Critical behavior of the n-vector model for $1 < n < 2$

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The Migdal-Kadanoff position-space renormalization-group scheme is used to study the critical behavior of the isotropic n-component-vector model in the previously unexplored region, $1 < n < 2$, $1 < d < 2$, of the n-d plane (d is the dimensionality of the space). We find a continuous phase transition at a finite temperature if $d \geq d_l(n)$. The lower critical dimension $d_l(n)$ increases continuously, but nonlinearly from 1 to 2 as n changes from 1 to 2. For $d_l(n) \leq d < 2$, the low-temperature phase is characterized by a power-law decay of the two-point correlation function with a temperature-independent exponent.

The isotropic n-component vector model has received a great deal of attention in the modern theory of phase transitions and critical phenomena.[1] The critical behavior of this model in several regions of the n-d plane (d is the dimensionality of space) is well understood. For all n, the critical behavior for $d \geq 4$ is correctly described by mean-field theory, and the critical properties for $d < 4$ have been calculated[1] as an expansion in $\epsilon = 4-d$. There is no finite-temperature phase transition for $n \geq 2$, $d < 2$. The behavior for $n > 2$ and $d \gtrsim 2$ has been studied[2] as an expansion in $\epsilon' = d-2$. Exact solutions are available at $d=1$ for all n,[3] for $n=-2$ and $n \to \infty$ at all d,[4] and at $d=2$, $n=1$.[5] For $n=d=2$, the phase transition is described by the Kosterlitz-Thouless theory.[6] However, not much information is available about the critical properties of this model in the region $1 < n < 2$, $1 < d < 2$. For any n, one may define a lower critical dimension $d_l(n)$ such that there is no finite-temperature phase transition for $d < d_l(n)$. Since $d_l(1) = 1$ and $d_l(n) = 2$ for $n \geq 2$, the lower critical dimension for $1 < n < 2$ is expected to lie between $d=1$ and $d=2$. However, the dependence of $d_l(n)$ on n in this region is not known. Furthermore, in view of the well-known result[7] that a system with continuous symmetry does not exhibit true long-range order for $d \leq 2$, it is interesting to enquire about the nature of the low-temperature phase (if it exists) for $1 < n < 2$, $1 < d < 2$.

A study of the critical behavior for $1 < n < 2$ may also be relevant in understanding the role of topological defects in the phase transition of this model. Few years ago, Cardy and Hamber[8] discussed the critical behavior of this model in the neighborhood of $n=d=2$, assuming analyticity of the renormalization-group (RG) equations in n and d. Their analysis suggests that there exists a line $d=d_c(n)$ in the n-d plane which passes through the point $(n,d) = (2,2)$, and has the following property: For $n > 2$, $d > 2$, topological defects play a crucial role in determining the nature of the phase transition if $d > d_c(n)$, and are unimportant if $d < d_c(n)$. The critical exponents are continuous but nonanalytic across this line. For $n < 2$, $d < 2$, $d_c(n)$ is identical to the lower critical dimension $d_l(n)$. The arguments of Cardy and Hamber, although intuitively appealing, are not conclusive. In particular, the RG equations used by them involve a "fugacity" parameter whose physical interpretation is unclear if $(n,d) \neq (2,2)$. Thus, it is important to check the predictions of their analysis by more direct calculations.[9] Also, their analysis suggests that a calculation of $d_l(n)$ for $n < 2$ would determine the slope of the line $d=d_c(n)$ at $(n,d) = (2,2)$ and thus would be useful in determining the role of topological defects in the phase transition in the physically interesting case, $(n,d) = (3,3)$. We also note that the critical properties of the n-vector model for nonintegral value of n is relevant to the statistical mechanics of a system of interacting loops,[10] with n playing the role of a "weight factor" associated with each loop.

In this paper, we present the results of a position-space renormalization-group study of the critical behavior of the n-vector model for arbitrary n, using the Migdal-Kadanoff[11] (MK) RG scheme. We concentrate in the region $1 < n < 2$, for which little information is available. The MK scheme has seen extensive use in studies of phase transitions. It provides qualitatively correct results for most systems, and is known to be reliable in determining the lower critical dimension. This method has been applied previously[11,12] to the n-vector model for $n=1$, 2, and 3. To our knowledge, our calculation is the first generalization to nonintegral values of n. The main results of our study are summarized below.

For $1 < n < 2$, the model exhibits a continuous phase transition at a finite temperature if $d \geq d_l(n)$. The lower critical dimension $d_l(n)$ changes continuously, but nonlinearly from 1 to 2 as n changes from 1 to 2. The slope of the line $d=d_l(n)$ is considerably less than unity ($\simeq 0.2$) at $n=d=2$. For $d_l(n) < d < 2$, the low-temperature phase is controlled by a stable fixed point at a nonzero temperature, indicating a power-law decay of correlations with a fixed exponent. (This is different from the behavior at $d=n=2$, where the exponent for the power-law decay of correlations in the low-temperature phase changes with temperature.[6]) The observed fixed-point structure is very similar to the behavior expected from the RG equations of Cardy and Hamber.[8]

We consider a O(n) model with fixed length spins and nearest-neighbor interactions on a d-dimensional hypercubic lattice, defined by the partition function

$$Z_N = \frac{1}{(S_n)^N} \int \prod_i d\Omega_i \prod_{\langle i,j \rangle} \exp[V(\cos\theta_{ij})] , \qquad (1)$$

where N is the number of spins, $S_n = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of a sphere in n dimensions, $\langle i,j \rangle$ represents distinct pairs of nearest-neighbor sites, $d\Omega_i$ denotes an integration over the n-dimensional spherical surface for the ith spin and θ_{ij} is the angle between the spin at sites i and j. The interaction for $n > 1$ is written in the general form

$$e^{V(\cos\theta)} = \sum_{l=0}^{\infty} B_{nl} A_l U_l^{\lambda}(\cos\theta) , \tag{2}$$

where the prefactors $B_{nl} = (2l + n - 2)/(n - 2)$ have been included for later convenience, the coefficients $\{A_l\}$ are non-negative and the U_l^{λ}'s are Gegenbauer polynomials [with $\lambda = (n-2)/2$] which form a complete set of orthogonal basis functions for n-dimensional spins ($n > 1$). The normalization of the U functions is chosen[13] so that

$$\int_{-1}^{+1} U_l^{\lambda}(x) U_{l'}^{\lambda}(x)(1-x^2)^{(n-3)/2} dx = C_{nl}\delta_{ll'} , \tag{3a}$$

$$C_{nl} = \frac{\sqrt{\pi}\,\Gamma(l+n-2)\Gamma((n-1)/2)}{(l+n/2-1)l!\,\Gamma(n-2)\Gamma((n-2)/2)} , \tag{3b}$$

where Γ represents the usual gamma function. The Gegenbauer polynomials become identical to Legendre polynomials for $n = 3$, and for $n = 2$, a trivial change of normalization reduces them to the usual cosine functions. The analytic continuation in n used in defining the model [Eq. (2)] for nonintegral values of n is essentially the same as the one used previously by several authors[3,14] to define the one-dimensional transfer matrix for arbitrary values of n [see discussion following Eq. (9) below]. For the starting reduced Hamiltonian, $V(\cos\theta) = K\cos\theta$, the coefficients $\{A_l\}$ have the form

$$A_l = \frac{2^{(n-2)/2}\Gamma(n/2)}{K^{(n-2)/2}} I_{n/2+l-1}(K) , \tag{4}$$

where I_ν is the modified Bessel function of the first kind of order ν. Since the RG transformation changes the form of the interaction, we treat the coefficients $\{A_l\}$ as independent parameters.

The MK transformation consists of two parts:[11] decimation in one-dimension and bond moving in the remaining $d-1$ dimensions. The effect of decimation on the model defined in Eq. (2) is easily obtained. Let i,j,k represent three consecutive sites along a particular direction. The effect of integrating out the spin at the middle site j is to produce an effective interaction, $V'(\cos\theta_{ik})$, between the spins at sites i and k, where

$$\exp[V'(\cos\theta_{ik})] = \frac{1}{S_n}\int d\Omega_j \exp[V(\cos\theta_{ij}) + V(\cos\theta_{jk})] . \tag{5}$$

The integral over the n-dimensional spherical surface can be written as one over $n-1$ angles:

$$\int d\Omega_j = \int_0^{\pi} (\sin\phi_j^{(1)})^{n-2} d\phi_j^{(1)} \int_0^{\pi} (\sin\phi_j^{(2)})^{n-3} d\phi_j^{(2)} \times \cdots \times \int_0^{2\pi} d\phi_j^{(n-1)} . \tag{6}$$

Choosing the (1) axis along $\mathbf{S}_i$ and the (2) axis in the $(\mathbf{S}_i, \mathbf{S}_k)$ plane, we have

$$\theta_{ij} = \phi_j^{(1)}, \quad \theta_{ik} = \phi_k^{(1)} , \tag{7}$$

$$\cos\theta_{jk} = \cos\phi_j^{(1)}\cos\phi_k^{(1)} + \sin\phi_j^{(1)}\sin\phi_k^{(1)}\cos\phi_j^{(2)} .$$

Using Eqs. (2) and (7), the integrand of Eq. (5) is written[13] as a sum of products of three U functions with arguments $\cos\phi_j^{(1)}$, $\cos\phi_j^{(2)}$, and $\cos\phi_k^{(1)}$, and the integrals over the angles are performed explicitly. The result is

$$e^{V'(\cos\theta_{ik})} = \sum_l B_{nl} A_l^2 U_l^{\lambda}(\cos\theta_{ik}) , \tag{8}$$

which implies that

$$R_d(b=2)[A_l] = A_l^2 , \tag{9}$$

where $R_d(b)$ represents the effect of decimation with a scale factor b. The above result shows that the parameters A_l are the eigenvalues of the one-dimensional transfer matrix for the model of Eq. (2). For $V(\cos\theta) = K\cos\theta$, the expression (4) for A_l reduces, as expected, to the form found in previous work.[3,14] The result (9) is easily generalized to arbitrary values of b:

$$R_d(b)[A_l] = A_l^b . \tag{10}$$

The effect of bond moving is to multiply the interaction $V(\cos\theta)$ by a factor b^{d-1}. For a general b, it is difficult to represent this as a transformation on the coefficients $\{A_l\}$. The situation simplifies if we choose $b^{d-1} = 2$. Then

$$e^{2V(\cos\theta)} = \sum_{l,l'} B_{nl} B_{nl'} A_l A_{l'} U_l^{\lambda}(\cos\theta) U_{l'}^{\lambda}(\cos\theta)$$

can be written as a series in $U_l^{\lambda}(\cos\theta)$ by using the standard rules[13] for decomposing the product of two U functions into a sum of single U functions. The result is a set of recursion relations of the form

$$R_m(b = 2^{1/(d-1)})[A_l] = \sum_{l_1,l_2} C(l; l_1, l_2) A_{l_1} A_{l_2} , \tag{11}$$

where R_m stands for bond moving and the coefficients $C(l; l_1, l_2)$ can be determined explicitly in terms of n, l, l_1, and l_2.

It is well known[11] that the transformations R_d and R_m do not commute for general $b > 1$. However, the transformations $R = R_d R_m$ and $R' = R_m R_d$ yield the same results for most of the quantities of interest here. In particular, it is easy to show that the existence of a nontrivial fixed point for R implies the same for R', i.e., both methods give the same result for the lower critical dimension $d_l(n)$. Also, we find that both R and R' yield the same values for the critical exponents. For this reason, we report only the results obtained from R. We have also considered the infinitesimal transformation, $b = 1 + \epsilon$, $\epsilon \to 0^+$, which is expected to provide more accurate results. In this limit, $R = R'$, and the effect of R is to change $V(\cos\theta)$ to $V'(\cos\theta)$, where

$$e^{V'(\cos\theta)}=e^{V(\cos\theta)}+\epsilon\left[(d-1)V(\cos\theta)e^{V(\cos\theta)}+\sum_l B_{nl}A_l(\ln A_l)U_l^{\lambda}(\cos\theta)\right]+O(\epsilon^2)\ . \tag{12}$$

This equation is reduced to a set of recursion relations for the coefficients $\{A_l\}$ by writing $V(\cos\theta)$ as an expansion in the U functions and then using the decomposition rules for products of two U functions. These recursion relations (with $\epsilon=10^{-2}$) and those for $b^{d-1}=2$ were analyzed numerically, keeping the first k $(10<k<26)$ coefficients A_l. Decreasing the value of ϵ or increasing the number of A_l's kept in the calculation did not produce any appreciable change in the results. We first tested the method for $n=2$ and 3 and found results in agreement with previous calculations.[12] For n, $d>2$, our results for nonintegral values of n interpolate smoothly between the known results for integral values of n. The results obtained in the previously unexplored region, $1<n<2$, $1<d<2$ are described in detail below. The transformations for $b=2^{1/(d-1)}$ with bond moving followed by decimation is referred to as I, and the one with $b=1+\epsilon$, $\epsilon\rightarrow 0^+$ is referred to as II in the following discussion.

The observed fixed-point structure is the same for transformations I and II. For dimensionalities $d_l(n)<d<2$, we find four fixed points: a zero-temperature, unstable fixed point $Z(A_l/A_0=1$ for all $l)$, an infinite-temperature, stable fixed point I $(A_l/A_0=0$ for all $l\geq 1)$, a low-temperature, stable fixed point L and a critical fixed point C with one unstable direction. The intersection of the critical surface of C with the line defined by Eq. (4) determines the critical value K_c of the coupling constant in the starting Hamiltonian, $V(\cos\theta)=K\cos\theta$. For $K<K_c$, the RG trajectories flow to the infinite-temperature fixed point I, whereas the trajectories for $K>K_c$ are attracted to the stable, low-temperature fixed point L. Thus, for $d_l(n)<d<2$, the model exhibits a continuous phase transition at $K=K_c$. The low-temperature phase is controlled by the finite-temperature fixed point L, and therefore, is characterized by a power-law decay of the two-point correlation function $g(r)$ with a temperature-independent exponent η', $g(r)\simeq r^{-(d-2+\eta')}$, $r\rightarrow\infty$, $K>K_c$. The exponent η' is presumably different from the usual one (η) that describes the algebraic decay of correlations at the critical point. Because of well-known difficulties[11] in determining magnetic exponents in the MK scheme, we did not attempt to calculate the values of η and η'. As d approaches $d_l(n)$ from above, the fixed points L and C move closer to each other in parameter space. They coalesce and become marginal at $d=d_l(n)$, and disappear for $d<d_l(n)$. Thus, for $d<d_l(n)$, all RG trajectories flow to the infinite-temperature fixed point, indicating the absence of any finite-temperature phase transition. In the other limit, as d approaches 2 from below, the fixed point L moves toward the zero-temperature fixed point Z. They merge and become marginal at $d=2$. For $d>2$, the fixed point L is not present and Z is stable, indicating, as expected, a conventional phase transition with long-range order in the low-temperature phase. The scenario described above is identical to that predicted by the RG equations written down by Cardy and Hamber.[8] This, we believe, is a significant result because our calculation does not involve the fugacity parameter of Ref. 8, whose meaning for $(n,d)\neq(2,2)$ is unclear.

The observed dependence of d_l on n is shown in Fig. 1. The values of $d_l(n)$ obtained from transformations I and II are identical within the numerical accuracy of our calculation. The fact that the calculated results for $d_l(n)$ do not depend on the scale factor b suggests that these results are perhaps close to being exact. The slope of $d_l(n)$ is large near $n=1$, and decreases continuously as n approaches 2. The slope at $n=2$ appears to be small but finite $(\simeq 0.2)$. However, we do not have reliable results very close to $n=d=2$ because three fixed points Z, L, and C come close to one another in this region, making a numerical analysis of the behavior very difficult. For this reason, we cannot rule out the possibility that the slope of $d_l(n)$ vanishes at $n=2$. We note that the heuristic arguments of Cardy and Hamber[8] predict a slope of $4/\pi^2\simeq 0.4$ at $n=d=2$. It appears safe to conclude that the slope of this line is substantially less than unity at $n=d=2$. As pointed out by Cardy and Hamber, this suggests that the point representing the three-dimensional Heisenberg model $(n=d=3)$ in the n-d plane lies above the line $d=d_c(n)$, indicating that topological defects (for $n=d=3$, the topological defects are point singularities) must be explicitly taken into account for a correct description of the critical behavior of this system. We are currently looking into this question.

Our numerical results for the critical coupling K_c, the effective temperature T_{eff}, defined as

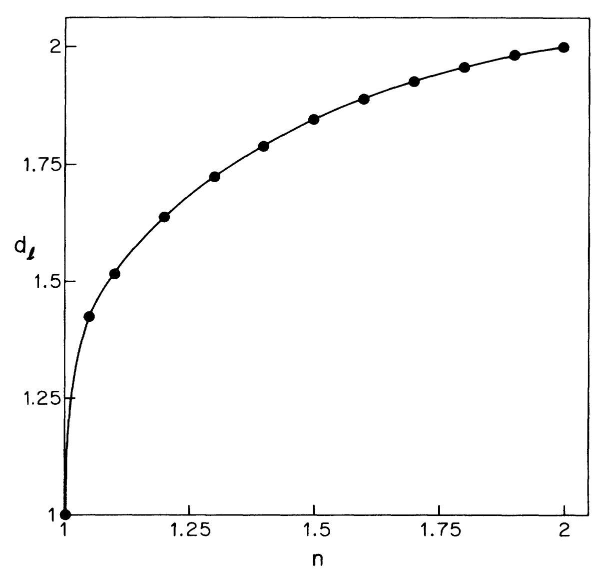

FIG. 1. The dependence of the lower critical dimension d_l on n for $1\leq n\leq 2$. The numerical uncertainty is less than the size of the data points shown.

TABLE I. Numerical results for the critical coupling K_c, the effective temperature $T_{\rm eff}$ at the fixed points C and L (see text) and the thermal eigenvalue λ_T at C for two values of n at different d. The first number in each column is the value obtained from transformation I, and the number in parentheses is the value obtained from transformation II. The calculated values of d_l are 1.957 and 1.638 for $n=1.8$ and 1.2, respectively.

n	d	K_c	$T_{\rm eff}(C)$	$T_{\rm eff}(L)$	$\lambda_T(C)$
1.8	2.0	0.74(1.01)	1.99(1.42)	0.0(0.0)	0.275(0.277)
	1.98	0.78(1.08)	1.86(1.33)	0.72(0.54)	0.197(0.198)
	1.96	0.84(1.20)	1.63(1.17)	1.28(0.93)	0.064(0.062)
1.2	2.0	0.38(0.56)	3.49(2.48)	0.0(0.0)	0.669(0.674)
	1.85	0.49(0.70)	3.00(2.15)	1.15(0.81)	0.559(0.563)
	1.75	0.59(0.86)	2.69(1.95)	1.60(1.15)	0.444(0.446)
	1.64	0.82(1.17)	2.22(1.60)	2.10(1.56)	0.077(0.036)

$$T_{\rm eff}^{-1} = -\left.\frac{d^2V(\cos\theta)}{d\theta^2}\right|_{\theta=0},$$

at the two fixed-points C and L, and the thermal eigenvalue λ_T for two values of n at different d are shown in Table I. A few features are worth pointing out. The calculated values of K_c and $T_{\rm eff}$ show a strong dependence on the scale factor b (the results obtained from transformations I and II are quite different) whereas the value of λ_T is much less sensitive to the choice of b. For fixed n, λ_T decreases as d is decreased and becomes very small as d approaches d_l, reflecting the marginal behavior at the lower critical dimension. The critical temperature, $T_c=1/K_c$, decreases with decreasing d as expected, but does not go to zero at $d=d_l$.

The behavior we find at $d=2$ for $1<n<2$ is somewhat puzzling. Recently, Nienhuis[15] analyzed the critical behavior of a somewhat different version of the two-dimensional n-vector model ($n \le 2$) by using a series of transformations which map the problem to a Coulomb gas system. He obtained a presumably exact analytic expression for the exponent ν which agrees with a conjecture made earlier by Cardy and Hamber.[8] His analysis also predicts a power-law decay of the two-point correlation function with a constant exponent in the low-temperature phase for $1<n<2$. Our calculation shows a similar behavior in the low-temperature phase for $d<2$. However, our results at $d=2$ are different. As mentioned earlier, the fixed point L which governs the algebraic decay of correlations in the low-temperature phase for $d<2$ merges with the zero-temperature fixed point at $d=2$. This fixed point is marginal at $d=2$, but higher order terms make it stable. We note that the existence of a stable, finite-temperature fixed point at $d=2$ would be inconsistent with the notion of continuity in d because no such fixed point is expected for any d greater than 2. The recursion relations of Cardy and Hamber,[8] which are based on assumptions of continuity and analyticity in n and d, do predict a behavior similar to what we find. On the other hand, it is difficult to reconcile the existence of an attractive zero-temperature fixed point with the expected destruction of long-range order at $d=2$. It appears that the MK approximation misses some subtle aspect (such as the line of fixed points at $n=d=2$) of the critical behavior at $d=2$ for $1<n<2$.

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