will be u and v, provided only that $J\neq 0$. The fact that this latter condition is satisfied, at least in the region around the critical point, can be seen by substituting the asymptotic forms in Eqs. (4) into Eq. (7). We find $J\simeq 1+O(t,h^2)$, so that at least in some finite region around the critical point, u and v in Eqs. (6) will be analytic. We conclude therefore that if u and v are determined in terms of g_t and g_h, in accordance with Eqs. (5)–(7), then u and v are indeed analytic in t and h, as required, and the formula for F_s in Eq. (3) will satisfy the phenomenology in Eq. (1) for the given u and v.

Consider now the converse. To this end suppose that the analytic functions u and v are now known—experimentally or otherwise—and ask if we can determine two analytic functions g_t and g_h in terms of u and v so that F_s in Eq. (3) reproduces the free energy obtained by use of Eq. (1). The answer is again affirmative provided u, v, g_t, and g_h are related by Eqs. (5) which now must be viewed as two partial differential equations for the unknowns g_t and g_h given u and v. Unfortunately, the situation for this case is not nearly as straightforward as above where we could solve for u and v directly to obtain Eqs. (6) and thus requires a somewhat more elaborate discussion.

To solve Eqs. (5) for g_t and g_h, note firstly that these two equations are linear and homogeneous and thus at best are determined only up to constant factors. Secondly, even disregarding this undetermined multiplicative factor, the solutions of Eqs. (5) are not unique; for given any solution of, say, the first of Eqs. (5), we can obtain another by adding to it an arbitrary function of a solution of the associated homogeneous equation obtained by setting the g_t term on the right-hand side of the first of Eqs. (5) to zero.[7] Thirdly, the similarity between Eqs. (5) and the singular part of Eq. (1)—i.e., with $A\equiv 0$—suggests the possibility that *analytic* solutions for g_t and g_h of Eqs. (5) may not exist at all! Thus to complete the analysis it is necessary to show unambiguously that there exist analytic solutions to Eqs. (5) and that these are unique up to an overall factor.

To demonstrate this fact that there exist such analytic solutions to Eqs. (5) let us make use of the analyticity and symmetry properties of u and v to expand them in powers of h:

$$u=u_0+h^2u_1+\cdots , \quad v=hv_0+h^3v_1+\cdots , \tag{8}$$

where $u_0,v_0,u_1,v_1,\cdots$ are analytic in t. Since the sought-for solutions for g_t and g_h must also be analytic, let us assume the corresponding expansions

$$g_t=g_t^0+h^2g_t^1+\cdots , \quad g_h=hg_h^0+h^3g_h^1+\cdots . \tag{9}$$

Our task thus becomes that of showing that $g_t^0,g_h^0,g_t^1,g_h^1,\cdots$ can be determined to be analytic in t and in a way so that Eqs. (5) are satisfied.

To proceed, we substitute Eqs. (8) and (9) into Eqs. (5). Equating equal powers of h on both sides of the resulting relations we find the set of ordinary differential equations

$$u_0\frac{d}{dt}g_t^0=\frac{1}{2-\alpha}g_t^0 ,$$
$$u_0\frac{d}{dt}g_h^0=\left[\frac{\Delta}{2-\alpha}-v_0\right]g_h^0 ,$$
$$u_0\frac{d}{dt}g_t^1=\left[\frac{1}{2-\alpha}-2v_0\right]g_t^1-\frac{1}{2-\alpha}\frac{u_1}{u_0}g_t^0 , \tag{10}$$
$$u_0\frac{d}{dt}g_h^1=\left[\frac{1}{2-\alpha}-3v_0\right]g_h^1-v_1g_h^0-\frac{u_1}{u_0}\left[\frac{\Delta}{2-\alpha}-v_0\right]g_h^0 ,$$

where in the last two equations we have simplified the right-hand sides slightly by use of the first two. These four relations and the corresponding higher-order ones are ordinary, linear, first-order differential equations and are easily solved given u_0, v_0, u_1, and v_1 as we presume here.

To display explicit solutions, it is convenient to introduce a function $\phi(a,b,t)$ of t, which depends on two real parameters a and b, and is defined by

$$u_0\frac{d\phi}{dt}=\left[\frac{1}{a}-bv_0\right]\phi . \tag{11}$$

This has the solution

$$\phi(a,b,t)=\exp\left[\int^t\frac{dt'}{u_0}\left[\frac{1}{a}-bv_0\right]\right] . \tag{12}$$

where under the integral u_0 and v_0 are taken to be functions of the dummy variable of integration t' and the integral is indefinite with the additive constant corresponding to an undetermined overall factor in ϕ. Making use of Eqs. (2), according to which $u_0\sim t/(2-\alpha)$, $v_0\sim\Delta/(2-\alpha)$ near the critical point, we find that ϕ has the form

$$\phi(a,b,t)=|t|^{(2-\alpha)/a-b\Delta}a_0(t) \tag{13}$$

with $a_0(t)$ an analytic function of t with $a_0(0)\neq 0$. For the special case when the exponent $[(2-\alpha)/a-b\Delta]$ is an integer, ϕ is analytic in t and the absolute value signs in Eq. (13) are to be dropped.

In terms of the function $\phi(a,b,t)$, the solutions of Eqs. (10) are as follows:

$$g_t^0=\phi(2-\alpha,0,t) ,$$
$$g_h^0=\phi((2-\alpha)/\Delta,1,t) ,$$
$$g_t^1=-\phi(2-\alpha,2,t)\int \frac{u_1}{u_0^2}\frac{1}{2-\alpha}\frac{g_t^0}{\phi(2-\alpha,2,t')}dt' , \tag{14}$$
$$g_h^1=-\phi(2-\alpha,3,t)\int \frac{dt'}{\phi(2-\alpha,3,t')} \times\left[v_1+\frac{u_1}{u_0}\left[\frac{\Delta}{2-\alpha}-v_0\right]\right]\frac{g_h^0}{u_0} ,$$

where the functions u_0, v_0, g_t^0, g_h^0, u_1, and v_1 in the integrands are functions of the dummy variable t' and the integrals themselves are indefinite. Reference to Eq. (13) shows that $g_t^0=ta_1(t)$ and $g_h^0=a_2(t)$ with a_1 and a_2 analytic functions of t with $a_1(0)\neq 0$ and $a_2(0)\neq 0$. Thus, as required, g_t^0 and g_h^0 are analytic in t. The overall factor in ϕ remains undetermined and corresponds to an undetermined multiplicative factor in $a_1(t)$ and $a_2(t)$. With regard to g_t^1 and g_h^1 in the third and fourth of Eqs. (14), brief reflection shows that these can also be made to be analytic by simply selecting the additive constants associated with the indefinite integrals to be zero. For since for small t, g_t^0 and u_0 both vanish linearly in t, whereas u_1, v_1, and g_h^0 are constant ($\neq 0$) for small t, and since $\phi(2-\alpha,2,t)\sim |t|^{1-2\Delta}$ and $\phi(2-\alpha,3,t)\sim |t|^{1-3\Delta}$, it follows that only if the additive constants associated with the integrals *fail* to vanish will g_t^1 and g_h^1 in Eqs. (14) be singular. Thus, in all cases, we have determined analytic solutions for g_t^0, g_h^0, g_t^1, and g_h^1 and thereby have obtained analytic solutions for g_t and g_h in Eqs. (5) through order h^3. Corresponding results can be obtained similarly in higher order.

III. ANALYTIC CORRECTIONS TO SCALING

In this section we apply the above results to derive certain formulas which relate, to all orders, the analytic corrections to scaling for certain thermodynamic quantities. Of particular interest is a formula for the temperature variation of the zero-field susceptibility which will be compared with experimental data in the following section. To simplify matters let us focus on the region $T\leq T_c$ for which $t\leq 0$. The corresponding formulas for $T\geq T_c$ can generally be obtained from these since most of the functions we deal with, $g_t^0, u_0, \cdots$, are analytic in t.

Consider again the phenomenological Eq. (1) for small h and suppose that u and v have been expanded as in Eq. (8). Since the analytic background term A is also analytic it may be expanded similarly as

$$A=A_0+A_1h^2+\cdots , \tag{15}$$

where A_0 and A_1 are analytic in t and where, without loss of generality, we may assume $A_0(0)=0$ since $A_0(0)$ corresponds to an irrelevant additive constant to the free energy, according to Eq. (1). Consider now Eq. (1) in the limit of small h. We derive four equations from this relation as follows: the first we obtain by setting $h=0$, and the remaining ones by differentiating Eq. (1) with respect to h i times ($i=1,2,3$) and setting $h=0$ after each differentiation. We find in this way the four relations

$$\begin{aligned} &u_0\frac{dF_0}{dt}-F_0=A_0 ,\\ &u_0\frac{dM_0}{dt}-M_0(1-v_0)=0 ,\\ &u_0\frac{d\chi_0}{dt}-\chi_0(1-2v_0)=2\left[A_1-u_1\frac{dF_0}{dt}\right] ,\\ &u_0\frac{d\psi_0}{dt}-\psi_0(1-3v_0)=-6M_0\left[v_1+\frac{u_1}{u_0}(1-v_0)\right] ,\end{aligned} \tag{16}$$

where F_0 is the zero-field free energy, $M_0\equiv-(\partial F/\partial h)_{h=0}$ is the spontaneous magnetization, $T\chi_0\equiv-(\partial^2F/\partial h^2)_{h=0}$ is the zero-field susceptibility, and $T^2\psi_0\equiv-(\partial^3F/\partial h^3)_{h=0}$ is the field derivative of the susceptibility in zero field. Note that the M_0 and ψ_0 equations are independent of the analytic background term $A=A_0+A_1h^2+\cdots$, while the F_0 and χ_0 equations are dependent on it.

A number of interesting consequences regarding the four zero-field thermodynamic functions, F_0, M_0, χ_0, follow from the differential Eqs. (16). First, as will be established in Appendix A, they have the respective analytic structures

$$\begin{aligned} &F_0(t)=A_F|t|^{2-\alpha}f_0(t)+a_0(t) ,\\ &M_0(t)=B_0|t|^{\beta}m_0(t) ,\\ &T\chi_0(t)=C_0|t|^{-\gamma}p_0(t)+|t|^{1-\alpha}D_\chi(t)+b_0(t) ,\\ &T^2\psi_0(t)=E_0|t|^{-\gamma-\Delta}q_0(t)+J_0|t|^{\beta-1}q_1(t) .\end{aligned} \tag{17}$$

In these formulas $\gamma=\Delta+\beta$; A_F, B_0, C_0, E_0, and J_0 are constants; $f_0(t)$, $m_0(t)$, $p_0(t)$, $q_0(t)$, and $q_1(t)$ are analytic in t normalized so that $f_0(0)=m_0(0)=p_0(0)=q_0(0)=q_1(0)=1$; and $a_0(t)$, $b_0(t)$, and $D_\chi(t)$ are analytic with $a_0(0)\neq 0$ and $D_\chi(0)\neq 0$. The analysis in Appendix A also shows that for the case of a logarithmic specific-heat singularity, corresponding to $\alpha=0$, the factor $|t|^{-\alpha}$ in the first and third of these relations are to be replaced by $\ln|t|$. Secondly, we establish in Appendix B that the four functions $f_0(t)$, $m_0(t)$, $p_0(t)$, and $q_0(t)$ which give the analytic corrections to scaling for the leading singular parts for F_0, M_0, χ_0, and ψ_0, respectively, are not independent but are related by

$$p_0(t)=\frac{m_0^2(t)}{f_0(t)} \tag{18}$$

and

$$q_0(t)=\frac{p_0^2(t)}{m_0(t)}=\frac{m_0^3(t)}{f_0^2(t)} , \tag{19}$$

where the second equality here follows from Eq. (18). These relations imply that if, as for certain two-dimensional Ising models, the leading analytic corrections to scaling, $f_0(t)$ and $m_0(t)$, are known, then $p_0(t)$, $q_0(t)$, and corresponding higher-order functions associated with the field derivatives of the free energy may also be computed directly. If we expand both sides of Eq. (18) and equate the coefficients of the linear and quadratic terms in t, we obtain among these coefficients two relations that are identical to those derived previously[6] by Aharony and Fisher directly from Eq. (3).

Of some interest is the fact, established in connection with the arguments presented in Appendix B, that certain combinations of the constants appearing in Eq. (17) are universal. Thus, in connection with the derivation of Eq. (18) we find the well-known result[8] that the quantity B_0^2/A_FC_0 is universal. Specifically, making use of the fact that the function Y_+ in Eq. (3) is universal, we find

$$\frac{B_0^2}{A_FC_0}=\left.\frac{(Y'_+)^2}{Y_+Y''_+}\right|_0 , \tag{20}$$

where the primes denote the derivative and the subscript 0 that the functions are to be taken with the argument zero. Similarly, the derivation of Eq. (19) in Appendix B leads to the universality relation

$$\frac{C_0^2}{B_0E_0}=\left.\frac{(Y''_+)^2}{Y'_+Y'''_+}\right|_0 . \tag{21}$$

As a final point it is interesting to note that it is also possible to relate F_{0s}, the singular part of F_0, and M_0, directly to g_t^0 and g_h^0. For, on comparing the first of Eqs. (10) and (B1) in Appendix B we conclude directly

$$F_{0s}=a_0\,|\,g_t^0\,|^{\,2-\alpha} , \tag{22}$$

with a_0 an undetermined constant which can be related to A_0 by comparison with the first of Eqs. (17). Similarly, by taking an appropriate linear combination of the first two of Eqs. (10) and the second of Eqs. (16), we obtain

$$M_0=B_0g_h^0\,|\,g_t^0\,|^{\,\beta} , \tag{23}$$

where the constant has been fixed by comparison with the second of Eqs. (17). By use of Eqs. (22) and (23), then we can express g_t^0 and g_h^0 directly in terms of M_0 and the singular part of the zero-field free energy.

IV. COMPARISON WITH EXPERIMENT

In this section we consider the possibility of comparing the results in Eqs. (17) and (18) with experiment. Because of the existence of the various unspecified analytic functions there, $f_0(t)$, $a_0(t)$, $m_0(t)$ $p_0(t)$, $D_\chi(t)$, $b_0(t)$, and so on, this comparison might be viewed as not being meaningful since there are so many free parameters available. Nevertheless, at least for Ni, we find that surprisingly few parameters are required so that Eqs. (17) and (18) reproduce experimental curves for M_0 and χ_0 for this material.

A comparison between the first two of Eqs. (17) and experiment was previously carried out[1] for nickel. Specifically, the limiting forms for $u_0(t)$ and $v_0(t)$ in Eqs. (2) were extended outside of the critical region by the formulas

$$u_0(t)=\frac{1}{2-\alpha}t(1+at) ,\qquad v_0=\frac{\Delta}{2-\alpha}(1+bt) , \tag{24}$$

where a and b are parameters to be determined by experiment. The substitution of these into the second of Eqs. (17) then gave for M_0 (normalized to unity at $T=0$)

$$M_0(t)=\frac{|\,t\,|^{\beta}}{(1-T/\xi T_c)^{\beta\xi}} , \tag{25}$$

where $\xi=1+b\Delta/\beta a$. For the choice[9] $b=-\beta/\Delta$ and $\xi=10$, corresponding to $a=-\frac{1}{9}$, excellent agreement was found for[10] $M_0(t)$ for Ni for which[11] $T_c=627.2$ K and[12] $\beta=0.3854$. The above form for M_0 agrees with experiment not only near T_c but throughout the temperature range $0<T\le T_c$. Unfortunately, it is not as easy to obtain experimental values for $f_0(t)$ as defined in the first of Eqs. (17). In principle, one could consider data for the specific heat of Ni and integrate twice to obtain F_0. But because of the fact that α is so small[13] ($\simeq -0.1$ for Ni), one cannot untangle so readily $f_0(t)$ from a_0 in the first of Eqs. (17). However, we can obtain $f_0(t)$ phenomenologically by integrating the first of Eqs. (16) by use of the form for $u_0(t)$ in Eq. (24). For later reference let us note here the result:

$$f_0(t)=(1+at)^{\alpha-2}=(1-a)^{\alpha-2}(1-T/\xi T_c)^{\alpha-2} . \tag{26}$$

Consider now the third of Eqs. (17) for the zero-field susceptibility. Here there are three undetermined analytic functions, namely, $p_0(t)$, $D_\chi(t)$, and $b_0(t)$. With an infinity of adjustable parameters thereby available, it would appear to be pointless to attempt a comparison with experiment for χ_0. However, because of Eq. (18), the quantity $p_0(t)$ is known to the same extent that $m_0(t)$ and $f_0(t)$ in the first two of Eqs. (17) are, and this suggests that at least close to T_c, where the $|\,t\,|^{-\gamma}$ term in the third of Eqs. (17) dominates, some meaningful comparison might be possible. This indeed turns out to be the case, at least for Ni and over a surprisingly large range of temperatures.

Let us consider the possibility of attempting a comparison with experiment by approximating χ_0 by the first of the three terms in the third of Eqs. (17). That is, by the relation

$$T\chi_0\simeq C_0\,|\,t\,|^{-\gamma}p_0(t) . \tag{27}$$

To proceed, we need to know both the constant C_0 and the analytic function $p_0(t)$. According to Eq. (18) the latter is the ratio $m_0^2(t)/f_0(t)$ and thus, in principle, can be determined from experimental measurements for the magnetization and specific heat. Unfortunately, as noted above, this does not work since $f_0(t)$ is difficult to obtain from specific-heat data because of its entanglement with the background term $a_0(t)$. To get around this difficulty let us make use of the procedure of Ref. 1 to assume Eqs. (24) and determine the parameters a and b from experimental values for $m_0(t)$ and thereby obtain the form for $f_0(t)$ in Eq. (26). In this way we obtain

$$p_0(t)=(1+at)^{2-\alpha-\beta\xi}\ , \tag{28}$$

with $\xi=1-1/a=10$. In the temperature region just above T_c, therefore, we would expect Eqs. (27) and (28), with an appropriate choice for the constant C_0 to reproduce experimental values for χ_0. Such a choice for C_0 is easily determined from the data of Weiss and Forrer[14] as analyzed by Kouvel and Fisher[11] near T_c. Extrapolating on a semilog plot of $T|t|^{\gamma}\chi_0$ against T as $T\to T_c$, we obtain the value

$$C_0=0.0033\pm0.0001\ \text{emu/g}\ . \tag{29}$$

Figure 1 shows experimental values of χ_0^{-1} for Ni as a function of T for $T_c<T<1.12T_c$ with $T_c=627.2$ K. Also shown are three theoretical curves. The top, or dashed curve, is the asymptotic scaling form, that is, the formula in Eq. (27) with $p_0(t)$ replaced by unity. The middle curve, which is manifestly closer to the data, is precisely Eq. (27) with $p_0(t)$ given by Eq. (28). We emphasize that the inclusion of $p_0(t)$ which involves *no* additional parameters improves the agreement with the experimental points. The third, or solid curve, is given by the expression

$$T\chi_0=C_0|t|^{-\gamma}p_0(t)+d_0\ , \tag{30}$$

with d_0 a constant and representing the first correction coming from the two terms that we have neglected in the third of Eqs. (17), namely, $|t|^{1-\alpha}D_\chi$ and $b_0(t)$. This solid curve in Fig. 1 represents Eq. (30) with the choice $d_0=0.0056$ and improves the agreement at higher temperatures. Furthermore, by taking a larger value for d_0, say, $d_0\cong0.01$, we can obtain virtually perfect agreement throughout the temperature interval $T_c\le T\le1.12T_c$. This is shown in Fig. 2 by the solid curve; the dashed curve represents, for comparison, Eq. (30) with the same values for C_0 and d_0 but with $p_0(t)=1$. Although the larger value for d_0 gives better agreement with experiment in the region $T\le1.12T_c$, the smaller value used in Fig. 1 seems to represent the experimental data better outside of this range.

This excellent agreement just above T_c in Figs. 1 and 2 between Eq. (30) and the experimental data, suggests that it might be worthwhile to attempt to extend this formula to higher temperatures. Figure 3 shows also the experimental points of Fallot[15] for χ_0^{-1} and the solid curve represents Eq. (30) with the parameters C_0 and d_0 having the same values as in Fig. 1. For comparison, we also include by the dashed curve the corresponding asymptotic scaling version of Eq. (30) obtained by replacing $p_0(t)$ by unity. Note that the agreement is good to excellent throughout the entire temperature interval, $T_c\le T\le3T_c$, although for the intermediate T values, $1.1T_c\lesssim T\lesssim1.9T_c$, the solid curve is slightly higher than the experimental points. If we utilize the value $d_0=0.01$ from Fig. 2 (keeping the same value for $C_0=0.0033$ as required by experiment for T near T_c), then although for the lower intermediate values the agreement is improved,

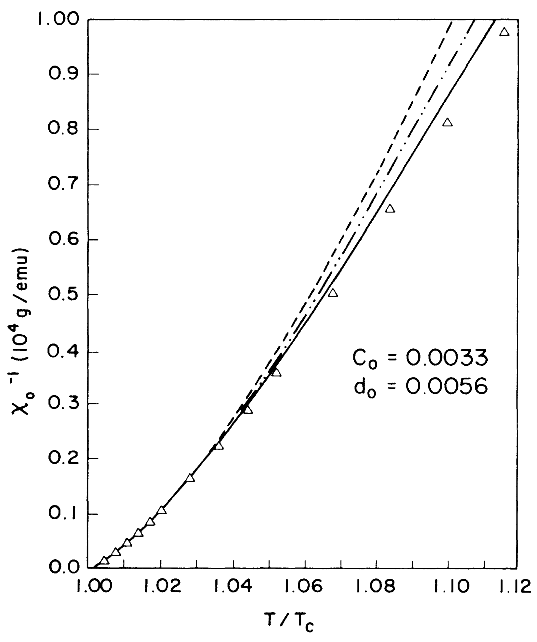

FIG. 1. Inverse susceptibility of Ni as a function of T/T_c. The theoretical curves are for Eq. (27) with $p_0(t)=1$ (dashed curve), Eq. (27) with $p_0(t)$ given by Eq. (28) (dashed-dotted curve), and for Eq. (30) with $p_0(t)$ given by Eq. (28) (solid curve). In this set of curves $C_0=0.0033$, $d_0=0.0056$, $a=-\frac{1}{9}$, and $\xi=10$. The values $\beta=0.3854$ and $\gamma=1.33$ are used here and in the three succeeding figures. The experimental points come from Ref. 14.

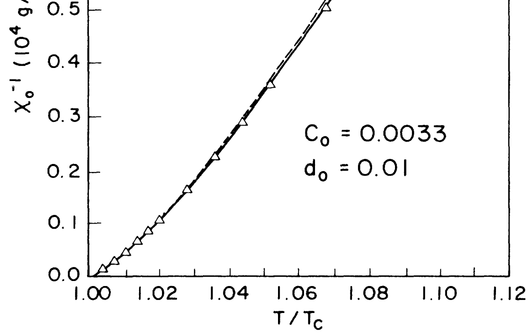

FIG. 2. Inverse susceptibility of Ni as function of T/T_c. The solid curve is the same as the solid curve in Fig. 1, but with $d_0=0.01$. The dashed curve is Eq. (30) with $d_0=0.01$ and $p_0(t)=1$. For both curves $C_0=0.0033$. The experimental points are the same as those in Fig. 1.

some disagreement is apparent for $T \gtrsim 2T_c$. Thus, disregarding this small descrepancy in the intermediate region, the agreement between the solid curve in Fig. 3 as given by Eq. (30) and the experimental points is surprisingly good. At least for Ni, then, it appears that the third of Eqs. (17), with the neglect of the singular $|t|^{1-\alpha}$ term and with the replacement of the analytic background term $b_0(t)$ by the constant d_0, agrees with experiment for the above values of C_0 and d_0 over an extensive temperature range—from 627.2 to 1882 K—above T_c. We emphasize that in Eq. (30) C_0 and d_0 are the *only* free parameters; the analytic function $p_0(t)$ has been completely determined by experimental values for the spontaneous magnetization in Eq. (28). Interestingly enough, M_0 is nonzero only for $T<T_c$, while we use Eq. (28) for $T>T_c$ for which experimental values for χ_0 are available.

In connection with the results in Figs. 1—3, it is of interest to consider the corresponding analysis for Ni carried out by Souletie and Tholence.[16] These authors fit the same data as in Figs. 1—3 by the scaling formula

$$\chi_0=\frac{C_0'}{T}|t'|^{-\gamma}+d_0' , \tag{31}$$

where[17] $t'=1-T_c/T\equiv t/(1+t)$ and with $C_0'=0.003\,59$ and $d_0'=2.43\times10^{-6}$. The point of interest here is that this formula also follows the experimental data fairly closely over the interval $T_c\le T\le 3T_c$ and if plotted would yield a curve very similar to that in Fig. 3. However, since near T_c, $t'\sim t$ and $C_0'\neq C_0$, we would expect our Eq. (30) to agree with the data for $T \gtrsim T_c$ a little better. This is shown in Fig. 4 in which the solid curve is our Eq. (30) [or equivalently in this region Eq. (27)] and the dashed curve, Eq. (31). If we attempt to rectify this difficulty by using for C_0' in Eq. (31) the value $C_0=0.0033$, and modifying the value for d_0' appropriately, it is possible for Eq. (31) to fit the data also in this region just above T_c. But then at higher temperatures the agreement is not good and Eq. (31) looks more like the dashed curve in Fig. 3. It is to be noted that Eqs. (17) and (18) would of course still be correct if we replaced the variable t by t'. We prefer to carry out the present analysis in terms of the variable t since then the obtaining of $m_0(t)$, $f_0(t)$, and thus $p_0(t)$ is much simpler than in terms of t', which is unbounded as $T\to0$.

V. SUMMARY AND CONCLUSIONS

In the first part of this paper we established the equivalence between the predictions of an earlier proposed phenomenology for a ferromagnetic system and that of the RG method with irrelevant variables neglected. Specifically, it was shown that if the analytic functions u and v of the phenomenology are related to the nonlinear scaling fields g_t and g_h in accordance with Eqs. (5) then these two approaches are essentially equivalent. Making use of the present formulation, then, we derived Eqs. (17), the first three of which were earlier obtained by Aharony and Fisher[6] by use of Eq. (3). Finally, we established that the analytic corrections to scaling for the leading singular

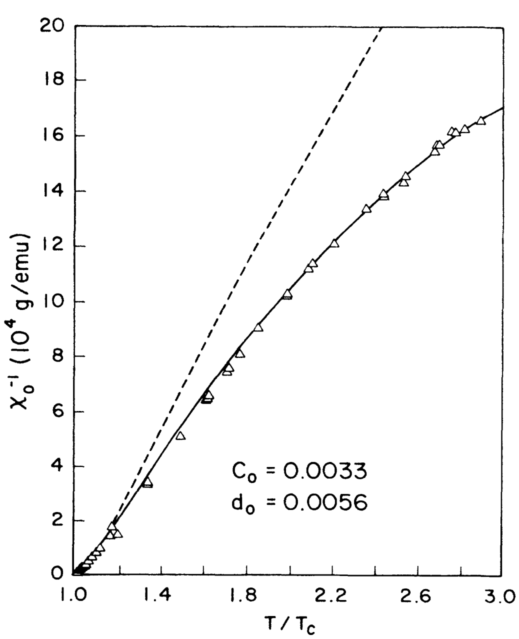

FIG. 3. Inverse susceptibility of Ni in the temperature range $T_c\le T\le 3T_c$. The experimental points for $T\gtrsim 1.2T_c$ are those of Ref. 15, while for $T_c\le T\le 1.12T_c$, those of Ref. 14. The solid curve is Eq. (30) with $C_0=0.0033$ and $d_0=0.0056$ and is the extension of the solid curve in Fig. 1 to higher temperatures. The dashed curve is Eq. (30) with $p_0(t)=1$.

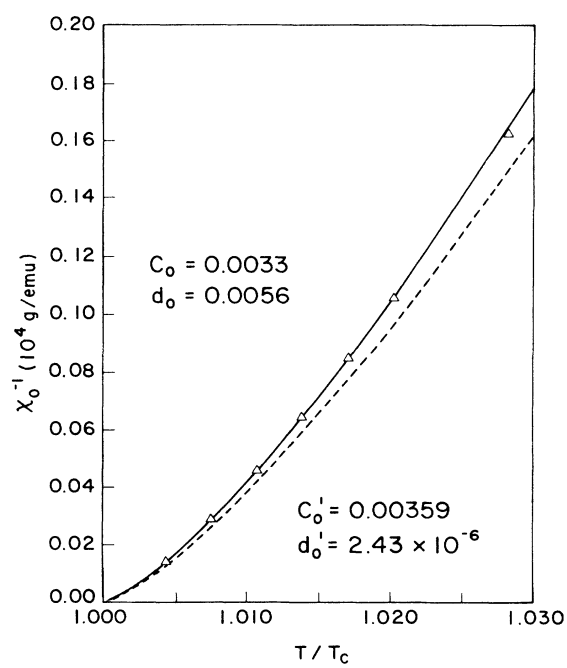

FIG. 4. Inverse susceptibility of Ni in the low-temperature range $T_c\le T\le 1.03T_c$. The solid curve is the same as that in Fig. 1 and the dashed curve is Eq. (31) with $C_0'=0.003\,59$ and $d_0'=2.43\times10^{-6}$ as given in Ref. 16.

parts of the various thermodynamic functions in Eqs. (17) are related to each other by Eqs. (18) and (19). Of particular theoretical interest here is the fourth of Eqs. (17) for which $J_0=0$ for $T>T_c$ so that ψ_0 is fully determined by magnetization and specific-heat data.

In the second part of this work we attempted to assess the usefulness of Eqs. (17) and (18) for analyzing experimental data. Of particular interest is the zero-field susceptibility of Ni for which ample experimental data exist. The dominant part of χ_0 near T_c is given by the first term $C_0|t|^{-\gamma}p_0(t)/T$ which when combined with data on the magnetization of Ni and, in principle, its specific heat, is completely determined by Eq. (18) up to the scale factor C_0. The difficulty associated with using specific-heat data to obtain $p_0(t)$ was overcome by use of the phenomenology of Ref. 1 and the agreement with experimental results for χ_0 are displayed in Figs. 1 and 2 for $T_c \le T \le 1.12T_c$. Evidentally, our prediction for the form and the presence of $p_0(t)$ is fully in accord with the data in this range. Furthermore, by approximating the remaining two terms $|t|^{1-\alpha}D_\chi(t)+b_0(t)$ by the constant d_0, we obtained Eq. (30) and were able to extend this agreement out to $3T_c$. This is shown in Fig. 3 and was unexpected and raises the question whether Eq. (30) can also be extended to other ferromagnetic materials of the three-dimensional Heisenberg universality class.

A final point of interest deals with the last of Eqs. (17) for $\psi_0(t)$, the field derivative of χ. In the region $T>T_c$, the constant $J_0=0$ so that $\psi_0(t)$ is completely determined up to the constant E_0. It would indeed be interesting to check this prediction of our phenomenology and of the RG without irrelevant variables.

APPENDIX A

To prove Eqs. (17) we proceed as follows. Making use of Eqs. (11)–(13) we may integrate the first of Eqs. (16) to

$$F_0(t)=\phi(1,0,t)\int_c^t \frac{dt'}{\phi(1,0,t')}\frac{A_0(t')}{u_0(t')} , \tag{A1}$$

with the lower limit c an undetermined constant of integration.[18] Since $\phi(1,0,t)=|t|^{2-\alpha}b(t)$ with $b(t)$ analytic, and since $A(0)=u_0(0)=0$, it follows that for $\alpha\neq 0$, the singular part of $F_0(t)$ comes exclusively from the lower limit and varies as $|t|^{2-\alpha}$. We note that unlike the analysis[6] which starts directly from Eq. (3), a separate treatment for the case $\alpha=0$ is not required here. For if $\alpha=0$, even though the factor $\phi(1,0,t)=t^2b(t)$ is analytic in t, the integral in (A1) is now found to have the $t^2\ln|t|$ singularity, known to characterize, for example, the two-dimensional Ising system. In either case, $F_0(t)$ in Eq. (17) will have precisely the structure given in Eq. (17) with $f_0(t)$ and $a_0(t)$ analytic and with the factor $|t|^{-\alpha}$ to be replaced by $\ln|t|$ for the case $\alpha=0$.

To obtain the second of Eqs. (17), we compare the second of Eqs. (16) with Eq. (11) and find

$$M_0(t)=\phi(1,1,t) . \tag{A2}$$

By use of Eq. (13) and the fact that $\beta=2-\alpha-\Delta$, this reduces directly to the second of Eqs. (17), a result previously obtained in a slightly different way.[1]

In a similar way, we obtain from the third of Eqs. (16) the solution

$$\chi_0(t)=\phi(1,2,t)\int_{c'}^t \frac{dt'}{\phi(1,2,t')}\frac{I_\chi(t')}{u_0(t')} , \tag{A3}$$

with $I_\chi=2[A_1(t)-u_1dF_0/dt]$. The structure for χ_0 in the third of Eqs. (17) then follows from the facts that $\phi(1,2,t)\simeq|t|^{2-\alpha-2\Delta}$, that $A_1(t)$ is analytic, and that $dF_0/dt\simeq|t|^{1-\alpha}$. For $\alpha=0$, the factor $|t|^{-\alpha}$ in the third of Eqs. (17) should be replaced by $\ln|t|$ just as for F_0.

Finally, the last of Eqs. (17) has the solution

$$\psi_0(t)=\phi(1,3,t)\int \frac{dt'}{\phi(1,3,t')}\frac{I_\psi(t')}{u_0(t')} , \tag{A4}$$

with $I_\psi(t)=-6M_0[v_1+u_1(1-v_0)/u_0]$. This time, since according to Eq. (13) $\phi(1,3,t)=|t|^{2-\alpha-3\Delta}d_0(t)$, we obtain the analyticity structure shown in the last of Eqs. (17). It is interesting to note that the analytic background term plays no role in determining the structure of M_0 or ψ_0.

APPENDIX B

The purpose of this appendix is to derive Eqs. (18) and (19) which relate to each other the analytic corrections to scaling for the leading singular terms in F_0, M_0, χ_0, and ψ_0. If we compare each of Eqs. (17) with the corresponding differential Eqs. (16), we see that the most singular parts of each of F_0, χ_0, and ψ_0 are the solutions of the respective homogeneous equations associated with Eqs. (16). If F_{0s}, χ_{0s}, and ψ_{0s} represent these most singular parts, we find from Eq. (16) that they satisfy the differential equations

$$\begin{aligned} u_0\frac{d}{dt}\ln F_{0s}&=1 ,\\ u_0\frac{d}{dt}\ln M_0&=1-v_0 ,\\ u_0\frac{d}{dt}\ln \chi_{0s}&=1-2v_0 ,\\ u_0\frac{d}{dt}\ln \psi_{0s}&=1-3v_0 , \end{aligned} \tag{B1}$$

where for convenience we have also included the corresponding M_0 equation.

To derive now Eq. (18) we multiply the second equation of (B1) by 2 and subtract from the result the sum of the first and the third of these relations. The result may be expressed by

$$u_0\frac{d}{dt}\ln(M_0^2/F_{0s}\chi_{0s})=0 ,$$

or equivalently that the quantity $M_0^2/F_{0s}\chi_{0s}$ is a t-independent constant. Making use of the first three of Eqs. (17), the exponent relation $2\beta=2-\alpha-\gamma$, and the fact that $f_0(0)=m_0(0)=p_0(0)=1$ to fix the constant, we obtain Eq. (18). Interestingly enough, the relation in Eq.

(18) can also be obtained directly from Eq. (3) by expanding g_t and g_h as in Eq. (9), and making appropriate comparisons of the results obtained by taking derivatives with respect to h. This incidentally also leads to the universality relation in Eq. (20).

The derivation of Eq. (19) follows along similar lines. This time we multiply the third of (B1) by 2 and subtract from it the result of adding the second to the fourth. The result may then be expressed as

$$u_0 \frac{d}{dt} \ln(\chi_{0s}^2 / M_0 \psi_{0s}) = 0 \, ,$$

so that the quantity $\chi_{0s}^2 / M_0 \psi_{0s}$ is a t-independent constant. The result in Eq. (19) then follows from this by use of Eqs. (17) and the fact that by our normalization $m_0(0) = p_0(0) = q_0(0) = 1$. As above, this relation can also be derived directly from Eqs. (3) and (9) and leads to the universal relations for the constants B_0, C_0, and E_0 in Eq. (21).

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$$\frac{dh}{dt} = \frac{v(t,h)}{u(t,h)}$$

will be a solution of the homogeneous equation.

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