^{3}He-^{4}He mixture as a weak link for Josephson effects in superfluid ^{3}He

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We discuss the Josephson current for a special weak-link arrangement in superfluid ^{3}He-B. The weak link is made out of a ^{3}He-^{4}He mixture which separates two regions of pure ^{3}He. For large separations ($D \gg \xi_m$) we find Josephson's current-phase relation $J = nv_c \sin\phi$, and calculate the critical velocity v_c.

I. INTRODUCTION

A solution of liquid ^{4}He in ^{3}He separates at low temperatures into two phases, an essentially pure ^{3}He liquid and a ^{4}He-rich mixture with about 6% dissolved ^{3}He. Pure ^{3}He is a Fermi liquid which exhibits p-wave superfluidity below 1 mK (at vapor pressure). The ^{3}He atoms in the mixture form a dilute Fermi liquid whose excitation spectrum is strongly influenced by the coupling to the background of superfluid ^{4}He. The dilute Fermi liquid does not become superfluid down to the lowest achieved temperatures, which are about 0.2 mK for cooling the mixture. Hence, superfluid ^{3}He in contact with the mixture is the equivalent to the well-studied system of a superconductor in contact with a normal metal (N-S contact). Because of this analogy one expects to find in the coupled system of two Fermi liquids, familiar effects from superconductivity such as the induced weak superfluidity in the normal liquid (proximity effect). Another somewhat more complicated arrangement is an S-N-S junction. Of special interest here are Josephson effects, where the Josephson coupling is provided by pair tunneling through the normal layer.

In the present paper we discuss some aspects of the Josephson coupling of two vessels filled with superfluid ^{3}He which are weakly linked by a bridge of ^{3}He-^{4}He mixture. Two realizations of such systems are sketched in Fig. 1. In Fig. 1(a) the weak link between the two superfluids is provided by pores filled with the ^{3}He-^{4}He mixture. In Fig. 1(b) superfluid ^{3}He floats on top of mixture, and is separated into two halves by a blade which slightly dips into the mixture and blocks any direct connection between the two halves. If the ^{4}He barrier were absent, one can, in principle, perform tunneling experiments by creating a weak link between two baths of ^{3}He. However, in this case the diameter of the pore connecting the two baths must be less than the Bardeen-Cooper-Schrieffer (BCS) coherence length $\xi_0 = \hbar v_F / 2\pi k_B T_c$ so that the superfluid state is destroyed inside the pore. The fabrication of such a small pore is not a simple matter. On the other hand, when the pores are filled by ^{3}He-^{4}He mixture as shown in Fig. 1(a), we shall show that the requirement for a significant critical current is that the thickness D of the barrier should not exceed a few times the thermal coherence length of the mixture, $\xi_M = \hbar v_F^M / 2\pi k_B T$, where v_F^M is the Fermi velocity of the mixture. The radius r of the pore is limited only by the condition that the pore will remain filled. If the surface tension between the ^{3}He and the mixture were the only factor, the condition is simply $r < D$, obtained by a minimization of the interface area. By going to low temperature ($T \ll T_c$), ξ_M can be made much larger than ξ_0, so that the requirement on the pore dimensions r and D is much less stringent.

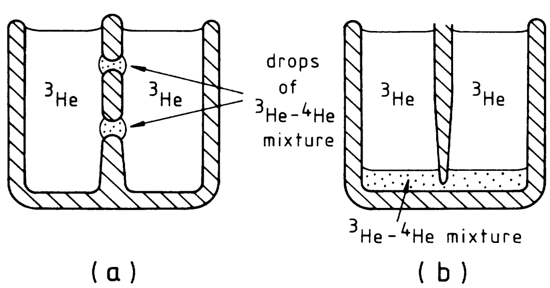

FIG. 1. Two types of weak-link structures.

Our main goal is a calculation of the critical current of the envisaged weak links. We first consider in Sec. II the idealized arrangement of two half spaces of superfluid ^{3}He separated by a thin planar film of ^{3}He-^{4}He mixture of constant thickness D. We calculate the dependence of the critical current density on the thickness D. In Sec. III we discuss qualitatively the generalizations of these results to more realistic situations, such as thin pores filled with the mixture, and summarize our results.

II. THE S-N-S JUNCTION

An idealized S-N-S junction is sketched in Fig. 2. S stands for superfluid ^{3}He; we assume it to be in its B phase represented by the Balian-Werthammer state. N stands for the ^{3}He-^{4}He mixture which we describe by a normal Fermi liquid. The ^{4}He background remains in its equilibrium state and is of no interest in the present con-

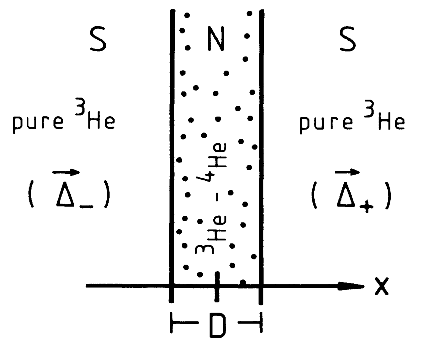

FIG. 2. S-N-S sandwich geometry.

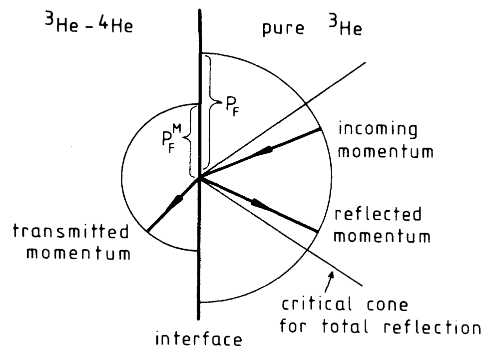

FIG. 3. Classical trajectories of quasiparticles at an interface.

text. The pairing interaction in the mixture is assumed to be small and will be neglected ($T_c^{\text{mixture}}=0$). The thickness D of the N region shall be a few times the thermal coherence length ξ_M of the mixture. At the temperatures of interest, the quasiparticle mean free path exceeds ξ_M by more than an order of magnitude, and we can neglect collision effects and work in the "clean limit" ($l \gg D$).

The critical current of a clean S-N-S junction with conventional superconductors has been the subject of various theoretical studies (see Likharev's review,[1] Sec. III D, for a detailed discussion). We can profit appreciably from the know-how developed in the theory of superconductivity. However, one should point out two complications, which are particular to an S-N-S contact in superfluid ^{3}He, and are consequences of its unconventional type of pairing.[2] First, the triplet character of the order parameter implies the existence of more than one "phase." In addition to the conventional phase whose variation across the bridge leads to a mass supercurrent through the junction, one has "magnetic phases." They are described by a three-dimensional (3D) rotation matrix $R_{\alpha i}$ with, e.g., the rotation axis and angle as degrees of freedom. The second complication originates from the anisotropic (p-wave) nature of the order parameter in ^{3}He. Reflections of quasiparticles at the N-S interfaces cause depairing effects which lead to distortions of the order parameter within a distance from the interface of the order of the coherence length ξ_0 ($=\hbar v_F/2\pi k_B T_c$). These depairing effects must be included in a calculation of Josephson effects in our S-N-S junction. The magnitude of the depairing effects depends critically on the reflection and transmission probabilities for quasiparticles at the N-S interfaces. Following Ref. 3, we assume a perfect interface without roughness and a step-function potential at the interface. The height of the potential step is determined by the differences in Fermi energies in pure and diluted ^{3}He. The reflection probability R is then given by

$$R=(v_{1x}-v_{2x})^2/(v_{1x}+v_{2x})^2 \ , \tag{1}$$

where v_{1x} and v_{2x} are the velocity components (perpendicular to the interface) of the incident and transmitted quasiparticles. The reflection probability is unity for quasiparticles hitting the interface at glancing angles below the critical angle of total reflection. The kinematics of scattering processes at the interface is sketched in Fig. 3.

We will calculate the Josephson supercurrent by solving the quasiclassical differential equations (Eilenberger's equations) for our S-N-S system:

$$[i\varepsilon_n \underline{\tau}_3-\underline{\Delta}(\hat{\mathbf{p}},x),\underline{g}(\hat{\mathbf{p}},x;\varepsilon_n)]+iv(x)\partial_x \underline{g}(\hat{\mathbf{p}},x;\varepsilon_n)=0 \ , \tag{2a}$$

$$[\underline{g}(\hat{\mathbf{p}},x;\varepsilon_n)]^2=-\pi^2 \ . \tag{2b}$$

We adopt the notation of Ref. 4. $\underline{g}$ and $\underline{\Delta}$ are 4×4 matrices which represent the quasiclassical propagator and the order-parameter field. ε_n is the Matsubara frequency [$\varepsilon_n=(2n-1)\pi T$], and $v(x)$ is the x component of the velocity of a quasiparticle moving in direction $\hat{\mathbf{p}}$ ($\hat{\mathbf{p}}$ is a unit vector). In terms of the Fermi velocities in pure (v_F) and diluted (v_F^M) ^{3}He we have

$$v(x)=\begin{cases} v_F\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}_x & \text{for } |x|>D/2 \ , \\ v_F^M\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}_x & \text{for } |x|<D/2 \end{cases}$$

($\hat{\mathbf{p}}_x$ is the unit vector in the x direction). The phase and spin orientation of the triplet order parameter shall be fixed at $x=\pm\infty$:

$$\underline{\Delta}(\hat{\mathbf{p}},x)\rightarrow\begin{cases} \underline{\Delta}_+(\hat{\mathbf{p}}) & \text{for } x\rightarrow+\infty \ , \\ \underline{\Delta}_-(\hat{\mathbf{p}}) & \text{for } x\rightarrow-\infty \ . \end{cases} \tag{3}$$

Elsewhere the order parameter has to be calculated self-consistently from the solution of Eilenberger's equations. At the two interfaces the quasiclassical propagators are, in general, discontinuous. The jumps in $\underline{g}$ are determined by the boundary conditions of Zaitsev[5] and Kieselmann:[6]

$$\underline{d}_+ + \underline{d}_- = 0 \ , \tag{4a}$$

$$\underline{d}_+\underline{s}_+^2=i\frac{1-R}{1+R}\left[\underline{s}_-,\underline{s}_+\left[\pi-\frac{i}{2}\underline{d}_+\right]\right] \ . \tag{4b}$$

Here, the indices $\pm$ refer to the two sides of an interface, and

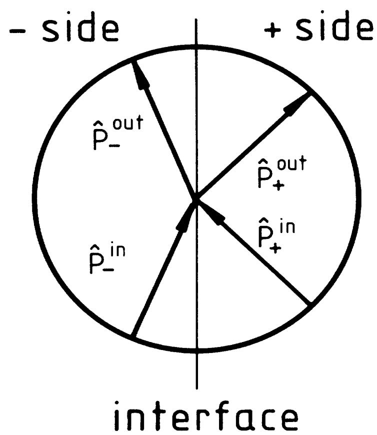

FIG. 4. The four momentum directions which occur in the boundary condition at a perfect interface.

$$\underline{d}_\pm = \underline{g}_\pm(\hat{\mathbf{p}}_\pm^{\text{out}}, x_I; \varepsilon_n) - \underline{g}_\pm(\hat{\mathbf{p}}_\pm^{\text{in}}, x_I; \varepsilon_n) ,$$

$$\underline{s}_\pm = \underline{g}_\pm(\hat{\mathbf{p}}_\pm^{\text{out}}, x_I; \varepsilon_n) + \underline{g}_\pm(\hat{\mathbf{p}}_\pm^{\text{in}}, x_I; \varepsilon_n)$$

are the differences and sums of propagators along incoming ($\hat{\mathbf{p}}_\pm^{\text{in}}$) reflected ($\hat{\mathbf{p}}_\pm^{\text{out}}$) momentum directions. x_I are coordinates of the interfaces ($x_I = - \pm D/2$), and R is the (momentum dependent) reflection coefficient. Our notation for the four momentum directions is explained in Fig. 4.

Having solved Eqs. (2a) and (2b) with the boundary conditions (3), (4a), and (4b), one can determine the supercurrent densities from

$$\mathbf{j} = 2N_M(E_F) \int \frac{d\Omega_p}{4\pi} T \sum_{\varepsilon_n} \mathbf{v}^M g(\hat{\mathbf{p}}, x; \varepsilon_n) \quad \text{(particle current)} \tag{5}$$

and

$$j_{\alpha i}^{\text{spin}} = N_M(E_F) \int \frac{d\Omega_p}{4\pi} T \sum_{\varepsilon_n} v_i^M (\mathbf{g}(\hat{\mathbf{p}}, x; \varepsilon_n))_\alpha \quad \text{(spin current)} . \tag{6}$$

g and $\mathbf{g}$ are the scalar and spin components of the Nambu matrix $\underline{g}$:

$$\underline{g} = \begin{bmatrix} g + \mathbf{g}\cdot\boldsymbol{\sigma} & (f + \mathbf{f}\cdot\boldsymbol{\sigma}) i\sigma_y \\ i\sigma_y (f + \mathbf{f}\cdot\boldsymbol{\sigma}) & g - \sigma_y \mathbf{g}\cdot\boldsymbol{\sigma}\sigma_y \end{bmatrix} . \tag{7}$$

In Eqs. (5) and (6) $N_M(E_F)$ denotes the quasiparticle density of states in the mixture, and $\mathbf{v}^M$ is the velocity of an excitation ($\mathbf{v}^M = v_F^M \hat{\mathbf{p}}$). Because of current conservation the Josephson currents can be calculated at an arbitrary point x within the layer of ^{3}He-^{4}He mixture.

It is no major problem to solve the quasiclassical equations for our S-N-S junction self-consistently on a computer and to calculate the current-phase relation for this system. Such a calculation has been performed successfully[3] for an N-S junction of diluted and pure ^{3}He. Here we prefer a more qualitative approach, because the highly simplified geometry of our model S-N-S junction does not warrant an involved calculation. We will take advantage of available numerical solutions for a contact between a saturated ^{3}He-^{4}He mixture and pure ^{3}He (N-S junction), and will construct from them the solution for the S-N-S junction. This is possible as long as the width D of the N layer is large compared to the thermal coherence length $\xi_M = \hbar v_F^M / (2\pi k_B T)$. In this case we can approximate the solution of the S-N-S problem in zeroth order by a superposition of two N-S solutions with interfaces at $\pm D/2$. The Josephson coupling first appears in next order [order $\exp(-D/\xi_M)$], and can be calculated by perturbation theory. In the following we present the details of this approach.

We are interested in calculating the quasiclassical propagator $\underline{g}(\hat{\mathbf{p}}, x; \varepsilon_n)$ in the region $-D/2 < x < D/2$ of diluted ^{3}He. The order parameter $\underline{\Delta}(\hat{\mathbf{p}}, x)$ vanishes in this region, and Eilenberger's equation (2a) can be solved easily. This differential equation has 16 independent fundamental solutions which can be characterized by their dependence on x. One finds eight constant solutions, $\underline{g}_0 - \underline{g}_7$; the most important one is the normal-state propagator

$$\underline{g}_0 = -i\pi \,\text{sgn}(\varepsilon_n) \underline{\tau}_3 . \tag{8}$$

It is unnecessary for our purpose to know the explicit form of the other seven constant solutions. We just note that the differential equation (2a) implies that

$$[\underline{g}_i, \underline{g}_0] = 0$$

for all constant solutions ($i = 0-7$). The other eight fundamental solutions exhibit an exponential x dependence. There are four solutions ($\underline{g}_8 - \underline{g}_{11}$) which decay exponentially in the positive x direction, and four solutions ($\underline{g}_{12} - \underline{g}_{15}$) with exponential decay in the negative x direction:

$$\underline{g}_8 \cdots \underline{g}_{11} = [\underline{\Delta}_i + \text{sgn}(v_x^M \varepsilon_n) \underline{\tau}_3 \underline{\Delta}_i] \exp(-x/\xi_n) ,$$

$$\underline{g}_{12} \cdots \underline{g}_{15} = [\underline{\Delta}_i - \text{sgn}(v_x^M \varepsilon_n) \underline{\tau}_3 \underline{\Delta}_i] \exp(+x/\xi_n) .$$

Here, $\xi_n = | v_x^M / 2\varepsilon_n |$ is an energy-dependent coherence length and v_x^M the x component of $\mathbf{v}^M$. The matrix $\underline{\Delta}_i$ stands for four conveniently chosen independent order-parameter matrices (one singlet, three triplet). Some algebraic relations which will be useful for the following calculations are

$$\{\underline{g}_0, \underline{g}_i\} = 0 \quad \text{for } i = 8-15 ,$$

$$\underline{g}_i \underline{g}_j = 0 \quad \text{for } i,j = 8-11 \text{ or } i,j = 12-15 .$$

The physical propagator in the N region is a specific linear combination of the 16 basic solutions. It is normalized according to Eq. (2b) and matches the physical propagator in the two S regions via the boundary conditions (4a) and (4b).

Our ansatz for this solution is

$$\underline{g}(\hat{\mathbf{p}}, x; \varepsilon_n) = \underline{g}_0(\varepsilon_n) + \underline{g}_{\text{left}}(\hat{\mathbf{p}}; \varepsilon_n) \exp[-(x + D/2)/\xi_n] + \underline{g}_{\text{right}}(\hat{\mathbf{p}}; \varepsilon_n) \exp[+(x - D/2)/\xi_n] + \delta\underline{g}(\hat{\mathbf{p}}; \varepsilon_n) . \tag{9}$$

Here $\underline{g}_0$ is the bulk normal-state solution, and $\underline{g}_{\rm left}$ and $\underline{g}_{\rm right}$ are given by the amplitude of the exponential tails of the exact solutions of two N-S problems with the S region on the left side $(x<-D/2)$ and on the right side $(x>D/2)$, respectively. $\delta\underline{g}(\hat{\mathbf{p}};\varepsilon_n)$ is a spatially constant correction of order $\exp(-D/\xi_n)$. Near $x=0$ the above ansatz keeps correctly the terms of order 1, $\exp(-D/2\xi_n)$ and $\exp(-D/\xi_n)$. The first neglected terms are of the order $\exp(-3D/2\xi_n)$. The leading terms $\underline{g}_0$, $\underline{g}_{\rm left}$, and $\underline{g}_{\rm right}$ carry no supercurrent; all the information on the Josephson current is contained in the correction term $\delta\underline{g}$. This term can be calculated conveniently from the normalization condition (2b) expanded through order $\exp(-D/\xi_n)$. One finds

$$\{\underline{g}_{\rm left},\underline{g}_{\rm right}\}\exp(-D/\xi_n)+\{\delta\underline{g},\underline{g}_0\}=0$$

and, using $[\delta\underline{g},\underline{g}_0]=0$, our general result for $\delta\underline{g}$:

$$\delta\underline{g}(\hat{\mathbf{p}};\varepsilon_n)=-\frac{i\,{\rm sgn}\varepsilon_n}{2\pi}\underline{\tau}_3\{\underline{g}_{\rm left}(\hat{\mathbf{p}};\varepsilon_n),\underline{g}_{\rm right}(\hat{\mathbf{p}};\varepsilon_n)\}\times\exp(-D/2\xi_n)\,. \tag{10}$$

This useful formula describes the induced weak superfluidity in the N layer in terms of the exponential tails of an N-S solution. It is instructive to study the consequences of Eq. (10) in the case of a piecewise constant order-parameter field as shown in Fig. 5(b), and reflection factor $R=0$. This simplified version of an S-N-S junction has been studied extensively.[7] It admits an easy analytic solution for $\underline{g}$. We find by solving the N-S contact the amplitudes of the exponential tails in the N regions:

$$\underline{g}_{\rm left}(\hat{\mathbf{p}};\varepsilon_n)=\frac{\pi}{|\varepsilon_n|+(\varepsilon_n^2+|\Delta_-|^2)^{1/2}}\times[\underline{\Delta}_-+{\rm sgn}(\varepsilon_n v_x^M)\underline{\tau}_3\underline{\Delta}_-]\,, \tag{11a}$$

$$\underline{g}_{\rm right}(\hat{\mathbf{p}};\varepsilon_n)=\frac{\pi}{|\varepsilon_n|+(\varepsilon_n^2+|\Delta_+|^2)^{1/2}}\times[\underline{\Delta}_+-{\rm sgn}(\varepsilon_n v_x^M)\underline{\tau}_3\underline{\Delta}_+]\,. \tag{11b}$$

Formula (10) then gives $\delta\underline{g}$ in terms of the order-parameter matrices on the + and − sides:

$$\delta\underline{g}(\hat{\mathbf{p}};\varepsilon_n)=\frac{i\pi}{[\,|\varepsilon_n|+(\varepsilon_n^2+|\Delta|^2)^{1/2}]^2}\times{\rm sgn}v_x^M[\underline{\Delta}_+,\underline{\Delta}_-]\exp(-D/\xi_n)+\text{odd terms in }\varepsilon_n\,. \tag{12}$$

We have set, for simplicity $|\Delta_+|=|\Delta_-|\equiv|\Delta|$. A phase difference ϕ in the order parameter $\Delta_+=\exp(i\phi)\Delta_-$ or equivalently,

$$\underline{\Delta}_+=\exp(i\phi\underline{\tau}_3/2)\underline{\Delta}_-\exp(-i\phi\underline{\tau}_3/2)$$

leads to

$$\delta\underline{g}(\hat{\mathbf{p}};\varepsilon_n)=\frac{2\pi|\Delta|^2\sin\phi}{[\,|\varepsilon_n|+(\varepsilon_n^2+|\Delta|^2)^{1/2}]^2}\times{\rm sgn}v_x^M\underline{\tau}_3\exp(-D/\xi_n)+\text{odd terms in }\varepsilon_n\,, \tag{13}$$

and consequently to a supercurrent proportional to $\sin\phi$ with the amplitude

$$j=\frac{4N_M(E_F)(v_F^M)^2|\Delta|^2}{\{\pi T+[(\pi T)^2+|\Delta|^2]^{1/2}\}^2D}\exp(-D/\xi_M)\sin\phi\,. \tag{14}$$

The result (14) is obtained from Eqs. (13) and (5) in the limit $D/\xi_M\gg1$. It agrees with the corresponding formula for S-N-S contacts in superconductivity, which reflects the similarity of ^{3}He-B and singlet superconductors in their nonmagnetic properties.

Next we consider magnetic Josephson currents. They are caused by a relative rotation of the vector order parameter Δ_+ and Δ_-. In Nambu-matrix notation the rotation is described by

$$\underline{\Delta}_+=\exp\left[-\frac{i}{2}\boldsymbol{\theta}\cdot\underline{\mathbf{S}}\underline{\tau}_3\right]\underline{\Delta}_-\exp\left[+\frac{i}{2}\boldsymbol{\theta}\cdot\underline{\mathbf{S}}\underline{\tau}_3\right]. \tag{15}$$

The vector $\boldsymbol{\theta}=\theta\hat{\mathbf{n}}$ specifies the rotation axis $\hat{\mathbf{n}}$ and angle θ. $\underline{\mathbf{S}}$ is the spin matrix in Nambu notation:

$$\underline{\mathbf{S}}=\begin{bmatrix}\boldsymbol{\sigma} & 0\\ 0 & -\sigma_y\boldsymbol{\sigma}\sigma_y\end{bmatrix}.$$

Insertion of (15) into Eq. (12), and some simple algebraic manipulations, lead to

$$\delta\underline{g}(\hat{\mathbf{p}};\varepsilon_n)=\frac{2\pi}{[\,|\varepsilon_n|+(\varepsilon_n^2+|\Delta|^2)^{1/2}]}{\rm sgn}v_x^M[\sin\theta(\hat{\mathbf{n}}\times\boldsymbol{\Delta}_-)\times\boldsymbol{\Delta}_-^*+(1-\cos\theta)(\hat{\mathbf{n}}\cdot\boldsymbol{\Delta}_-)\hat{\mathbf{n}}\times\boldsymbol{\Delta}_-^*]\cdot\underline{\tau}_3\underline{\mathbf{S}}\times\exp(-D/\xi_n)+\text{odd terms in }\varepsilon_n\,. \tag{16}$$

Using Eq. (6), one can calculate the spin-Josephson current in the limit $D/\xi_M\gg1$:

$$j_{\alpha x}^{\rm spin}=\frac{2N_M(E_F)(v_F^M)^2}{\{\pi T+[(\pi T)^2+|\Delta|^2]^{1/2}\}^2D}\exp(-D/\xi_M)\{\sin\theta[\hat{\mathbf{n}}\times\boldsymbol{\Delta}_-(\hat{\mathbf{p}}_x)]\times\boldsymbol{\Delta}_-^*(\hat{\mathbf{p}}_x)+(1-\cos\theta)\hat{\mathbf{n}}\cdot\boldsymbol{\Delta}_-(\hat{\mathbf{p}}_x)\hat{\mathbf{n}}\times\boldsymbol{\Delta}_-^*(\hat{\mathbf{p}}_x)\}_\alpha\,. \tag{17}$$

We note that the current depends only on the pairing amplitude for quasiparticles moving in the x direction, i.e., perpendicular to the interfaces. This simple result is a consequence of our limit $D \gg \xi_M$. Equation (17) can also be written directly in terms of $\Delta_+(\hat{\mathbf{p}}_x)$ and $\Delta_-(\hat{\mathbf{p}}_x)$:

$$j_{\alpha x}^{\rm spin} = \frac{2N_M(E_F)(v_F^M)^2}{\{\pi T + [(\pi T)^2 + |\Delta|^2]^{1/2}\}^2} \exp(-D/\xi_M) \times \mathrm{Re}[\Delta_+(\hat{\mathbf{p}}_x) \times \Delta_-^*(\hat{\mathbf{p}}_x)]_\alpha \ . \tag{18}$$

It is obvious from Eq. (18) that one obtains maximum Josephson-spin current if the Δ vectors ("d" vectors in Leggett's[2] notation) of the two ^{3}He baths are orthogonal (for momentum in the x direction). It is useful to introduce a critical velocity v_c defined by

$$j_{\max} = n v_c \ . \tag{19}$$

Here, n is the density of ^{3}He particles in the mixture. From Eq. (14) we find, using $N_M(E_F)(v_F^M)^2 = \frac{3}{2} n / m^*$:

$$v_c = \frac{6\hbar}{m^* D} \exp(-D/\xi_M) \frac{|\Delta|^2}{\{\pi T + [(\pi T)^2 + |\Delta|^2]^{1/2}\}} \approx \frac{6\hbar}{m^* D} \exp(-D/\xi_M) \quad (\text{for } T \ll |\Delta|) \ . \tag{20}$$

The maximum spin current, Eq. (18), is given in terms of v_c by the suggestive formula

$$j_{\max}^{\rm spin} = \frac{\hbar}{2} n v_c \ . \tag{21}$$

Our results so far are obtained for an undistorted step-function order parameter as shown in Fig. 5(b). In reality, the order-parameter field will be distorted by depairing effects at the interface and by the proximity effect between ^{3}He and the mixture.[3] These effects have been included in a recent numerical calculation of the order parameter near the interface between ^{3}He and the mixture.[3] The result is shown in Fig. 5(a). It is important for our calculation that this deformation of the order parameter does not affect the matrix structure of the amplitudes $\underline{g}_{\rm left}(\hat{\mathbf{p}}_x;\varepsilon_n)$ and $\underline{g}_{\rm right}(\hat{\mathbf{p}}_x;\varepsilon_n)$. This is a consequence of the x independence of the *direction* of the vector order parameter $\Delta(\hat{\mathbf{p}}_x,x)$. Only the *magnitude* $|\Delta(\hat{\mathbf{p}}_x,x)|$ shows spatial dependences. Equations (2a) and (2b) then imply that the exact solutions $\underline{g}_{\rm left}$ and $\underline{g}_{\rm right}$ differ from the results (11a) and (11b) simply by a renormalization factor Z, which fully takes into account the order-parameter distortions near the interfaces. Z can be obtained by solving Eqs. (2a) and (2b) numerically. The renormalization factor has already been calculated by Ashauer[3] as a by-product of her self-consistent calculation of the order parameter. We use Ashauer's results for the temperature dependence of Z which is shown in Fig. 6. Inclusion of the order-parameter distortion leads to a reduction of the Josephson currents by a factor Z^2, since $\underline{g}_{\rm left}$ and $\underline{g}_{\rm right}$ each carry a factor Z in Eq. (10). Hence, we find the critical velocity:

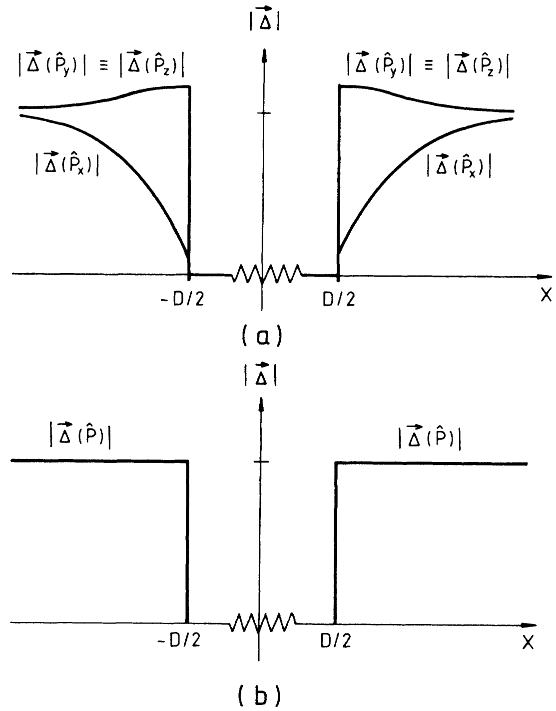

FIG. 5. Order parameter for an S-N-S sandwich geometry; (a) shows the order parameter as calculated in Ref. 3, and (b) the step-function approximation.

$$v_c = Z^2 \frac{6\hbar}{m^* D} \exp(-D/\xi_M) \times \frac{|\Delta|^2}{\{\pi k_B T + [(\pi k_B T)^2 + |\Delta|^2]^{1/2}\}^2} \ . \tag{22}$$

For clarity we have included $\hbar$ and k_B in formula (22). In terms of v_c the Josephson current density and spin current density across the layer of normal fluid are given by

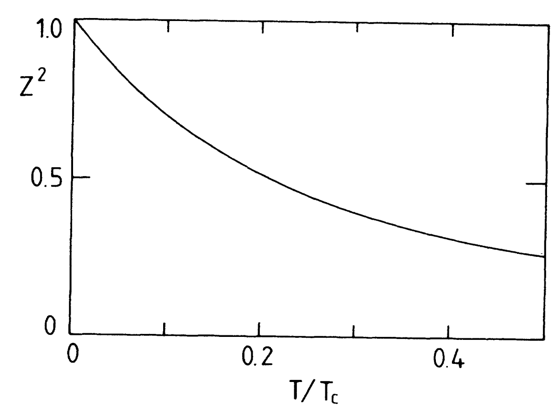

FIG. 6. Temperature dependence of the reduction factor Z^2 of the critical velocity in the weak link.

$$j = nv_c \sin\phi = nv_c \,\mathrm{Im}[\boldsymbol{\Delta}_+(\hat{\mathbf{p}}_x)\cdot\boldsymbol{\Delta}_-^*(\hat{\mathbf{p}}_x)/|\Delta|^2]\ , \tag{23a}$$

$$\mathbf{j}_{\rm spin} = n\frac{\hbar}{2}v_c \,\mathrm{Re}[\boldsymbol{\Delta}_+(\hat{\mathbf{p}}_x)\times\boldsymbol{\Delta}_-^*(\hat{\mathbf{p}}_x)/|\Delta|^2]\ . \tag{23b}$$

$\boldsymbol{\Delta}_+$ and $\boldsymbol{\Delta}_-$ are the asymptotic order parameters faraway from the junction. The vectors represented by boldface in (23b) refer to spin directions, and $|\Delta|$ is the bulk energy gap in the B phase. Equations (22), (23a) and (23b) are our final results for the Josephson currents in an S-N-S junction between ^{3}He and a saturated ^{3}He-^{4}He mixture.

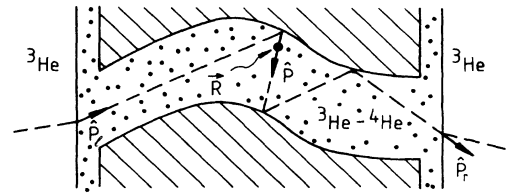

FIG. 7. Sketch of a classical trajectory in a weak link with specular walls.

III. DISCUSSION

Our calculations of the Josephson current in superfluid ^{3}He were done for a very simple geometry, namely an S-N-S sandwich where a planar interlayer of ^{3}He-^{4}He mixture separates two baths of pure ^{3}He. For a thick interlayer ($D \gg \xi_M$) we find Josephson's current-phase relations with a critical current density of the order $(6n\hbar/m^*D)\exp(-D/\xi_M)$. [See Eqs. (22), (23a), and (23b).]

We will discuss in this section the relevance of these results for more realistic weak-link geometries like narrow pores filled with ^{3}He-^{4}He mixture. The most important additional effect which needs to be incorporated for such more complicated geometries is quasiparticle scattering off the walls in the constrictions.

The easiest case is specular walls. Here the appropriate boundary condition is the continuity of $\underline{g}(\hat{\mathbf{p}},\mathbf{R};\varepsilon_n)$ along any classical trajectory $\mathbf{R}(s)$.[4] A typical classical orbit $\mathbf{R}(s)$ is sketched in Fig. 7. $\hat{\mathbf{p}}(s)$ is the direction of the velocity at $\mathbf{R}(s)$. One finds from the general quasiclassical equations that superfluidity penetrates into the mixture along such classical trajectories. This process is described by Eq. (2a) if x is interpreted as the spatial variable along the trajectory. The solution of the quasiclassical equations on a given trajectory can be found by the methods of Sec. II. We obtain, e.g., for the interference term $\delta\underline{g}$, in generalization of Eq. (10),

$$\delta\underline{g}(\hat{\mathbf{p}},\mathbf{R};\varepsilon_n) = -\frac{i\,\mathrm{sgn}\varepsilon_n}{2\pi}\underline{\tau}_3\{\underline{g}_{\rm left}(\hat{\mathbf{p}}_l,\mathbf{R}_l;\varepsilon_n),\underline{g}_{\rm right}(\hat{\mathbf{p}}_r,\mathbf{R}_r;\varepsilon_n)\}\exp[-L(\hat{\mathbf{p}})/|2v_F^M\varepsilon_n|]\ . \tag{24}$$

Here, $L(\hat{\mathbf{p}})$ is the length of the trajectory measured from one interface to the other, $(\hat{\mathbf{p}}_l,\mathbf{R}_l)$ and $(\hat{\mathbf{p}}_r,\mathbf{R}_r)$ are the directions and positions of the trajectory at its intersections with the left and right interfaces, respectively, and $\underline{g}_{\rm left}$ ($\underline{g}_{\rm right}$) describe, as in Sec. II, the induced superfluidity in the mixture at the respective interface.

Because of the exponential L dependence in Eq. (24) the main contribution to the Josephson current at point R comes from trajectories through $\mathbf{R}$ with length L near the minimal length $L^{(0)}$. Figure 8 shows typical minimal trajectories for the "razor-blade junction" in Fig. 1(b). For $L^{(0)} \gg \xi_M$ we can expand the argument of the exponential around the direction $\hat{\mathbf{p}}^{(0)}$ of the minimal trajectory, and evaluate the supercurrent from formula (5). We find

$$\mathbf{j}(\mathbf{R}) = \hat{\mathbf{p}}^{(0)} n v_c \,\mathrm{Im}[\boldsymbol{\Delta}_+(\hat{\mathbf{p}}_r^{(0)})\cdot\boldsymbol{\Delta}_-^*(\hat{\mathbf{p}}_l^{(0)})/|\Delta|^2]\ , \tag{25}$$

where

$$v_c = Z^2\frac{6\hbar}{m^*L^{(0)}}G\exp(-L^{(0)}/\xi_M) \times\frac{|\Delta|^2}{\{\pi k_BT+[(\pi k_BT)^2+|\Delta|^2]^{1/2}\}^2}\ . \tag{26}$$

The unit vectors $\hat{\mathbf{p}}_{l,r}^{(0)}$ are the directions of the minimal trajectory at the left and right interfaces (see Fig. 8). A comparison with our results for a sandwich geometry [Eqs. (22) and (23a)] shows that we simply have to replace the thickness D by the minimal length $L^{(0)}$, and the normal ($\hat{\mathbf{p}}_x$) to the interlayer by the path directions $\hat{\mathbf{p}}^{(0)}$, $\hat{\mathbf{p}}_l^{(0)}$, $\hat{\mathbf{p}}_r^{(0)}$. In addition, the renormalization factor Z^2 will be modified. It describes the effects of the order-parameter distortions near the interfaces which will, in general, depend on the specific geometry of the junction. The factor G is a dimensionless quantity of order unity. It carries information on the classical trajectories in the link, and is defined by

$$G = \frac{\int(d\Omega_p/4\pi)\exp[-L(\hat{\mathbf{p}})/\xi_M]}{\int(d\Omega_p/4\pi)\exp[-L^{(0)}/|(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}^{(0)})\xi_M|]}\ . \tag{27}$$

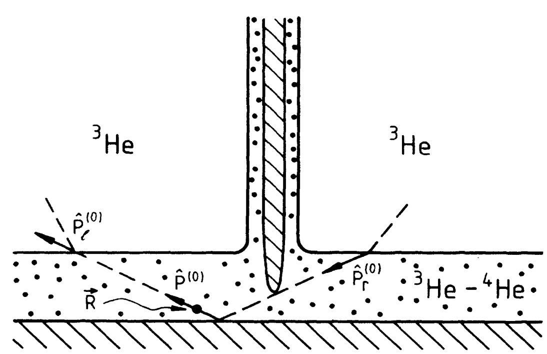

FIG. 8. Minimal trajectory (dashed line) through point $\mathbf{R}$. $\hat{\mathbf{p}}_l^{(0)}$ and $\hat{\mathbf{p}}_r^{(0)}$ are directions of the minimal trajectory at the interfaces between bulk ^{3}He and the mixture.

A straight pore, for example, yields $G = 1$, independent of the shape of its cross section.

We finally comment on the effects of rough walls on the Josephson current. Roughness reduces the critical current in two ways. Firstly, backward scattering of quasiparticles will, in general, increase the average length of a trajectory in the weak link, and hence reduce the factor $\exp(-L/\xi_M)$. Secondly, because of anisotropic pairing, we have depairing effects at rough walls which reduce the induced superfluidity in the mixture, and consequently also the critical current. A quantitative analysis of the effects of roughness probably needs a numerical solution of the quasiclassical equations. Numerical studies have been done successfully for superfluid ^{3}He-B near rough surfaces.[7,8] Similar calculations for a Josephson junction are feasible but certainly outside the scope of this paper. A simple lower bound for the critical current may be obtained by assuming that any scattering of a quasiparticle at rough walls destroys completely its memory for superfluid coherence. In this model, only trajectories which do not hit a wall contribute to the Josephson coupling. Formally, this effect can be described by a modified geometry factor G which is obtained by setting in Eq. (27) the path length $L(\hat{\mathbf{p}}) = \infty$ whenever the path hits a wall. This factor G depends on the particular geometry of the weak link. For example, $G = 0$ in the razor-blade geometry of Fig. 1(b). The reason is that all classical trajectories which link the two superfluid regions must hit the bottom of the container.

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