PHYSICAL REVIEW B VOLUME 35, NUMBER 7 1 MARCH 1987

Melnikov's method for nonperiodic perturbations and the bifurcations in a Josephson junction

Majid Taki
Observatoire de Nice, Université de Nice, Boite Postale No. 139, 06003 Nice Cédex, France
(Received 14 August 1986)

We consider a Josephson junction with an applied dc current and nonperiodic time-dependent magnetic field. Despite this nonperiodicity, one may use Melnikov's method to derive the bifurcation curves analytically. We give the critical theoretical value I_c of the dc current separating qualitatively different regions, essentially one region where the solutions (Josephson phase difference ϕ) rotate (infinite period) and another region where they oscillate (finite period). Numerical simulations are given and the results are in good agreement with the theoretical predictions.

I. INTRODUCTION

Nonlinear dynamics of the Josephson oscillator has been extensively studied in both theory and experiment. Huberman, Crutchfield, and Packard[1] first showed that the rf-biased Josephson junction should exhibit chaotic and bifurcating solutions for a variety of junction parameters. This in turn led to a number of simulations[2,3] showing the effects which should be observed in a real Josephson junction. In particular, several mechanisms leading to the chaos are found through the quasiperiodic responses and the period-doubling bifurcation. The lack of a global theory leads almost all authors to use the perturbational methods to study these phenomena. More recently Bartuccelli *et al.*[4] applied Melnikov's method for prediction of Smale horseshoe chaos in the rf-driven Josephson junction and showed that the Melnikov technique provides a good, but slightly low, estimate of the chaos threshold. The Smale horseshoe chaos[4] is due essentially to the periodicity in time of the perturbation considered by Bartuccelli *et al.* (infinite intersections between stable and unstable manifolds in the sections of Poincaré). In this paper, we use the Melnikov's method but with a *nonperiodic perturbation.* More precisely, we consider a Josephson junction with applied dc-current and nonperiodic time-dependent magnetic field. In spite of this nonperiodicity the Melnikov technique is still valid if the perturbation is bounded. The paper is organized as follows. In Sec. II we review the Josephson junction equations and give the theoretical problem in which we are interested. In Sec. III the Melnikov's method is reviewed. Section IV presents our technique to derive the bifurcation curves. Section V contains numerical simulations, their comparison with the analytical predictions and conclusions.

II. THE JOSEPHSON JUNCTION IN MAGNETIC FIELD

A "small" (compared with the Josephson penetration depth) Josephson junction may be represented by an equivalent circuit[5-7] whose balance equation is

$$C\frac{dV}{dt}+\frac{1}{R}V(t)+I_1\sin[\phi(t)]=I_{\mathrm{dc}}\ , \tag{2.1}$$

where CdV/dt is the displacement current through the capacitor C, $(1/R)V(t)$ is the current through the resistor R, and $I_1\sin[\phi(t)]$ is the Josephson supercurrent. $V(t)$ is the actual voltage developed across the device and is related to the phase difference ϕ by the Josephson equation $\partial\phi/\partial t=2eV/\hbar$. Therefore the junction phase difference ϕ obeys the well-known pendulumlike kinetic equation:

$$C\frac{\hbar}{2e}\frac{\partial^2\phi}{\partial t^2}+\frac{\hbar}{2e}\frac{1}{R}\frac{\partial\phi}{\partial t}+I_1\sin\phi=I_{\mathrm{dc}}\ . \tag{2.2}$$

I_1 is the maximum Josephson current, and will be allowed to vary with time in the present paper, according to

$$I_1(t)=I_1\{1-\epsilon a[1-\mathrm{sech}(bt)]\}\ . \tag{2.3}$$

Hence we assume the maximum Josephson current to jump from an initial value I_1 to a final one equal to $I_1(1-\epsilon a)$ with $(0<\epsilon\ll 1)$. Note that there is no restriction concerning b.

Experimentally, such a time variation of I_1 can be produced, either by a controlled modulation of an external magnetic field dependence of the maximum Josephson current[8] or by modulating tunneling properties of a Pb-CdS-Pb (CdS is cadmium sulfide) light-sensitive Josephson junction through a laser illumination process.[9] The detailed technical processes will not be considered in this paper.

We define $\tau=\omega_j t$ and $\beta_j=(1/\omega_j)(1/RC)$, where $\omega_j=[2e/\hbar I_1/C]^{1/2}$ is the plasma frequency.[7] By this choice and denoting $\partial\phi/\partial\tau\equiv\dot\phi$ and $\partial^2\phi/\partial\tau^2\equiv\ddot\phi$, Eq. (2.2) can be written, in dimensionless form, as

$$\ddot\phi+\beta_j\dot\phi+\{1-\epsilon a[1-\mathrm{sech}(bt)]\}\sin\phi=\epsilon I\ , \tag{2.4}$$

where $\epsilon I=I_{\mathrm{dc}}/I_1$ is the normalized current.

In what follows we take $\beta_j=0$ and focus on the theoretical question: What should be the dependence of I in a and b so that the oscillating solutions of Eq. (2.4) bifurcate to the rotating ones? In other words, what is the maximum $I_c(a,b)>0$ such that the phase difference ϕ performs exactly 2π rotation when t goes from 0 to infinity? This defines a critical value $I_c(a,b)$.

The aim of this work is to use Melnikov's method to solve this problem despite *the nonperiodicity of the perturbation (2.3).* We deal with this problem exactly like bifurcations of planar homoclinic cycles. We emphasize that, in the Melnikov's method, we replace a periodic perturbation by a bounded one.

III. MELNIKOV'S METHOD: PERTURBATIONS OF PLANAR HOMOCLINIC ORBITS

Consider a system of ordinary differential equations of the form[10]

$$\frac{dX}{dt}=f(X)+\epsilon g(X,t) \tag{3.1}$$

where $X=(x,y)\in R^2$ and g periodic in time of period T. Assume that, for $\epsilon=0$, the system (3.1) possesses a homoclinic orbit to hyperbolic saddle point, say p_0 (or homoclinic cycles), and that the eigenvalues of the linearized problem around of p_0 are real and of opposite sign. Therefore we define the Melnikov function by

$$M(t_0)=\int_{-\infty}^{+\infty} f(X_0(t-t_0))\wedge g(X_0(t-t_0),t) \times \exp\left[\int_0^{t-t_0} \mathrm{tr}[D_X f(X_0(s))]ds\right]dt$$

where X_0 is the solution corresponding to the homoclinic orbit (or cycle) of the unperturbed system. Here the D_X denotes the partial derivative with respect to X and the wedge product is defined as $X_1\wedge X_2=x_1y_2-x_2y_1$. And we have $d(t_0)=\epsilon M(t_0)+O(\epsilon^2)$ where $d(t_0)$ is the separation distance between the stable and unstable perturbed manifolds. Therefore, $M(t_0)$ provides a good measure [to $O(\epsilon^2)$] of $d(t_0)$.

In particular, if the system (3.1) is Hamiltonian one may transform it as

$$\dot{x}=\frac{\partial H}{\partial y}+\epsilon\frac{\partial G}{\partial y}\ ,$$
$$\dot{y}=-\frac{\partial H}{\partial x}-\epsilon\frac{\partial G}{\partial x}\ ,$$

where H is the unperturbed Hamiltonian and G is the perturbation Hamiltonian. The dot denotes the derivative with respect to time. In this case, the Melnikov function reduces to

$$M(t_0)=\int_{-\infty}^{+\infty}\{H(X_0(t-t_0)),G(X_0(t-t_0),t)\}dt$$

where

$$\{H,G\}=\frac{\partial H}{\partial x}\frac{\partial G}{\partial y}-\frac{\partial H}{\partial y}\frac{\partial G}{\partial x}$$

denotes the Poisson bracket.[11]

IV. HOMOCLINIC CYCLE BIFURCATIONS IN JOSEPHSON JUNCTIONS

The equation in which we are interested is

$$\ddot{\phi}+[1-\epsilon a(1-\mathrm{sech}(bt))]\sin\phi=\epsilon I\ . \tag{4.1}$$

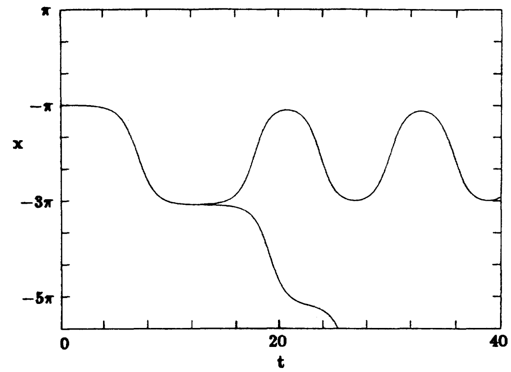

FIG. 1. Time evolution of the phase difference x for $a=0.2$, $b=0.1$. Upper curve, $I=0.0059$; lower curve, $I=0.0060$. The upper curve corresponds to the oscillating solution while the lower curve corresponds to the rotating one. The numerical bifurcation value of I is $I_N\simeq 0.0060$ which is in good agreement with the theoretical one $I_c\simeq 0.0062$ given by Eq. (4.4).

We put $X(x=\phi,y=\dot{\phi})$ and write (4.1) as a first-order system:

$$\dot{x}=y\ ,$$
$$\dot{y}=-\sin x+\epsilon\{a(1-\mathrm{sech}(bt))\sin x+I\}\ . \tag{4.2}$$

The unperturbed Hamiltonian of the system (4.2) is

$$H(x,y)=\frac{y^2}{2}+1-\cos x\ ,$$

and

$$G(x,y,t)=a(1-\mathrm{sech}(bt))\cos x-Ix$$

is the perturbation Hamiltonian.

In the standard form, the system (4.2) reads

$$\dot{x}=\frac{\partial H}{\partial y}\ ,$$
$$\dot{y}=-\frac{\partial H}{\partial x}-\epsilon\frac{\partial G}{\partial x}\ .$$

Only when $\epsilon=0$, the system (4.2) is a true autonomous phase problem, when $\epsilon\neq 0$, it is one of the problems treated in the Sec. III above.

When $\epsilon=0$, the phase plane of (4.2) is simply that of the pendulum. The heteroclinic orbits for the unperturbed system ($\epsilon=0$) are given by:

$$[x^{\pm}(t),y^{\pm}(t)]=[\pm 2\arctan[\sinh(t)],\pm 2\,\mathrm{sech}(t)]\ .$$

We now introduce the Melnikov function for the system (4.2):

$$\begin{aligned}M(t_0,a,b,I)&=-\int_{-\infty}^{+\infty}\dot{x}^{\pm}\left[\left[\frac{a}{\cosh[b(t+t_0)]}-a\right]\sin x^{\pm}-I\right]dt\\&=+I\int_{-\infty}^{+\infty}\dot{x}^{\pm}dt-a\int_{-\infty}^{+\infty}\frac{\dot{x}^{\pm}(\sin x^{\pm})}{\cosh[b(t+t_0)]}dt\\&=\pm 2\pi I-aJ\ ,\end{aligned}$$

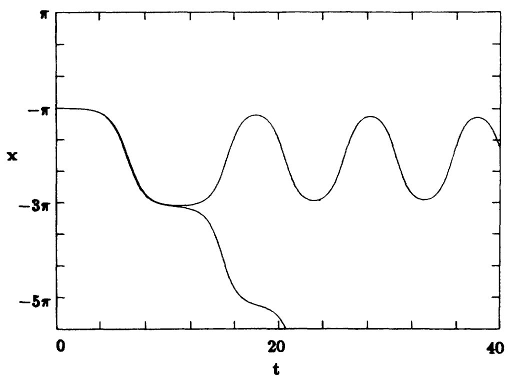

FIG. 2. Time evolution of the phase difference x for $a=0.4$, $b=0.1$. Upper curve, $I=0.0110$; lower curve, $I=0.0120$. The value given by Eq. (4.4) is $I_c \simeq 0.0124$.

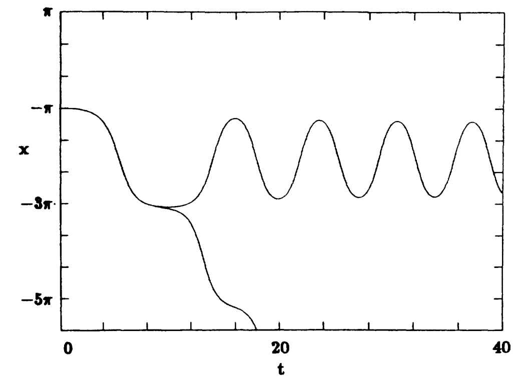

FIG. 4. Time evolution of the phase difference x for $a=1.2$, $b=0.1$. Upper curve, $I=0.030$; lower curve, $I=0.031$. The value given by Eq. (4.4) is $I_c \simeq 0.037$.

where

$$J=\int_{-\infty}^{+\infty}\frac{\dot{x}^{\pm}(\sin x^{\pm})}{\cosh[b(t+t_0)]}dt ,$$

$$\int_{-\infty}^{+\infty}\dot{x}^{\pm}(\sin x^{\pm})dt=0 .$$

By integrating by parts:

$$\begin{aligned}J&=\int_{-\infty}^{+\infty}\frac{\dot{x}^{\pm}(\sin x^{\pm})}{\cosh[b(t+t_0)]}dt\\&=-2b\int_{-\infty}^{+\infty}\frac{\sinh[b(t+t_0)]}{\cosh^2(t)\cosh^2[b(t+t_0)]}dt\\&=4\int_{-\infty}^{+\infty}\frac{\sinh(t)}{\cosh^3(t)\cosh[b(t+t_0)]}dt .\end{aligned}$$

When $b=1$, we calculate J:

$$J=-\pi\frac{\sinh(t_0/2)}{\cosh^3(t_0/2)} .$$

Then

$$M_{b=1}(t_0,a,I)=\pm 2\pi I+a\pi\frac{\sinh(t_0/2)}{\cosh^3(t_0/2)} .$$

Here the Melnikov function is independent of ϵ and time dependent.

We will now concentrate only on the time-independent bifurcation curves of the I-parametrized family curves. If we define

$$R(t_0)=\frac{1}{2}\frac{\sinh(t_0/2)}{\cosh^3(t_0/2)} ,$$

with

$$\max_{t\in R}[R(t_0)]=\frac{2\sqrt{3}}{9} ,$$

then it follows from Melnikov's technique that if $I/a<2\sqrt{3}/9$, the stable and unstable perturbed manifolds intersect for ϵ sufficiently small, and if $I/a>2\sqrt{3}/9$, no intersection can occur at all. In between (transition from intersection of manifolds to no intersection), homoclinic bifurcation takes place. Thus, there is a bifurcation in the I-a plane, tangent to

$$I_c \simeq \frac{2\sqrt{3}}{9}a \tag{4.3}$$

at $I=a=0$.

If $b\neq 1$, we approximate the expression J, and we have

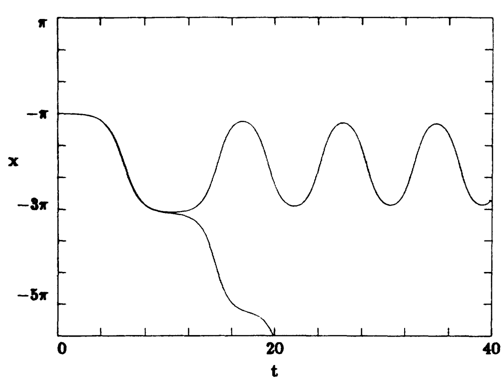

FIG. 3. Time evolution of the phase difference x for $a=0.6$, $b=0.1$. Upper curve, $I=0.0160$; lower curve, $I=0.0170$. The value given by Eq. (4.4) is $I_c \simeq 0.0186$.

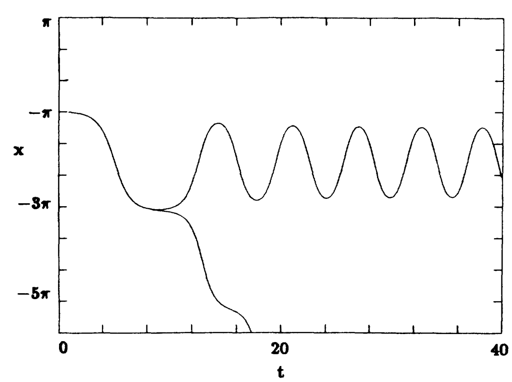

FIG. 5. Time evolution of the phase difference x for $a=1.8$, $b=0.1$. Upper curve, $I=0.042$; lower curve, $I=0.043$. The value given by Eq. (4.4) is $I_c \simeq 0.050$.

$$I_c \simeq \begin{cases} \dfrac{ab}{\pi} & \text{if } b \le 2 \quad (4.4) \\ \dfrac{2a}{\pi} & \text{if } b > 2\,. \quad (4.5) \end{cases}$$

For $b=1$, (4.4) is a good approximation of (4.3).

V. NUMERICAL RESULTS AND CONCLUSION

We use a Runge-Kutta scheme to compute numerically the system (4.2), with the initial conditions

$$x(0) = -\pi\,,$$
$$\dot{x}(0) = 0\,.$$

For different values of a and b we vary I from 10^{-7} to 10^{-1}. Numerically there exists one value I_N of I for which the oscillating solutions bifurcate to the rotating ones. More precisely, for all values than I_N we have oscillating solutions. Otherwise $(I > I_N)$, no oscillating solutions can occur and we have only the rotating phenomena. In between (transition from oscillating solutions to rotating ones), a bifurcation takes place. In Fig. 1 we fix $a=0.2$, $b=0.1$, and show the solutions $x(t)$ of (4.2). Two significative curves are selected for $I=0.0059$ and $I=0.0060$. The numerical bifurcation value is $I_N \simeq 0.0060$ which is in excellent agreement with the theoretical value $I_c \simeq 0.0062$!

Figures 2–5 are obtained in the same conditions with different values of a and b. For each figure the analytical value I_c is given.

We conclude that Melnikov's method provides an excellent instrument to calculate the bifurcation curves analytically, even if the perturbations are not periodic. But our main hypothesis is that the perturbation is bounded. The generalization of this method to infinite dimensional evolution equations was carried out by Holmes and Marsden.[12] It can be used for partial differential equations. In particular, Holmes[13] has applied these methods to the sine-Gordon equation. In fact, his work can be successfully used to study the soliton dynamics in the long Josephson junctions.

ACKNOWLEDGMENTS

The author wishes to thank J. C. Fernandez and G. Reinisch whose criticism and encouragement have helped him in preparing this work. He also thanks O. Legrand and T. Passot for useful and stimulating discussions.

[1]B. A. Huberman, J. P. Crutchfield, and N. H. Packard, Appl, Phys. Lett. **37**, 750 (1980).
[2]N. F. Pedersen and A. Davidson, Appl. Phys. Lett. **39**, 830 (1981).
[3]R. L. Kantz and Monaco, J. Appl. Phys. **57**, 875 (1985).
[4]M. Bartuccelli, P. L. Christiansen, N. F. Pedersen, and M. P. Soerensen, Phys. Rev. B **33**, 4686 (1986).
[5]D. E. McCumber, J. Appl. Phys. **39**, 2503 (1968); **39**, 3113 (1968).
[6]W. C. Stewart, Appl. Phys. Lett. **12**, 277 (1968).
[7]W. J. Jonhson, Ph.D. thesis, University of Wisconsin, Madison, 1968.
[8]A. Barone and G. Paterno, *Physics and Applications of the Josephson Effect* (Wiley-Interscience, New York, 1982), Chap. 3.4.
[9]F. Andreozzi, A. Barone, R. Cristiano, M. Russo, and G. Peluso, IEEE Trans. Magn. **MAG–19**, 983 (1983).
[10]V. K. Melnikov, Trans. Moscow Math. Soc. **12**, 1 (1963).
[11]J. Gukenheimer and P. J. Holmes, *Nonlinear Oscillation, Dynamical System and Bifurcation of Vector Fields,* Vol. 12 of *Applied Mathematical Sciences* (Springer-Verlag, Berlin, 1983).
[12]P. J. Holmes and J. E. Marsden, Archive for Rational Mechanics and Analysis **76**, 135 (1981).
[13]P. J. Holmes, in *Dynamical Systems and Turbulence,* Vol. 898 of *Springer Lecture Notes in Mathematics,* edited by D. A. Rand and L. -S. Young (Springer-Verlag, Berlin, 1981).