PHYSICAL REVIEW B VOLUME 35, NUMBER 1 1 JANUARY 1987

Rotational symmetry breaking in Heisenberg spin glasses: A microscopic approach

G. Kotliar
Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

H. Sompolinsky
Department of Physics, Bar-Ilan University, Ramat-Gan, Israel

A. Zippelius
Institut für Festkörperforschung, Kernforschungsanlage Jülich GmbH, D-5170 Jülich, Federal Republic of Germany
(Received 2 May 1986)

We study the consequences of rotational symmetry breaking in isotropic vector spin glasses. Starting from a microscopic model we identify the underlying symmetries of time-dependent (replica-dependent) SO(m) rotations. The hydrodynamic triad theory is confirmed and the spin-wave stiffness is related to a generalized transverse susceptibility. An expansion around mean-field theory is used to calculate the stiffness approximately.

I. INTRODUCTION

The low-temperature properties of magnetic systems with quenched random exchange (spin glasses) have been studied extensively in recent years. Most theoretical work is based on the spin-glass (SG) model of Edwards and Anderson,[1] who suggested that such systems might undergo a phase transition to a SG state, in which the local magnetic moments acquire nonzero thermal expectation values $\langle \mathbf{S}(\mathbf{x})\rangle \neq 0$. In the mean-field (MF) limit, the model does show a sharp phase transition at a finite freezing temperature T_g.

The Hamiltonian of the Edwards-Anderson (EA) model is isotropic in spin space, so that a global rotation of all spins around any axis in spin space is an exact symmetry of the model. We are interested here in the manifestations of the breaking of this rotational symmetry in the SG phase, which we assume to exist for $T < T_g$. We limit our considerations to a SG phase, in which the average magnetization vanishes and the system remains macroscopically isotropic. Thus, *locally,* the rotational symmetry is spontaneously broken in the SG state, but there is no *global* symmetry breaking.

The local magnetic moments point in noncollinear directions. Therefore a global rotation of all spins around *any* axis generates a new state,[2] which can be distinguished from the unrotated state. The SO(m) symmetry of the model is *completely broken.* Due to the isotropy of the model, the rotated and the unrotated state have the same free energy and are connected by a flat path in phase space. Figure 1 gives a qualitative picture of the free-energy surface. There are many valleys separated by free-energy barriers[3] and each valley is infinitely degenerate with a manifold of states generated by a uniform rotation.

Given the spontaneous breaking of rotational symmetry in the SG state, we expect to find low-energy excitations corresponding to almost uniform rotations. These have been analyzed within a phenomenological theory by Halperin and Saslow.[2] We briefly review the hydrodynamic theory. Below T_g the system will be found in one of many possible equilibrium states. One such state is singled out arbitrarily, denoted by α and specified by the local magnetic moments $\{\langle \mathbf{S}(x)\rangle_\alpha\}$. Another equilibrium state α' is generated by a uniform rotation R of all spins $\{\vec{\mathbf{R}}\langle \mathbf{S}(\mathbf{x})\rangle_\alpha\}$.

In general, for an m-component spin system, we represent a rotation $R(\theta)=\exp(-\sum_\delta \theta^\delta T^\delta)$ by the $[m(m-1)]/2$ generators T^δ of SO(m) rotations. For clarity of presentation it is sometimes useful to specialize to Heisenberg spins. In that case, $R(\theta)$ are the familiar three-dimensional rotation matrices. For example, rotations around the $\hat{\mathbf{z}}$ axis are achieved by

$$R_{ij}(\theta^z)=\begin{bmatrix} \cos\theta^z & \sin\theta^z & 0 \\ -\sin\theta^z & \cos\theta^z & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{1.1}$$

where i and j denote Cartesian spin components. Of particular interest are infinitesimal rotations, for which $R_{ij}(\theta)$ reduces to

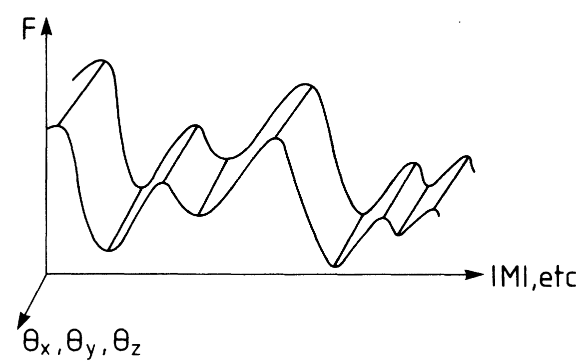

FIG. 1. Schematic picture of the free-energy landscape, to visualize that a uniform rotation by an angle θ generates a degenerate manifold for each valley.

$$R_{ij}(\theta)=\delta_{ij}+\epsilon_{kij}\theta^k \ . \tag{1.2}$$

(We adopt a summation convention over repeated indices.) Here ϵ_{kij} denotes the antisymmetric tensor $\epsilon_{xyz}=1$ and $\epsilon_{yxz}=-1$ for all cyclic permutations. The relative angle of the two configurations α and α' can then be represented as

$$\theta^i=\frac{1}{2q_{\rm EA}}\epsilon_{ijk}\langle S_j(\mathbf{x})\rangle_{\alpha'}\langle S_k(\mathbf{x})\rangle_\alpha \ , \tag{1.3}$$

with

$$q_{\rm EA}=\frac{1}{mN}\sum_{\mathbf{x}}\langle \mathbf{S}(\mathbf{x})\rangle_\alpha\cdot\langle \mathbf{S}(\mathbf{x})\rangle_\alpha \ .$$

This concept can be generalized to a situation, in which $\theta^i=\theta^i(\mathbf{r})$ is a slowly varying function of space. Halperin and Saslow define a fluctuating variable

$$\theta^i(\mathbf{r})=\frac{1}{2q_{\rm EA}n}\sum_{\mathbf{x}\in V}\epsilon_{ijk}\langle S_j(\mathbf{x})\rangle S_k(\mathbf{x}) \ , \tag{1.4}$$

whose expectation value specifies the local orientation of a rotated state with respect to the reference state α. $\theta^i(\mathbf{r})$ is a coarse-grained variable: V denotes a hydrodynamic volume, which contains a large number of spins n, but is small compared to the macroscopic length scale.

The free energy of the twisted state is expected to be higher than the free energy of the equilibrium state. For a long-wavelength twist, Halperin and Saslow suggested the following form:

$$\delta F_\theta=\frac{\rho_s}{2}\int d^dx\sum_i|\nabla\theta^i|^2 \ .$$

The stiffness constant ρ_s is a scalar, since the SG state is macroscopically isotropic. In the hydrodynamic theory, ρ_s enters as an unknown parameter, which is assumed to be finite.

The rotation angles $\theta^i(\mathbf{r})$ are coupled to the magnetization density via Poisson bracket relations. The resulting dynamic excitations have a linear dispersion $\omega\sim ck$ with spin-wave velocity $c^2=\rho_s/\chi$, with χ the uniform magnetic susceptibility. The damping is predicted to vanish like k^2, so that the excitations are well defined in the long-wavelength limit. The theory has been extended to include net ferromagnetic interactions[4] as well as anisotropies and nonzero remanent magnetization and external field.[5]

The hydrodynamic theory is based on the assumption of a finite exchange stiffness in the SG state. An upper bound to ρ_s is obtained by considering a rigid twist. The resulting bare stiffness for a nearest-neighbor hypercubic lattice in d dimension is

$$\rho_s^0=\frac{2|\varepsilon_0|}{md}a^{2-d} \ . \tag{1.5}$$

Here, ε_0 is the ground-state energy per spin, a denotes the lattice constant and we assume fixed length spins $\mathbf{S}^2=m$. However, the true stiffness may be considerably reduced even at zero temperature due to fluctuations in the strength of the bonds J_{ij}. Indeed, an important question is whether or not the stiffness of the SG state is at all finite.

This problem has been addressed in various numerical simulations. At zero temperature, ρ_s can be deduced from the energy difference of an equilibrium state and a state with a macroscopic rotational gradient imposed.[6–8] For a nearest-neighbor planar SG, a finite stiffness was found in $d=2$ and 3. As compared to the bare stiffness, its value is reduced by $\sim 50\%$. For Heisenberg spins in $d=3$ the stiffness at $T=0$ was found to be $\sim\frac{1}{3}$ of its bare value ρ_s^0.[9,10]

Information about the stiffness is contained in the spectrum of the Hessian matrix.[8,11] Due to the presence of other low-energy excitations in spin glasses, it is difficult to obtain reliable estimates for ρ_s in that way.

Several authors[11,12] have solved the microscopic equations of motion numerically for small samples. Propagating modes with a linear dispersion were found to exist for planar spins[11] in three dimensions, but not for Heisenberg spins.[12] Note, however, that the relatively small sample size severely restricts the range of wave numbers, which are accessible to such a study.

Inelastic neutron scattering offers one possibility to detect spin-wave excitations experimentally. So far no propagating modes with linear dispersion have been observed.[13] Several reasons might account for this failure: Possibly, the stiffness is so small that the energy range of spin-wave excitations is beyond the resolution of current experiments. The excitations might be overdamped due to relaxation processes which have not been taken into account in the hydrodynamic theory. The analysis of neutron scattering data is further complicated by the appearance of quasielastic components, which are superimposed on the spin-wave spectrum.

Recently, several authors[14] have observed ferromagnetic spin waves in reentrant systems. Anomalies in the linewidth and position of the spectra are observed simultaneously with the occurrence of a quasielastic peak, which was taken as evidence for a SG phase.

Another possibility of observing low-energy excitations is to measure the low-T magnetic specific heat and resistivity.[15] In most cases, the observed specific heat is dominated by a linear term, which suggests that other low-energy excitations are dominant at low T.

It is our aim to establish the hydrodynamic picture in a microscopic theory. We want to check it for consistency and provide a framework for a systematic calculation of its parameters. We concentrate here on a discussion of ρ_s and address the following questions: Is there a finite stiffness at zero temperature, and how does it compare to ρ_s^0? What is ρ_s at finite T, and in particular, what is its critical behavior? Even though our discussion is restricted to the isotropic model, we point out that the framework is general enough to also consider perturbations of the rotational symmetry by weak uniform external fields and random anisotropy. This will be discussed elsewhere. A short summary of our work has been reported previously.[16]

II. A MICROSCOPIC DEFINITION OF THE EXCHANGE STIFFNESS CONSTANT

We consider an EA model[1] for an m-component SG,

$$H=-\frac{1}{2}\sum_{\mathbf{x},\mathbf{x}'} J_{\mathbf{x}\mathbf{x}'}\mathbf{S}(\mathbf{x})\cdot\mathbf{S}(\mathbf{x}') . \tag{2.1}$$

The exchange couplings $\{J_{\mathbf{x}\mathbf{x}'}\}$ are Gaussian variables with zero mean and variance $[J^2_{\mathbf{x}\mathbf{x}'}]=J^2K(|\mathbf{x}-\mathbf{x}'|)$. Here K is a nonrandom, function of $|\mathbf{x}-\mathbf{x}'|$, normalized such that $\sum_{\mathbf{x}'}K(|\mathbf{x}-\mathbf{x}'|)=1$. In real spin glasses K is relatively short ranged. However, very little is known, at present, about the low-temperature properties of the short-range EA model. We will therefore assume that the SG phase of the system has the same qualitative features as the MF model, where the low-T phase is well understood.[17–19] Whether this assumption is valid for the three-dimensional system is unclear at present.

In the MF model $J_{\mathbf{x}\mathbf{x}'}$ is *infinitely ranged* with $K(|\mathbf{x}-\mathbf{x}'|)=1/N$,[20] N being the total number of spins. MF theory predicts a second-order phase transition from a paramagnetic to a SG phase with a nonzero value of the EA order parameter, $q_{\mathrm{EA}}=[\langle\mathbf{S}(\mathbf{x})\rangle\cdot\langle\mathbf{S}(\mathbf{x})\rangle]/m$. The bold square brackets [ ] denote ensemble average over the $J_{\mathbf{x}\mathbf{x}'}$. A full description of the SG phase must take into account the existence of many equilibrium states which are degenerate in the thermodynamic limit but have finite free-energy differences in a finite system.[21]

The degenerate SG phases which are not related by a global rotation are separated by free-energy barriers which diverge in the $N\to\infty$ limit. Thus, finite time (i.e., quasiequilibrium) quantities are obtained as thermal averages which are restricted to a single SG state ("valley") and the manifold of states which are related to it by a global rotation. On the other hand, true equilibrium corresponds to an average over all the valleys.

In order to calculate the exchange stiffness constant in the SG phase, we concentrate on the following ensemble averaged correlation function,

$$\chi^{\delta}(\mathbf{k})=\frac{\beta}{4N}\sum_{\mathbf{x},\mathbf{x}'}[\langle\mathbf{S}(\mathbf{x})\cdot\overleftrightarrow{\mathbf{T}}^{\delta}\langle\mathbf{S}(\mathbf{x})\mathbf{S}(\mathbf{x}')\rangle\overleftrightarrow{\mathbf{T}}^{\delta}\cdot\langle\mathbf{S}(\mathbf{x}')\rangle]e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} , \tag{2.2}$$

where δ denotes the axis of rotation. In the limit of long wavelength this will correspond to $\chi^{\delta}(\mathbf{k})=Tq^2_{\mathrm{EA}}\langle\theta^{\delta}(\mathbf{k})\theta^{\delta}(-\mathbf{k})\rangle$. Hence, the stiffness constant can be defined as

$$\rho_s=vq^2_{\mathrm{EA}}\lim_{\mathbf{k}\to 0}[k^2\chi^{\delta}(\mathbf{k})]^{-1} , \tag{2.3}$$

where v is the volume of a unit cell. $\chi^{\delta}(\mathbf{k})$ is a quasiequilibrium quantity. All thermal averages in Eq. (2.2) are restricted to a single valley.

The susceptibility $\chi^{\delta}(\mathbf{k})$ can be intuitively thought of as a response function to a twisting field. We envisage a Heisenberg system where $\{\langle\mathbf{S}(\mathbf{x})\rangle\}$ is one of its equilibrium states. We add a small longitudinal field $\mathbf{h}_L(\mathbf{x})=h_L\langle\mathbf{S}(\mathbf{x})\rangle$. We assume that h_L is small enough so that $\langle\mathbf{S}(\mathbf{x})\rangle$ is not changed by the field, and the only effect of $\mathbf{h}_L$ is to split the degeneracy between the state $\{\langle\mathbf{S}(\mathbf{x})\rangle\}$ and the rotated states $\{\overleftrightarrow{\mathbf{R}}\langle\mathbf{S}\rangle\}$. Next, we add a small transverse field $\mathbf{h}_T(\mathbf{x})$

$$h^i_T(\mathbf{x})=h_T\epsilon_{zij}\langle S_j(\mathbf{x})\rangle\cos(\mathbf{k}\cdot\mathbf{x}) ,$$

where $h_T/h_L\ll 1$. If $\mathbf{k}=\mathbf{0}$ the only effect of $\mathbf{h}_T$ is to rotate the state $\langle\mathbf{S}\rangle\to R\langle\mathbf{S}\rangle$ where R is a uniform rotation by an angle $\theta_z=h_T/h_L$ around the z axis. If $\mathbf{k}$ is nonzero (but small), the resultant state will have a nonuniform long-wavelength angle of rotation $\theta_z(\mathbf{x})$ relative to the original state. The transverse response $[\sum_{\mathbf{x}}\hat{\mathbf{z}}\cdot\delta\langle\mathbf{S}(\mathbf{x})\rangle\times\langle\mathbf{S}(\mathbf{x})\rangle]/\delta h_T$ is just

$$\chi^{z}(\mathbf{k})=\frac{\beta}{4N}\sum_{\mathbf{x},\mathbf{x}'}\epsilon_{zij}\epsilon_{zkl}[\langle S_i(\mathbf{x})S_k(\mathbf{x}')\rangle\langle S_j(\mathbf{x})\rangle\langle S_l(\mathbf{x}')\rangle]\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')] . \tag{2.4}$$

The bulk of the paper is devoted to a microscopic calculation of $\chi^{\delta}(k)$ and ρ_s. Two independent approaches have been used previously to calculate ensemble averaged quantities in SG models. One is based on the dynamic formulation and the other one on the replica trick. In Secs. III and IV we will use the dynamic approach. We first formulate the dynamic model and discuss its invariance properties (Sec. III) and then (Sec. IV) present an approximate calculation of ρ_s. In Sec. V we develop a framework for a microscopic calculation of ρ_s within the replica formalism and use it to obtain an explicit MF expression for ρ_s. A summary and discussion of the results will be presented in Sec. VI.

III. INVARIANCE UNDER TIME-DEPENDENT ROTATIONS

In the dynamic approach[22,23] we calculate ρ_s as the static limit of a dynamic response function. In principle, the dynamic approach is more general and allows one to consider time-dependent perturbations and address the problem of propagating modes and their damping. Here we do not consider finite-time dynamics but rather concentrate on static or quasistatic quantities. We have therefore chosen the simplest microscopic dynamics, namely a purely relaxational equation of motion[24]

$$\Gamma_0^{-1}\partial_t \mathbf{S}(\mathbf{x},t) = -\beta \frac{\delta(H_{\rm EA}+V)}{\delta \mathbf{S}(\mathbf{x},t)} + \xi(\mathbf{x},t) \ . \tag{3.1}$$

Here, Γ_0 denotes a constant kinetic coefficient and the thermal noise $\xi(\mathbf{x},t)$ is a Gaussian random variable with variance

$$\langle \xi_i(\mathbf{x},t)\xi_j(\mathbf{x}',t')\rangle = \delta_{ij}\delta_{\mathbf{x}\mathbf{x}'}\delta(t-t')\frac{2}{\Gamma_0} \ .$$

Instead of working with fixed-length spins it is more convenient to consider a soft-spin version of the EA model by introducing a potential $V(\mathbf{S}^2)$, that provides a local constraint on the length of the spin. For a discussion of finite-time dynamics, it is certainly important to include precessional terms in the equation of motion. As far as equilibrium or quasiequilibrium properties are concerned, we do not expect that they will be modified by the precession of spins at finite frequencies.

The *quenched* average over the random exchange can be performed as for the Ising case ($m=1$).[23] It is convenient to introduce auxiliary response fields $\hat{\mathbf{S}}_i(\mathbf{x},t)$ and define a generating functional $Z\{J_{xx'}\}$ for correlations and response functions. Since $Z\{J_{xx'}\}$ is properly normalized, we can perform the quenched average directly on $Z\{J_{xx'}\}$ with the following result:

$$\begin{aligned} Z = [Z\{J_{xx'}\}] = \int D\{S,\hat{S}\} \exp \Bigg[& L_0(S,\hat{S}) + \sum_{\mathbf{x}} \int dt\,[l_i(\mathbf{x},t) i\hat{S}_i(\mathbf{x},t) + \hat{l}_i(\mathbf{x},t) S_i(\mathbf{x},t)] \\ & + \frac{2\beta^2 J^2}{z} \sum_{\mathbf{x},\mathbf{x}'} K_{xx'} \int dt\, dt' [i\hat{S}_i(\mathbf{x},t) S_i(\mathbf{x}',t) i\hat{S}_j(\mathbf{x},t') S_j(\mathbf{x}',t') \\ & + i\hat{S}_i(\mathbf{x},t) S_i(\mathbf{x}',t) i\hat{S}_j(\mathbf{x}',t') S_j(\mathbf{x},t')] \Bigg] \ , \end{aligned} \tag{3.2}$$

where L_0 is purely local,

$$L_0(S,\hat{S}) = \int dt \sum_{\mathbf{x}} i\hat{S}_i(\mathbf{x},t) \left[-\Gamma_0^{-1}\partial_t S_i(\mathbf{x},t) - \frac{\delta\beta(H_{\rm EA}+V)}{\delta S_i(\mathbf{x},t)} + i\Gamma_0^{-1}\hat{S}_i(\mathbf{x},t) \right] - \int dt \sum_{\mathbf{x}} \frac{1}{2} \frac{\delta^2(\beta V)}{\delta S_i(\mathbf{x},t)\delta S_i(\mathbf{x},t)} \ . \tag{3.3}$$

The physical quantities of interest are the averaged multiple-spin correlation and response functions. These are obtained from Z by functional differentiation with respect to the fields $l_i(\mathbf{x},t)$ and $\hat{l}_i(\mathbf{x},t)$,

$$\frac{\delta^n \delta^m \ln Z}{\delta l_{i_1}(\mathbf{x}_1 t_1) \cdots \delta l_{i_n}(\mathbf{x}_n t_n) \delta \hat{l}_{j_1}(\mathbf{x}^{1'},t_1') \cdots \delta \hat{l}_{j_m}(\mathbf{x}_m',t_M')} = [\langle i\hat{S}_{i_1}(\mathbf{x}_1 t_1) \cdots i\hat{S}_{i_n}(\mathbf{x}_n t_n) S_{j_1}(\mathbf{x}_1' t_1') \cdots S_{j_m}(\mathbf{x}_m',t_m')\rangle] \tag{3.4}$$

The SG phase is characterized by time-persistent parts in both the local correlations

$$q_{ij}(t,t') \equiv \lim_{\Gamma_0(t-t')\to\infty} [\langle S_i(\mathbf{x},t) S_j(\mathbf{x},t')\rangle] \tag{3.5}$$

and response functions

$$\Delta_{ij}'(t,t') = \lim_{\Gamma_0(t-t')\to\infty} [\langle i\hat{S}_i(\mathbf{x},t') S_j(\mathbf{x},t)\rangle] \ . \tag{3.6}$$

One possible state of the system is given by

$$q_{ij}(t,t') = \delta_{ij} q(t-t') \ , \tag{3.7}$$

$$\Delta_{ij}'(t,t') = \delta_{ij}\Delta'(t-t') \tag{3.8}$$

Before we go on to discuss the rotational degeneracy of this state, we briefly recall some of the properties of the SG state, which have been derived within the dynamic mean-field theory (MFT).

The simplest MFT does not allow for an anomalous response, i.e., $\Delta'=0$. In this case the frozen correlations have a constant, finite value on the longest time scales. This solution is identical to the solution of Sherrington and Kirkpatrick (SK),[20] which is known to be unstable below T_c.[25]

Another MF solution has been proposed by Sommers.[26] To rederive it within the dynamic approach,[23,24] one has to assume the existence of one microscopic time scale τ_Δ, such that for $\omega \gg 1/\tau_\Delta$,

$$G(\omega) = \frac{1}{m} \sum_i \left[\frac{\partial \langle S^i(\mathbf{x})\rangle}{\partial \beta h_i(\mathbf{x})} \right] = 1-q \ ,$$

whereas for $\omega \ll 1/\tau_\Delta$, $G(\omega)=1-q+\Delta$. This solution was shown to be stable on finite time scales, but is unstable on the longest time scales.[27,28] These results are straightforward generalizations of the corresponding calculations for the Ising case.

In the stable MF solution one takes into account the relaxation of both q and Δ with a distribution of macroscopic time scales.[18] These are due to "hopping" processes over energy barriers, which diverge in the thermodynamic limit. The relaxation times have a hierarchical structure that can be parametrized by $x \in [0,1]$, such that $\tau_x \ll \tau_y$, if $y < x$. The transitions between different states give rise to a slow decay of frozen correlations

$$q(x) = \frac{1}{m} [\langle \mathbf{S}(\mathbf{y},0)\cdot \mathbf{S}(\mathbf{y},\tau_x)\rangle]$$

and an anomalous contribution,

$$\Delta'(x)=\frac{1}{m}[\langle i\hat{\mathbf{S}}(y,0)\cdot\mathbf{S}(y,\tau_x)\rangle]$$

to the low-frequency response

$$G(x)=1-q(1)+\Delta(x)=1-q(1)+\int_1^x dy\,\Delta'(y)\ .$$

In the finite time limit—denoted by τ_1—the frozen correlations are maximal and there is no anomalous response $\Delta(1)=0$. On the longest time scale τ_0, frozen correlations have completely decayed $q(0)=0$ and the response has reached its equilibrium value $G(0)=1-q(1)+\Delta(0)$.

In the absence of external fields the Lagrangian is trivially invariant under a uniform rotation of all the spins. More importantly, the spins at time t can be uniformly rotated with respect to their direction at some other, arbitrary time t', provided the rotation is slow on the microscopic time scale set by Γ_0^{-1}. The transformation

$$\begin{aligned}S(\mathbf{x},t)&\to R(\theta_t)S(\mathbf{x},t)\ ,\\ \hat{S}(\mathbf{x},t)&\to R(\theta_t)\hat{S}(\mathbf{x},t)\end{aligned} \tag{3.9}$$

leaves the dynamic Lagrangian invariant, provided $\Gamma_0^{-1}\dot{\theta}\to 0$. This can be easily checked with the explicit expression for L [Eqs. (3.2) and (3.3)]. The nonlocal part of L (3.2) only involves scalar products of two spin operators at the *same* time. The same is true for the local part L_0 (3.3), except for the term $i\hat{S}_i(x,t)\Gamma_0^{-1}\partial_t S_i(x,t)$. If we require $\Gamma_0^{-1}\dot{\theta}\to 0$, then the transformation defined in Eq. (3.9) becomes an exact symmetry of L.

In this limit, *nontrivial* dynamics for $\theta(t)$ survives, namely variations of $\theta(t)$, on macroscopic time scales $\{\tau_x\}$, which are associated with the hopping over macroscopic free-energy barriers. Rotations on time scales $\{\tau_x\}$ leave L invariant, reflecting the *absence* of barriers to uniform rotations.

The invariance properties of the Lagrangian imply a manifold of degenerate spin-glass states, which are related to one another by time-dependent rotations. In Eqs. (3.7) and (3.8), we have singled out that particular state, which does not contain a slow time-dependent rotation. The most general order parameters have the form

$$\begin{aligned}\overleftrightarrow{q}(t,t')&=\lim_{\Gamma_0(t-t')\to\infty}[\langle[\overleftrightarrow{\mathbf{R}}(\theta_t)\cdot\mathbf{S}(\mathbf{x},t)]\cdot[\overleftrightarrow{\mathbf{R}}(\theta_{t'})\mathbf{S}(\mathbf{x},t')]\rangle]\\ &=\overleftrightarrow{\mathbf{R}}(\theta_t-\theta_{t'})q(t-t')\end{aligned} \tag{3.10a}$$

$$\begin{aligned}\overleftrightarrow{\Delta}'(t,t')&=\lim_{\Gamma_0(t-t')\to\infty}[\langle[\overleftrightarrow{\mathbf{R}}(\theta_t)\cdot i\hat{\mathbf{S}}(\mathbf{x},t)][\overleftrightarrow{\mathbf{R}}(\theta_{t'})\cdot\mathbf{S}(\mathbf{x},t')]\rangle]\\ &=\overleftrightarrow{\mathbf{R}}(\theta_t-\theta_{t'})\Delta'(t-t')\ .\end{aligned} \tag{3.10b}$$

As discussed in Sec. II, the spin-wave stiffness ρ_s is related to a quasiequilibrium susceptibility $\chi^\delta(k)$. In the dynamic approach, this amounts to a particular choice of time separations in a dynamic four-spin correlation, namely,

$$\chi^z(\mathbf{x},\mathbf{x}')=\frac{1}{4T}\int^{\le\tau_1}d(t_2-t_4)[\langle S_i(\mathbf{x},t_1)S_j(\mathbf{x},t_2)S_k(\mathbf{x}',t_3)i\hat{S}_l(\mathbf{x}',t_4)\rangle]\epsilon_{zij}\epsilon_{zkl}\ , \tag{3.11}$$

with time ordering $t_1<t_3<t_4<t_2$ and time separations $t_3-t_1\sim\tau_1$, $t_4-t_3\sim\tau_1$, and (t_2-t_4) integrated up to time scales τ_1. For this ordering, the correlations factorize according to

$$\chi^z(\mathbf{x},\mathbf{x}')=\frac{1}{4T}\int^{\le\tau_1}d(t_2-t_4)[\langle S_i(\mathbf{x},t_1)\rangle\langle S_k(\mathbf{x}',t_3)\rangle\langle S_j(\mathbf{x},t_2)i\hat{S}_l(\mathbf{x}',t_4)\rangle]\epsilon_{zij}\epsilon_{zkl}\ . \tag{3.12}$$

We can then use the fluctuation dissipation theorem to relate the unaveraged response function

$$\int^{\le\tau_1}d(t_2-t_4)\langle S_j(\mathbf{x},t_2)i\hat{S}_l(\mathbf{x}',t_4)\rangle=\langle S_j(\mathbf{x},t_2)S_l(\mathbf{x}',t_2)\rangle-\langle S_j(\mathbf{x},t_2)\rangle\langle S_l(\mathbf{x}',t_2)\rangle$$

to the corresponding correlation function. The disconnected part does not contribute in Eq. (3.12), so that the definition in (3.12) is indeed the same as the preceding definition of Eq. (2.2).

IV. CALCULATION OF ρ_s

The four-spin interactions appearing in Eq. (3.2) are conveniently decoupled by a Gaussian transformation to local two-spin operators

$$\begin{aligned}Z&=\int D\{Q\}e^{L\{Q\}}\\ &=\int D\{Q\}\exp\left[-\frac{\beta^2J^2}{2}\sum_{1,2}\sum_{\mathbf{x},\mathbf{x}'}K_{\mathbf{x}\mathbf{x}'}^{-1}Q^{\alpha\beta}(\mathbf{x},1,2)Q^{\gamma\delta}(\mathbf{x},1,2)A^{\alpha\beta\gamma\delta}+\ln\int D\{\Phi\}\exp(L_1\{\Phi,Q\})\right]\ ,\end{aligned} \tag{4.1}$$

with

$$L_1=L_0+\frac{\beta^2J^2}{2}\sum_{1,2}\sum_{\mathbf{x}}Q^{\alpha\beta}(\mathbf{x},1,2)\Phi^\alpha(\mathbf{x},1)\Phi^\beta(\mathbf{x},2)\ , \tag{4.2}$$

with L_0 given in Eq. (3.3). Here we have introduced the following notation:

$$\Phi^\alpha(\mathbf{x},1)=(i\hat{S}_{i_1}(\mathbf{x},t_1),S_{i_1}(\mathbf{x},t_1))$$

is a two-component vector where the argument $1=(t_1,i_1)$ denotes the time argument and the Cartesian spin component. Summation over greek indices is implied and

$$\sum_{1,2}=\int_{-\infty}^{+\infty}dt_1\int_{t_1}^{\infty}dt_2\sum_{i_1,i_2} .$$

The matrix A has components

$$A^{1122}=A^{2211}=A^{1221}=A^{2112}=1$$

and zero otherwise. The expectation values of the Q fields are related to spin correlations and response functions via

$$\langle Q^{11}(\mathbf{x},1,2)\rangle=\frac{1}{N}\sum_{\mathbf{x}}\langle\Phi^2(\mathbf{x},1)\Phi^2(\mathbf{x},2)\rangle , \tag{4.3}$$

$$\langle Q^{21}(\mathbf{x},1,2)\rangle=\frac{1}{N}\sum_{\mathbf{x}}\langle\Phi^1(\mathbf{x},1)\Phi^2(\mathbf{x},2)\rangle \tag{4.4}$$

and similar for higher-order cumulants. Here we concentrate on $\chi^z(k)$, which is given by

$$4T\chi^z(\mathbf{x}-\mathbf{x}')=\int^{\leq\tau_1}d(t_2-t_4)[\langle Q_{ij}^{11}(\mathbf{x},t_1,t_2)Q_{kl}^{12}(\mathbf{x}',t_3,t_4)\rangle-\langle Q_{ij}^{11}(\mathbf{x},t_1,t_2)\rangle\langle Q_{kl}^{12}(\mathbf{x}',t_3,t_4)\rangle]\epsilon_{zij}\epsilon_{zkl} \tag{4.5}$$

for the particular ordering of external times $t_1<t_3<t_4<t_2$ and times separation $(t_3-t_1)\sim\tau_1$, $(t_4-t_3)\sim\tau_1$. The integration over (t_2-t_4) extends up to time scales τ_1.

In the MF approximation we replace the fields Q by their values at the saddle point

$$\frac{\delta L\{\bar{Q}\}}{\delta Q^{\alpha\beta}(x,1,2)}=0 . \tag{4.6}$$

To go beyond MF theory we expand the Lagrangian $L\{Q\}$ in fluctuations $\delta Q^{\alpha\beta}(x,1,2)=Q^{\alpha\beta}(x,1,2)-\bar{Q}^{\alpha\beta}(1,2)$. Up to quadratic order this expansion reads

$$L\{\delta Q\}=\frac{1}{2}\sum_{1,2}\sum_{\mathbf{x},\mathbf{x}'}\delta Q^{\alpha\beta}(\mathbf{x},1,2)\frac{\delta^2 L}{\delta Q^{\alpha\beta}(\mathbf{x},1,2)\delta Q^{\gamma\delta}(\mathbf{x}',3,4)}\delta Q^{\gamma\delta}(\mathbf{x}',3,4) , \tag{4.7}$$

with

$$-\frac{\delta^2 L}{\delta Q^{\alpha\beta}(\mathbf{x},1,2)\delta Q^{\gamma\delta}(\mathbf{x}',3,4)}=K_{\mathbf{x}\mathbf{x}'}^{-1}\beta^2J^2A^{\alpha\beta\gamma\delta}\delta(1-3)\delta(2-4)-\beta^4J^4\langle\Phi^\alpha(\mathbf{x},1)\Phi^\beta(\mathbf{x},2)\Phi^\gamma(\mathbf{x}',3)\Phi^\delta(\mathbf{x}',4)\rangle_{\mathrm{MF}}$$
$$+\beta^4J^4\langle\Phi^\alpha(\mathbf{x},1)\Phi^\beta(\mathbf{x},2)\rangle_{\mathrm{MF}}\langle\Phi^\gamma(\mathbf{x}',3)\Phi^\delta(\mathbf{x}',4)\rangle_{\mathrm{MF}} . \tag{4.8}$$

To calculate ρ_s we use the quadratic approximation to $L\{Q\}$ given in Eq. (4.7). Within the Gaussian approximation to $L\{Q\}$ the correlations of Q fields are obtained by solving the integral equation,

$$\frac{\delta^2 L}{\delta Q^{\alpha\beta}(\mathbf{x},1,2)\delta Q^{\gamma\delta}(\mathbf{x}',3,4)}\langle\delta Q^{\gamma\delta}(\mathbf{x}',3,4)\delta Q^{\nu\mu}(\mathbf{y},5,6)\rangle=\delta(\mathbf{x}-\mathbf{y})\delta(1-5)\delta(2-6)\delta_{\alpha\nu}\delta_{\beta\mu} \tag{4.9}$$

It is instructive to first calculate the stiffness for the SK solution with constant q and $\Delta=0$. Naively, one might expect a k^{-2} divergence of $\chi^\delta(k)$ also for this theory and one might even hope to find the correct stiffness ρ_s for T sufficiently close to T_g. These expectations are not justified by our calculations, as we now show. We explicitly solve the integral equation (4.9) for components $(\alpha\beta\nu\mu)=(2222)$ and (2212),

$$\frac{\delta^2 L}{\delta Q^{22}(\mathbf{x},1,2)\delta Q^{11}(\mathbf{x}',3,4)}\langle\delta Q^{11}(\mathbf{x}',3,4)\delta Q^{12}(\mathbf{y},5,6)\rangle+\frac{\delta^2 L}{\delta Q^{22}(\mathbf{x},1,2)\delta Q^{21}(\mathbf{x}',3,4)}\langle\delta Q^{21}(\mathbf{x}',3,4)\delta Q^{12}(\mathbf{y},5,6)\rangle=0 , \tag{4.10}$$

$$\frac{\delta^2 L}{\delta Q^{22}(\mathbf{x},1,2)\delta Q^{11}(\mathbf{x}',3,4)}\langle\delta Q^{11}(\mathbf{x}',3,4)\delta Q^{22}(\mathbf{y},5,6)\rangle$$
$$+\frac{\delta^2 L}{\delta Q^{22}(\mathbf{x},1,2)\delta Q^{21}(\mathbf{x}',3,4)}\langle\delta Q^{21}(\mathbf{x}',3,4)\delta Q^{22}(\mathbf{y},5,6)\rangle=\delta(\mathbf{x}-\mathbf{y})\delta(1-5)\delta(2-6) . \tag{4.11}$$

The rotational susceptibility is obtained from Eq. (4.10) if the external times are ordered as $t_1<t_5<t_6<t_2$ and the limit $t_5-t_1\to\infty$, $t_6-t_5\to\infty$ is taken and t_2 is integrated over all times $t_2>t_6$. The equation then reads

$$\left[\frac{K^{-1}(k)}{T^2}-\frac{1}{3T^4}[G_T(G_T+2G_L)]\right]\chi^z(k)=\frac{[m^2G_T]}{3T^3}G_{1221}(k) \tag{4.12}$$

with

$$G_{1221}(\mathbf{x}-\mathbf{y})=\frac{1}{4T^2}\lim_{t_2-t_1\to\infty}\int dt_3dt_4\langle Q_{ij}^{12}(\mathbf{x},t_1,t_2)Q_{kl}^{21}(\mathbf{y},t_3,t_4)\rangle\epsilon_{zij}\epsilon_{zkl} \; . \tag{4.13}$$

Here m, G_T, and G_L denote the local unaveraged magnetization and transverse and longitudinal response functions of MFT:

$$m=\sqrt{3}\left[\tanh(z\sqrt{3})-\frac{1}{z\sqrt{3}}\right],\quad G_L=\frac{\partial m}{\partial z},\quad \text{and } G_T=\frac{m}{z} \; .$$

The average refers to the Gaussian static noise z with variance $[z^2]=2\beta^2q$. In particular,

$$[G_Tm^2]=\text{const}\times\int_0^\infty dz\,z^2e^{-z^2/2\beta^2q}\frac{m^3}{z} \; , \tag{4.14}$$

$$[G_T(G_T+G_L)]=\text{const}\times\int_0^\infty dz\,z^2e^{-z^2/2\beta^2q}\left[\frac{m^2}{z^2}+2\frac{m}{z}\frac{\partial m}{\partial z}\right]=3T^2 \; , \tag{4.15}$$

with

$$\text{const}=\sqrt{\pi/2}(2\beta^2q)^{3/2} \; ,$$

where the last equality follows by partial integration. In the limit of long wavelength, Eq. (4.12) yields

$$\chi^z(k)=\frac{[m^2G_T]2d}{3T(ka)^2}G_{1221}(k) \; . \tag{4.16}$$

To be specific, we have used

$$K(k)\simeq K(0)-\tfrac{1}{2}(k_xa)^2=1-\frac{1}{2d}(\mathbf{k}a)^2 \; , \tag{4.17}$$

which is appropriate to a nearest-neighbor hypercubic lattice in d dimensions. The spin-wave susceptibility $\chi^z(k)$ diverges in the limit of long wavelength, as we expect. Whether or not the spin-wave stiffness is finite depends on the long-wavelength behavior of $G_{1221}(k)$. To obtain the latter function we have to solve Eq. (4.11) for time separation $t_2-t_1\to\infty$ and t_5 and t_6 integrated freely. In this limit the second term does not contribute and we find

$$G_{1221}(k)=\frac{T^2}{4}[T^2K^{-1}(k)-\tfrac{1}{3}[G_T(G_T+2G_L)]]^{-1}$$
$$\simeq\frac{d}{2(ka)^2} \; . \tag{4.18}$$

So the divergence of the rotational susceptibility is actually stronger than k^{-2}

$$\chi^z(k)=\frac{[m^2G_T]}{3T}\frac{d^2}{(ka)^4} \tag{4.19}$$

and ρ_s vanishes for the SK solution. Note that the coefficient of the k^{-4} divergence vanishes as $|T-T_g|$ as the transition is approached. To see whether the vanishing of ρ_s is peculiar to the SK solution or a genuine feature of the SG state, we clearly have to go beyond this simple theory. Possibly, the existence of a finite stiffness is related to the existence of an anomalous response (or replica symmetry breaking). If so, then Sommers's solution should provide the simplest MF theory with a finite ρ_s. For this solution the nonequilibrium $\chi^z(k)$ requires all time separations to be small as compared to τ_Δ. The calculation of $\chi^z(k)$ is rather involved, so we have used a truncated Lagrangian (quartic approximation), which gives the correct behavior near T_g. The details of the calculation are delegated to Appendix A, the result is

$$\chi^z(k)=\frac{dq^2}{2\Delta q(ka)^2} \; . \tag{4.20}$$

Therefore the stiffness $\rho_s=2J^2a^2\Delta q/d$ is finite. This result, that the SG stiffness is finite, also holds for the marginally stable MF solution, which involves a distribution of q and Δ, as was discussed above. However, the explicit calculation of ρ_s for the marginally stable phase is difficult to perform in the dynamic approach. In the following section we present a calculation of ρ_s using the replica formalism, which provides a more elegant method of calculation.

V. REPLICA THEORY

A. The replica theory

In the replica formalism, the ensemble averaged free energy is obtained as the limit

$$-\beta f=\frac{1}{N}[\ln Z]=\frac{1}{N}\lim_{n\to 0}([Z^n]-1) \; . \tag{5.1}$$

The quantity Z^n is calculated by analytical continuation from positive integer n, in which case Z^n is just the partition function of n copies (or replicas) of the system. Using standard transformations, one traces out the spin degrees of freedom in favor of continuous fields $Q_{\alpha\beta}^{ij}(\mathbf{x})$ where α and β are replica indices $\alpha,\beta=1,\ldots,n$, and i,j are the spin Cartesian coordinates $i,j=1,\ldots,m$. The result is

$$-\beta f=\frac{1}{N}\lim_{n\to 0}\frac{1}{n}\left[\int DQ_{\alpha\beta}\,e^{-\beta F\{Q\}}-1\right] \; , \tag{5.2}$$

where

$$\beta F\{Q\}=\frac{\beta^2J^2}{4}\sum_{\mathbf{x},\mathbf{x}'}K^{-1}(\mathbf{x}-\mathbf{x}')\sum_{\substack{\alpha,\beta\\ i,j}}Q^{ij}_{\alpha\beta}(\mathbf{x})Q^{ji}_{\alpha\beta}(\mathbf{x}')$$
$$-\ln \mathrm{Tr}_{\{\mathbf{S}_\alpha\}}\exp\left\{\frac{\beta^2J^2}{2}\sum_{\alpha,\beta}\sum_{\mathbf{x}}S^i_\alpha(\mathbf{x})\times Q^{ij}_{\alpha\beta}(\mathbf{x})S^j_\beta(\mathbf{x})\right\} . \quad (5.3)$$

The fields $Q_{\alpha\beta}$ can be restricted to have the symmetry $Q^{ij}_{\alpha\beta}=Q^{ji}_{\beta\alpha}$. The order parameters of the spin-glass phase are

$$\langle Q^{ij}_{\alpha\beta}\rangle=\frac{1}{N}\left[\sum_{\mathbf{x}}\langle S^i_\alpha(\mathbf{x})S^j_\beta(\mathbf{x})\rangle\right],\quad \alpha\neq\beta . \quad (5.4)$$

The rotational invariance of the system is manifested in the invariance of the free-energy functional (5.3) under global rotations of each of the replicated systems separately, i.e.,

$$F\{Q_{\alpha\beta}(\mathbf{x})\}=F\{R_\alpha Q_{\alpha\beta}(\mathbf{x})R^{-1}_\beta\} .$$

Here, $\{R_\alpha\}^n_{\alpha=1}$ is a set of n arbitrary O(m) rotations, $R_\alpha=R(\{\theta^\sigma_\alpha\})$ [recall that $\sigma=1,\dots,m(m-1)/2$ and $\alpha=1,\dots,n$]. Studies of the replica theory (in the MF limit) usually assume $\theta^\sigma_\alpha=0$ or $R_\alpha=I$.[29] In this case the order parameter (5.4) takes the form

$$\langle Q^{ij}_{\alpha\beta}\rangle=q_{\alpha\beta}\delta^{ij},\quad \alpha\neq\beta . \quad (5.5)$$

Equation (5.5) is a manifestation of the fact that none of the pure SG states contain *spontaneous* or *external bulk* anisotropy. In fact, this holds also for a system with *random* anisotropies. In this case, Eq. (5.5) is the *only* allowed symmetry of the order parameter. On the other hand, in the case of an isotropic system, Eq. (5.3), the most general symmetry of the order parameter is

$$\langle Q_{\alpha\beta}\rangle=q_{\alpha\beta}R_\alpha R^{-1}_\beta,\quad \alpha\neq\beta , \quad (5.6)$$

where R_α is an arbitrary replica-dependent rotation and $q_{\alpha\beta}$ is the following invariant quantity

$$q_{\alpha\beta}=\frac{1}{m}\left\{\sum^m_{i,j=1}\langle Q^{ij}_{\alpha\beta}\rangle^2\right\}^{1/2},\quad \alpha\neq\beta \quad (5.7)$$

B. Replica symmetry breaking in vector SG

In the MF replica theory, $Q^0_{\alpha\beta}=\langle Q_{\alpha\beta}\rangle_{\mathrm{MF}}$ is calculated by the saddle-point equations

$$\frac{\delta F}{\delta Q^{ij}_{\alpha\beta}}=0 . \quad (5.8)$$

The marginally stable solution of Eqs. (5.8) below T_g, breaks the replica symmetry, i.e., $q_{\alpha\beta}$ [Eq. (5.7)] depends on the replica indices (α,β).[29] In Parisi's replica symmetry breaking scheme, the matrix $q_{\alpha\beta}$ consists of a hierarchy of blocks, as shown schematically in Fig. 2. This structure allows for the parametrization of $q_{\alpha\beta}$ by a single (monotonically increasing) function $q(x)$, $0\le x\le 1$. This function is equivalent to the function $q(x)$ in the dynamic MF theory (Sec. III) if one chooses the "gauge" $d\Delta/dx=-x$.[18]

Extending the current physical interpretation[19,21] of the Ising order function to the vector case we consider two physical copies of the system with the same exchange couplings,

$$H=-\sum_{\mathbf{x},\mathbf{y}}J_{\mathbf{xy}}[\mathbf{S}_1(\mathbf{x})\cdot\mathbf{S}_1(\mathbf{y})+\mathbf{S}_2(\mathbf{x})\cdot\mathbf{S}_2(\mathbf{y})] . \quad (5.9)$$

The overlap between the two systems is defined as

$$Q^{ij}_{12}=\frac{1}{N}\sum_{\mathbf{x}}\langle S^i_1(\mathbf{x})\rangle\langle S^j_2(\mathbf{x})\rangle . \quad (5.10)$$

The probability distribution of Q^{ij}_{12} equals

$$P(Q)=\sum_{a,b}P_aP_b\delta(Q-Q_{ab}) , \quad (5.11)$$

where

$$Q^{ij}_{ab}=\frac{1}{N}\sum_{\mathbf{x}}\langle S^i(\mathbf{x})\rangle_a\langle S^j(\mathbf{x})\rangle_b .$$

Here, P_a and P_b are the Boltzmann distributions of the two pure SG states $\langle\mathbf{S}\rangle_a$ and $\langle\mathbf{S}\rangle_b$. The summation over the pure states (a) and (b) in Eq. (5.11) includes all the "rotationally" degenerate states. Thus by applying a global rotation to, say, the states (a), one observes that $P(Q)=P(RQ)$, where R is an arbitrary rotation. Furthermore, the fact that each of the pure states is, on the average, isotropic, implies that the overlap Q must be of the form,

$$Q_{ab}=q_{ab}R_aR^{-1}_b . \quad (5.12)$$

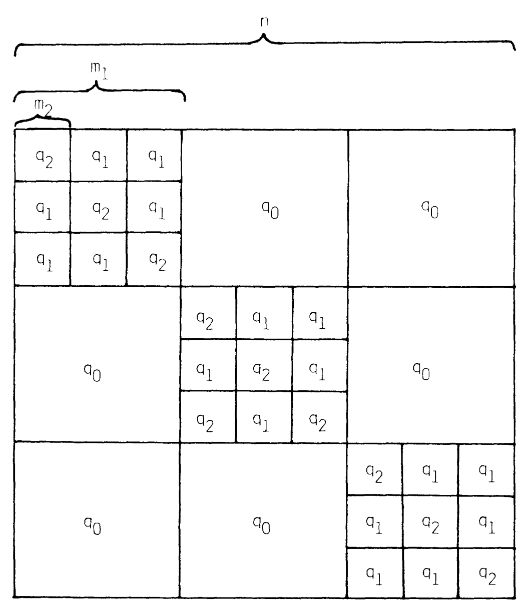

FIG. 2. Parisi's replica symmetry breaking scheme for the matrix $q_{\alpha\beta}$.

Thus $P(Q)$ is a function only of the single invariant q_{ab},

$$P(q)=\frac{1}{N}\sum_{a,b}P_aP_b\delta(q-q_{ab}) \tag{5.13}$$

where

$$q_{ab}=\frac{1}{m}\left[\sum_{i,j=1}^{m}\left[\frac{1}{N}\sum_{\mathbf{x}}\langle S^i(\mathbf{x})\rangle_a\langle S^j(\mathbf{x})\rangle_b\right]^2\right]^{1/2} \tag{5.14}$$

is invariant under the rotation of one pure state relative to the other. The quantity (5.13) is related to the replica order function $q(x)$ via $P(q)=dx/dq$, just as in the Ising case. In MF theory, $P(q)$ has a δ-function piece at $q_{\max}=q(1)$, implying that $q(1)$ represents the EA single-valley order parameter, $q(1)=q_{aa}$. The quantity $q(1)$ is "self-averaging," i.e., it does not exhibit sample-to-sample fluctuations in the $N\to\infty$ limit.[21,30] The definition (5.13) and (5.14) of the overlap distribution in a vector SG is convenient from a computational point of view. Alternatively, one can eliminate the rotational degrees of freedom by applying an external field.

C. Single-valley stiffness constant in the replica theory

The calculation of $\chi^z(k)$ [Eq. (2.4)] can be performed in the replica theory using four-spin correlations with three different replica indices [which corresponds to the three thermal averages that appear in (2.4)]. However, we have found it considerably simpler to evaluate within the replica theory, the following correlation function,

$$G^z(\mathbf{k})=\frac{-\beta}{8N}\sum_{\mathbf{x},\mathbf{x}'}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\epsilon_{zij}\epsilon_{zkl}\times[C_{ik}(\mathbf{x},\mathbf{x}')C_{jl}(\mathbf{x},\mathbf{x}')] , \tag{5.15}$$

where

$$C_{ik}(\mathbf{x},\mathbf{x}')=\langle S_i(\mathbf{x})S_k(\mathbf{x}')\rangle .$$

The two quantities $G^z(\mathbf{k})$ and $2\chi^z(\mathbf{k})$ differ by an amount which remains finite as $k\to 0$, see Appendix C.

Thus, in analogy with Eq. (2.3), ρ_s can be defined as

$$\rho_s=vq_{\rm EA}^2\lim_{\mathbf{k}\to 0}[k^2G^z(\mathbf{k})]^{-1} . \tag{5.16}$$

As we are interested in the single-valley ρ_s, the thermal averages of C_{ik} and C_{jl} must be restricted to the *same pure SG phase*. This can be done by considering the replica propagator,

$$G^z_{\alpha\beta}(\mathbf{k})=\frac{\beta}{8}\epsilon_{zij}\epsilon_{zkl}\langle Q^{ij}_{\alpha\beta}(\mathbf{k})Q^{kl}_{\alpha\beta}(-\mathbf{k})\rangle . \tag{5.17}$$

Using Parisi's theory, $G^z_{\alpha\beta}(\mathbf{k})$ is parametrized by a function $G^z(\mathbf{k};x)$ where $0\le x\le 1$, where x represents the scale of the smallest block to which both α and β belong. The propagator $G^z(\mathbf{k};x)$ is a continuous function of x for $0\le x\le\bar{x}$ and is constant $G^z(\mathbf{k};x)=G^z(\mathbf{k},1)$ for $\bar{x}<x<1$, just as $q(x)$.

The physical interpretation of $G^z(\mathbf{k};x)$ is similar to that of $q(x)$. Defining the overlap of spatial correlation functions of a pair of pure SG states

$$G^z_{ab}(\mathbf{k})\equiv\frac{\beta}{8N}<\sum_{\mathbf{x},\mathbf{x}'}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\epsilon_{zij}\epsilon_{zkl}\times\langle S^i(\mathbf{x})S^k(\mathbf{x}')\rangle_a\langle S^j(\mathbf{x})S^l(\mathbf{x}')\rangle_b \tag{5.18}$$

then, the probability distribution of $G^z_{ab}(\mathbf{k})$

$$P(G^z(\mathbf{k}))=\sum_{a,b}P_aP_b\delta(G^z(\mathbf{k})-G^z_{ab}(k)) =\left|\frac{dG_z(\mathbf{k};x)}{dx}\right|^{-1} , \tag{5.19}$$

where $G^z(\mathbf{k};x)$ is the x-dependent propagator calculated in the replica theory. In particular, $G^z(\mathbf{k};1)$ corresponds to the single-valley propagator in which all thermal averages are restricted to the same pure SG phase. Thus, the correct definition of the single-valley ρ_s in the replica theory is

$$\rho_s=v^2q^2(1)\lim_{\mathbf{k}\to 0}[k^2G^z(\mathbf{k};1)]^{-1} , \tag{5.20}$$

where $G^z(\mathbf{k};1)$ is the propagator $G^z_{\alpha\beta}(\mathbf{k})$, Eq. (5.17), but with α and β restricted to belonging to the same *smallest*-scale block. In order to calculate ρ_s, it suffices to calculate the infrared divergence of $G^z(\mathbf{k};1)$. This is done in the following subsection by relating this propagator to spin-wave fluctuations in the fields $Q_{\alpha\beta}(\mathbf{x})$.

D. Spin-wave fluctuations in the replica theory

We identify spin-wave fluctuations in $Q_{\alpha\beta}$ as fluctuations which correspond to a long-wavelength modulation in the replica-dependent rotation angles. We consider a small perturbation of the state $\langle Q^{ij}_{\alpha\beta}\rangle=q_{\alpha\beta}\delta^{ij}$, by space-dependent rotations around the z axis,

$$\delta Q_{\alpha\beta}(\mathbf{x})=q_{\alpha\beta}(R[\delta\theta^z_\alpha(\mathbf{x})]R^{-1}[\delta\theta^z_\beta(\mathbf{x})]-I) \tag{5.21a}$$

$$\simeq q_{\alpha\beta}(\delta\theta^z_\alpha-\delta\theta^z_\beta)\epsilon_z , \tag{5.21b}$$

where ϵ_z stands for the antisymmetric tensor ϵ_{zij}. There are $3(n-1)$ independent fluctuations of this form (in the Heisenberg case). The reason for the factor $n-1$ is that a replica-independent rotation does not induce fluctuations in $\langle Q_{\alpha\beta}\rangle$. It is useful to construct a basis of orthonormal eigenvectors for the subspace of fluctuations of the form (5.21). To do so we consider the following eigenvalue problem in replica space,

$$\sum_\beta\left[\left[\sum_\sigma q^2_{\alpha\sigma}\right]\delta_{\alpha\beta}-q^2_{\alpha\beta}\right]U_\beta(\lambda)=\beta^{-1}\epsilon(\lambda)U_\alpha(\lambda) , \quad \lambda=1,\ldots,n-1 . \tag{5.22}$$

Here $U_\alpha(\lambda)$ is the eigenvector associated with the λth eigenvalue $\beta^{-1}\epsilon(\lambda)$. The $n-1$ eigenvectors satisfy the orthonormality condition,

$$\sum_\alpha U_\alpha(\lambda)U_\alpha(\lambda')=\delta_{\lambda\lambda'} , \tag{5.23}$$

and the additional constraint,

$$\sum_\alpha U_\alpha(\lambda)=0 , \tag{5.24}$$

which excludes the trivial eigenvector $U_\alpha(\lambda)=U$. A general angle fluctuation around the z axis can now be written as

$$\delta\theta^z_\alpha(\mathbf{x}) = \sum_\lambda \delta\theta^z_\lambda(\mathbf{x}) U_\alpha(\lambda) . \tag{5.25}$$

We define the following $n-1$ orthonormal vectors $\{Q_{\alpha\beta}(\lambda)\}$ in the space of the matrices $Q_{\alpha\beta}$,

$$Q_{\alpha\beta}(\lambda) = \epsilon_z q_{\alpha\beta}[U_\alpha(\lambda) - U_\beta(\lambda)]/[2\beta^{-1}\epsilon(\lambda)]^{1/2} . \tag{5.26}$$

Equations (5.22) and (5.23) guarantee the normalization,

$$\sum_{\substack{i,j \\ \alpha,\beta}} Q^{ij}_{\alpha\beta}(\lambda) Q^{ij}_{\alpha\beta}(\lambda') = \delta_{\lambda\lambda'} .$$

A general spin-wave fluctuation, (5.21), can be expressed as

$$\delta Q_{\alpha\beta}(\mathbf{x}) = \sum_{\lambda=1}^{n-1} \delta q^z_\lambda(\mathbf{x}) Q_{\alpha\beta}(\lambda) . \tag{5.27}$$

We can use $\{Q_{\alpha\beta}(\lambda)\}$ to project out the spin-wave part of a general fluctuation. The amplitudes $\{\delta q^z_\lambda(\mathbf{x})\}$ associated with a general fluctuation $\delta Q_{\alpha\beta}(\mathbf{x})$ will be

$$\delta q^z_\lambda(\mathbf{x}) = -\tfrac{1}{2} \sum_{\alpha,\beta} \sum_{i,j} \delta Q^{ij}_{\alpha\beta}(\mathbf{x}) Q^{ij}_{\alpha\beta}(\lambda) . \tag{5.28}$$

The amplitudes of the fluctuating angles $\{\delta\theta^z_\lambda(\mathbf{x})\}$ of a general fluctuation will then be

$$\delta\theta^z_\lambda(\mathbf{x}) = \delta q^z_\lambda(\mathbf{x})/[2\beta^{-1}\epsilon(\lambda)]^{1/2} . \tag{5.29}$$

The spin-wave propagators are,

$$\begin{aligned} G^z_\lambda(\mathbf{k}) &\equiv \langle \delta q^z_\lambda(\mathbf{k}) \delta q^z_\lambda(-\mathbf{k}) \rangle \\ &= -\tfrac{1}{4} \sum_{\substack{\alpha,\beta,\gamma,\delta \\ i,j,k,l}} Q^{ij}_{\alpha\beta}(\lambda) Q^{kl}_{\gamma\delta}(\lambda) \langle \delta Q^{ij}_{\alpha\beta}(\mathbf{k}) \delta Q^{kl}_{\gamma\delta}(-\mathbf{k}) \rangle \end{aligned} \tag{5.30}$$

and

$$\langle \delta\theta^z_\lambda(\mathbf{k}) \delta\theta^z_\lambda(-\mathbf{k}) \rangle = G^z_\lambda(\mathbf{k})/2\beta^{-1}\epsilon(\lambda) \tag{5.31}$$

represent pure spin-wave fluctuations. They diverge as $|k|^{-2}$ below T_g.

Equation (5.31) defines $n-1$ propagators which for any finite n, diverge as $k \to 0$ due to rotational symmetry. Since the replica free energy must be proportional to n, there must exist an additional propagator whose mass vanishes as $n \to 0$. In the following subsection we will make the connection between $G^z_\lambda(\mathbf{k})$ and the physical correlation function, $G^z(\mathbf{k};1)$.

E. Eigenvectors of $q_{\alpha\beta}$ and their relations to ρ_s

Let us first discuss the replica symmetric state given by $q_{\alpha\beta} = q(1-\delta_{\alpha\beta})$. In this case all the $n-1$ eigenvectors of Eq. (5.22) are degenerate with the eigenvalue $\beta^{-1}\epsilon = nq^2$, indicating clearly that the propagators $\langle \delta\theta_\lambda(\mathbf{k}) \delta\theta_\lambda(-\mathbf{k}) \rangle$ diverge more strongly than k^{-2} in the $n \to 0$ limit. Indeed, one finds in this case another mode which becomes massless in the $n \to 0$ limit. Together these give rise to $G^z(\mathbf{k};1) \sim |\mathbf{k}|^{-4}$, i.e., $\rho_s = 0$, see Appendix C. Let us turn to the SG phase with broken replica symmetry. We assume that $q_{\alpha\beta}$ has the hierarchical block structure of Parisi's symmetry breaking scheme, as shown schematically in Fig. 2. The diagonalization of such a matrix is presented in detail in Appendix B. They consist of bands of degenerate modes. The bands can be parametrized by the scales x of the block sizes. The xth band contains $-ndx/x^2$ eigenvectors U_λ which vary only on the scale x. The eigenvalue of Eq. (5.22) associated with the xth band is,

$$\epsilon(x) = \beta \left[x q^2(x) - \int_0^x q^2(y) dy \right] . \tag{5.32}$$

This eigenvector structure suggests that the single-valley response functions are associated with $x=1$ modes. Indeed, an explicit eigenmode decomposition of the propagator $G^z(\mathbf{k};1)$ shows (see Appendix C) that it has nonzero projection only on the $x=1$ mode, i.e.,

$$\begin{aligned} G^z(\mathbf{k};1) &\cong \frac{\beta q^2(1)}{2\epsilon(1)} G^z_{\lambda=1}(\mathbf{k}) \\ &= q^2(1) \langle \delta\theta^z(\mathbf{k}) \delta\theta^z(-\mathbf{k}) \rangle . \end{aligned} \tag{5.33}$$

Note that Eq. (5.33) represents only the massless part of $G^z(\mathbf{k};1)$. Other contributions to $G^z(\mathbf{k};1)$ come from *massive* ("longitudinal") modes. From Eqs. (5.20) and (5.33) we conclude that

$$\begin{aligned} \rho_s &= \lim_{k\to 0} v^2 [\beta k^2 \langle \delta\theta_1(\mathbf{k}) \delta\theta_1(-\mathbf{k}) \rangle]^{-1} \\ &= [2v^2 \epsilon(1) \beta^{-1}] \lim_{k\to 0} [k^2 G^z_1(\mathbf{k})]^{-1} , \end{aligned} \tag{5.34}$$

where $G^z_1(\mathbf{k})$ is the spin-wave propagator defined in Eq. (5.30) with $\lambda = 1$. The result (5.34) is quite general. It relies only on the symmetries of the replica free energy $F\{Q_{\lambda\beta}\}$ and on the assumption that the SG phase is described by a continuous Parisi order function. To actually evaluate ρ_s one needs an explicit evaluation of $G^z_\lambda(\mathbf{k})$ in the $k \to 0$ limit. In the following subsection we will evaluate this propagator using a Gaussian expansion around MF theory.

F. Expansion around MF theory

Expanding $F\{Q\}$ in powers of $\delta Q_{\alpha\beta} = Q_{\alpha\beta} - Q^0_{\alpha\beta}$ where Q^0 is the saddle-point value, $F\{Q\}$ takes the form,

$$\begin{aligned} \delta F\{Q\} = &\frac{\beta^2}{2} \sum_{\substack{\alpha,\beta \\ x,y \\ (\alpha<\beta)}} K^{-1}(\mathbf{x}-\mathbf{y}) \sum_{i,j} \delta Q^{ij}_{\alpha\beta}(\mathbf{x}) \delta Q^{ij}_{\alpha\beta}(\mathbf{y}) \\ &-\beta^4/2 \sum_{\substack{\alpha,\beta,\sigma,\delta \\ (\alpha<\beta,\sigma<\delta) \\ i,j,k,l}} \sum_{\mathbf{x}} C^{ijkl}_{\alpha\beta\sigma\delta} \delta Q^{ij}_{\alpha\beta}(\mathbf{x}) \delta Q^{kl}_{\sigma\delta}(\mathbf{x}) \\ &+ O(\delta Q^3) , \end{aligned} \tag{5.35}$$

where

$$C^{ijkl}_{\alpha\beta\sigma\delta} = \langle S^i_\alpha S^j_\beta S^k_\sigma S^l_\delta \rangle_{\mathrm{MF}} - \langle S^j_\alpha S^j_\beta \rangle_{\mathrm{MF}} \langle S^k_\sigma S^l_\delta \rangle_{\mathrm{MF}} .$$

Restricting ourselves to quadratic order in δQ, it is easy to separate out the contribution of the spin-wave (SW) fluctuations. This can be done by substituting the ansatz (5.21a) in Eq. (5.35) and retaining only quadratic orders in the angles δQ. One obtains

$$\delta F_{\rm SW}=\frac{T}{2}\sum_{\substack{k,\sigma\\ \alpha,\beta}} q_{\alpha\beta}^2[\delta\theta_\alpha^\sigma(\mathbf{k})-\delta\theta_\beta^\sigma(-\mathbf{k})]^2[K^{-1}(\mathbf{k})-K^{-1}(0)]$$

$$=\frac{T^2}{2}\sum_{\mathbf{k},\sigma,\lambda}\epsilon(\lambda)[K^{-1}(\mathbf{k})-K^{-1}(0)]\delta\theta_\lambda^\sigma(\mathbf{k})\delta\theta_\lambda^\sigma(-\mathbf{k})$$

$$=\frac{T}{2}\sum_{\lambda,\sigma}\delta q_\lambda^\sigma(\mathbf{k})\delta q_\lambda^\sigma(-\mathbf{k})[K^{-1}(\mathbf{k})-K^{-1}(0)]\ . \quad (5.36)$$

Here the index σ stands for all $m(m-1)/2$ rotation "directions." From Eq. (5.36) we obtain,

$$\langle\delta\theta_\lambda(\mathbf{k})\delta\theta_\lambda(-\mathbf{k})\rangle=\frac{T^2}{\epsilon(\lambda)[K^{-1}(\mathbf{k})-K^{-1}(0)]}\ , \quad (5.37)$$

and using the expansion of $K(\mathbf{k})$ yields the final result

$$\rho_s=J^2a^{2-d}\epsilon(1)d^{-1}$$

$$=\frac{J^2a^{2-d}}{dT}\left[q^2(1)-\int_0^1 q^2(x)dx\right]\ . \quad (5.38)$$

In terms of the order functions $q(x)$ and $\Delta(x)$, Eq. (5.38) reads

$$\rho_s=2J^2a^{2-d}d^{-1}\int_0^1|\Delta'(x)|q(x)dx\ . \quad (5.39)$$

VI. CONCLUSIONS

It was our main concern to give a microscopic basis for the phenomenological theory of hydrodynamic fluctuations in isotropic spin glasses. We have focused here on the spin-wave stiffness, which enters the hydrodynamic theory as an unknown parameter. In a microscopic approach the spin-wave stiffness is related to a correlation function of the microscopic spin variables

$$\rho_s=vq_{\rm EA}^2\lim_{k\to 0}[k^2\chi^z(k)]^{-1}\ ,$$

$$\chi^z(k)=\frac{1}{4NT}\sum_{\mathbf{x},\mathbf{x}'}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\epsilon_{zij}\epsilon_{zkl}\times[\langle S_i(\mathbf{x})S_k(\mathbf{x}')\rangle\langle S_j(\mathbf{x})\rangle\langle S_l(\mathbf{x}')\rangle]\ . \quad (6.1)$$

This correlation function has been calculated approximately. Expanding around the marginally stable MF solution, we find a finite nonequilibrium stiffness,

$$\rho_s=2J^2(a^{2-d}/d)\int_0^1|\Delta'(x)|q(x)dx\ . \quad (6.2)$$

As the critical point is approached we expect ρ_s to vanish in the following way:

$$\rho_s\simeq|T_g-T|^\mu \quad (6.3)$$

MFT predicts close to T_g: $|\Delta'(x)|=q(x)q'(x)$ and

$$q(1)=\left[1-\frac{T}{T_g}\right]+O\left[\left|1-\frac{T}{T_g}\right|^2\right]\ .$$

Substituting these results into Eq. (6.2) we find the MF value of the exponent $\mu_{\rm MF}=3$.

At the critical point one normally expects that the symmetry is restored, i.e., that the singularity of the longitudinal and the transverse SG susceptibility is the same. This would imply $\chi^z(k)\sim k^{\eta-2}$ for the critical behavior, whereas the hydrodynamic result is $\chi^z(k)\sim(q_{\rm EA}^2/\rho_s k^2)$. If one assumes scaling, then the two behaviors can be matched at $k\sim\xi\sim|T-T_g|^{-\nu}$ yielding $\mu=\nu(d-2)$. This relation should hold as long as normal scaling laws are obeyed, which means for $d\le d_c$. Above d_c, dangerous irrelevant variables exist and cause a breakdown of hyperscaling laws. In the SG problem, $d_c=6$, which suggests that $\mu=2$ in $d=6$. On the other hand, one normally expects that μ should take on its' MF value in $d=d_c$.

The reason for this apparent violation of scaling has been discussed by Fisher and Sompolinsky,[31] who showed that additional dangerously irrelevant variables appear in the SG state. The stiffness constant vanishes for the replica symmetric solution and has a singular dependence on one of the additional dangerously irrelevant variables. This singular dependence is responsible for the discrepancy of the MF value $\mu=3$ and the Josephson relation in $d=6$. If the singular dependence is taken into account, then μ is found to change $[\mu=\frac{1}{2}(d-2)]$ already for $6\le d\le 8$, yielding $\mu=2$ in $d=6$.

It is of interest to see how fluctuations in the exchange reduce the value of the stiffness at zero temperature, as compared to its bare value ρ_s^0 [Eq. (1.5)], which results from a rigid twist. For that purpose we express ρ_s in terms of $q(1)$ and the ground-state energy per spin $\varepsilon(0)$. In MFT,

$$|\varepsilon(0)|=\frac{J^2m}{2T}\left[1-\int_0^1 q^2(x)dx\right]$$

and therefore

$$\rho_s=\frac{2a^{2-d}}{md}\left[|\varepsilon(0)|-\frac{J^2m}{2T}[1-q^2(1)]\right]\ . \quad (6.4)$$

For very small temperatures, we know that $q(1)\simeq 1-\eta(T/J)$ decreases linearly with T. Further information is available from computer experiments, which yield $\eta\simeq 0.6$ and $[\varepsilon(0)/2]\sim 0.9$ for $m=2$ (Ref. 32) and $\eta\sim 0.7$ and $[\varepsilon(0)/3]\sim 0.9$ for $m=3$ (Ref. 33). This implies a strong reduction in the stiffness constant even at zero temperature

$$\lim_{T\to 0}\frac{\rho_s}{\rho_s^0}=1-\frac{m\eta}{|\varepsilon(0)|}\sim\begin{cases}0.3 & \text{for } m=2\ ,\\ 0.2 & \text{for } m=3\ .\end{cases} \quad (6.5)$$

In the spherical limit ($m=\infty$), the stiffness vanishes completely, since $\eta=\varepsilon=1$.[33] Note that even for infinite dimension, the true stiffness differs from its bare value.

One of the main results of the above theory is the close relationship between the finite stiffness constant in the SG state and the breaking of replica symmetry. Note that Eq. (5.39) implies the vanishing of ρ_s in a replica symmetric phase. At the moment, it is not clear whether this relation holds only in MF theory or is a more general feature of SG condensation. Thus an intriguing open question is, whether short range SG's at low d can have a finite ρ_s without the features of replica symmetry breaking.

Finally, it should be stressed again that in the present work we have concentrated on the *single-valley* stiffness constant. The calculation of the long-time limit (or true equilibrium value) of the transverse susceptibility $\chi^z(k)$ is much more complicated. Approximate MF solutions (such as Sommers solution, see Appendices A and C) indicate that the equilibrium transverse susceptibility diverges stronger than k^{-2} in the SG phase. This would imply the complete relaxation of ρ_s on the longest time scale. But this conclusion still has to be confirmed by a more detailed analysis.

ACKNOWLEDGMENTS

This work was supported in part by the Fund of Basic Research administered by the Israel Academy of Science and Humanities and by the US-Israel Binational Science Foundation. One of us (A.Z.) would like to thank Bar-Ilan University for its hospitality during the final stages of this work. G.K. was supported by the National Science Foundation under Grant No. DMR85-21377.

APPENDIX A: STIFFNESS CONSTANT FOR SOMMERS'S SOLUTION

In this appendix we present the details of the calculation for Sommers's solution, which leads to the result of Eq. (4.20). The dynamic definition of $Q_{11}(t_1,t_2)$ and $Q_{12}(t_1,t_2)$ implies that these fields have nonzero expectation values even in the paramagnetic phase. We want to use a truncated Lagrangian, which correctly describes the critical behavior of the full model. To achieve a consistent expansion in $\tau=|(T-T_c)/T_c|$, we first have to redefine the fields, such that they correspond to order parameters:

$$Q_{11}(x,1,2)=\delta_{t_1,t_2}\delta_{i_1,i_2}+\tilde{Q}_{11}(x,1,2)\ , \tag{A1}$$

$$Q_{12}(x,1,2)=\delta(t_1-t_2)\delta_{i_1,i_2}+\tilde{Q}_{12}(x,1,2)\ . \tag{A2}$$

Here we have assumed, that the microscopic dynamics is infinitely fast on the time scales of interest. The Lagrangian is expanded in the fields up to quartic terms:

$$\begin{aligned} L=&-\frac{\beta^2}{2}\sum_{1,2}\sum_{x,x'}K_{xx'}^{-1}Q^{\alpha\beta}(x,1,2)Q^{\gamma\delta}(x',1,2)A^{\beta\alpha\gamma\delta}+\frac{1}{2!}\Gamma_2^{\alpha\beta\gamma\delta}(1,2,3,4)Q^{\alpha\beta}(x,1,2)Q^{\gamma\delta}(x,3,4)\\ &+\frac{w}{3!}\Gamma_3^{\alpha\beta\gamma\delta\nu\mu}(1,2,3,4,5,6)Q^{\alpha\beta}(x,1,2)Q^{\gamma\delta}(x,3,4)Q^{\nu\mu}(x,5,6)\\ &+\frac{y}{4!}\Gamma_4^{\alpha\beta\gamma\delta\nu\mu\rho\epsilon}(1,2,3,4,5,6,7,8)Q^{\alpha\beta}(x,1,2)Q^{\gamma\delta}(x,3,4)Q^{\nu\mu}(x,5,6)Q^{\rho\epsilon}(x,7,8)\ . \end{aligned} \tag{A3}$$

Here and in the following we put $J=1$ and denote $\tilde{Q}$ by Q again. $\Gamma_n^{\alpha\beta\cdots}(1,2,\ldots,n)$ are the correlations of the local Lagrangian L_0, which are connected with respect to pairs, for example,

$$\Gamma_2^{\alpha\beta\gamma\delta}(1,2,3,4)=\langle\Phi^\alpha(1)\Phi^\beta(2)\Phi^\gamma(3)\Phi^\delta(4)\rangle_{L_0}\beta^4-\langle\Phi^\alpha(1)\Phi^\beta(2)\rangle_{L_0}\langle\Phi^\gamma(3)\Phi^\delta(4)\rangle_{L_0}\beta^4\ . \tag{A4}$$

We have formally introduced coupling constants w and y to set up a systematic expansion in τ. Eventually these parameters will be set equal to unity.

The equation of state is obtained as the solution of the saddle point equation $\delta L\{\bar{Q}\}/\delta Q_{\alpha\beta}=0$. With the ansatz

$$\frac{1}{m}\sum_i\bar{Q}_{ll}^{ii}(x,t_2-t_1)=q\ ,$$

$$\frac{1}{m}\sum_i Q_{21}^{ii}(x,\omega)=\begin{cases}\tilde{G}=1-q & \text{for } \omega\gg 1/\tau_\Delta\ ,\\ G=1-q+\Delta & \text{for } \omega\ll 1/\tau_\Delta\ ,\end{cases} \tag{A5}$$

we obtain Sommer's solution

$$q=\tau+\frac{5}{m+2}\tau+O(\tau^3) \tag{A6}$$

and

$$\Delta=\frac{4}{m+2}\tau^2+O(\tau^3)\ . \tag{A7}$$

This expansion is exact up to and including terms of $O(\tau^2)$.

The next step is an expansion in fluctuations around the saddle-point solution up to quadratic order. Using the truncated Lagrangian of Eq. (A3), the MF local four-spin correlations of Eq. (4.8) are approximately given by

$$\begin{aligned}&\beta^4\langle\Phi_\alpha(1)\Phi_\beta(2)\Phi_\gamma(3)\Phi_\delta(4)\rangle_{\rm MF}-\beta^4\langle\Phi_\alpha(1)\Phi_\beta(2)\rangle_{\rm MF}\langle\Phi_\gamma(3)\Phi_\delta(4)\rangle_{\rm MF}\\ &\quad=\Gamma_2^{\alpha\beta\gamma\delta}(1,2,3,4)+w\Gamma_3^{\alpha\beta\gamma\delta\nu\mu}(1,2,3,4,5,6)\bar{Q}_{\nu\mu}(5,6)+\frac{y}{2!}\Gamma_4^{\alpha\beta\gamma\delta\nu\mu\rho\epsilon}(1,2,3,4,5,6,7,8)\bar{Q}_{\nu\mu}(5,6)\bar{Q}_{\rho\epsilon}(7,8)\ .\end{aligned} \tag{A8}$$

We are going to use these results to solve Eq. (4.9) for the spin-wave stiffness ρ_s. We first consider Eq. (4.11) for time ordering $t_5 < t_1 < t_6 < t_2$ with $t_2 - t_1 = \Delta t$ fixed and integrate over t_5 freely and over t_6, such that $t_2 - t_6 \ll \tau_\Delta$. The first term gives the following contribution:

$$\epsilon_{zmn}\epsilon_{zij}\frac{1}{N}\sum_{\mathbf{x},\mathbf{x}'} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\int dt_3 dt_4 dt_5 \int^{<} dt_6 [T^2 K_{xx'}^{-1}\delta_{ik}\delta_{jl}\delta(t_1-t_3)\delta(t_2-t_4) - \langle S_i(t_1)S_j(t_2)i\hat{S}_k(t_3)i\hat{S}_1(t_4)\rangle_{\mathrm{MF}}]$$
$$\times[\langle S_k(\mathbf{x},t_3)S_1(\mathbf{x},t_4)i\hat{S}_m(\mathbf{x}',t_5)i\hat{S}_n(\mathbf{x}',t_6)\rangle] = \left[T^2\frac{(ka)^2}{2d} - \frac{\Delta}{T^2}\right]F_1(k,\Delta t)\;, \tag{A9}$$

with

$$F_1(k,\Delta t) = \epsilon_{zkl}\epsilon_{zmn}\frac{1}{N}\sum_{\mathbf{x},\mathbf{x}'} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\int dt_5 \int^{<} dt_6 [\langle S_k(\mathbf{x},t_1)S_l(\mathbf{x},t_2)i\hat{S}_m(\mathbf{x}',t_5)i\hat{S}_n(\mathbf{x}',t_6)\rangle]\;. \tag{A10}$$

Here, $\int^{<} dt_6$ denotes an integration range $t_2 - t_6 \ll \tau_\Delta$. The first term also gives rise to an inhomogeneous term which combines with the right-hand side of Eq. (4.11)

$$\int dt_5 \int^{<} dt_6 \epsilon_{zij}\epsilon_{zmn}\langle S_i(t_1)S_j(t_2)S_m(t_5)S_n(t_6)\rangle_{\mathrm{MF}} = 2\left[T^2 + \frac{\Delta}{T^2}\right] \tag{A11}$$

The second term contributes

$$\epsilon_{zij}\epsilon_{zmn}\int dt_3 \int dt_4 \langle S_i(t_1)S_j(t_2)S_k(t_3)i\hat{S}_l(t_4)\rangle_{\mathrm{MF}}[\langle i\hat{S}_k(\mathbf{x},t_3)S_l(\mathbf{x},t_4)i\hat{S}_m(\mathbf{x}',t_5)i\hat{S}_n(\mathbf{x}',t_6)\rangle] = \left[\frac{q}{T^2} + O(\tau^2)\right]F_2(k)\;, \tag{A12}$$

with

$$F_2(k) = \epsilon_{zkl}\epsilon_{zmn}\frac{1}{N}\sum_{\mathbf{x},\mathbf{x}'} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\int dt_3 dt_5 \int^{<} dt_6 [\langle i\hat{S}_k(\mathbf{x},t_3 S_l(\mathbf{x},t_4)i\hat{S}_m(\mathbf{x}',t_5)i\hat{S}_n(\mathbf{x}',t_6)\rangle] \tag{A13}$$

Collecting the various terms together, we obtain

$$\left[T^2\frac{(ka)^2}{2d} - \frac{\Delta}{T^2}\right]F_1(k,\Delta t) - q/T^2 F_2(k) = 2\left[T^2 + \frac{\Delta}{T^2}\right]\;. \tag{A14}$$

Note that this implies that $F_1(k,\Delta t) = F_1(k)$ is independent of Δt.

To solve for $F_1(k)$ and $F_2(k)$ we need a second equation, which is provided by Eq. (4.9) for components $(\alpha\beta\nu\mu) = (1222)$. We integrate this equation freely over t_1 and t_5, and integrate over a restricted range $t_2 - t_6 \ll \tau_\Delta$. The first term gives rise to an inhomogeneity

$$\int dt_1 dt_5 \int^{<} dt_6 \epsilon_{zij}\epsilon_{zmn}\langle i\hat{S}_i(t_1)S_j(t_2)i\hat{S}_m(t_5)i\hat{S}_n(t_6)\rangle_{\mathrm{MF}} = \frac{8T^2}{5}q\Delta \tag{A15}$$

as well as to the following term:

$$\epsilon_{zij}\epsilon_{zmn}\int dt_1 dt_5 \int^{<} dt_6 \int dt_3 dt_4 \frac{1}{N}\sum_{\mathbf{x},\mathbf{x}'} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}[T^2 K_{xx'}^{-1}\delta_{ik}\delta_{lj}\delta(t_1-t_3)\delta(t_2-t_4) - \langle i\hat{S}_i(t_1)S_j(t_2)S_k(t_3)i\hat{S}_l(t_4)\rangle_{\mathrm{MF}}\delta_{xx'}]$$
$$\times[\langle i\hat{S}_k(x,t_3)S_l(x,t_4)i\hat{S}_m(x',t_5)i\hat{S}_n(x',t_6)\rangle] = \left[T^2\frac{(ka)^2}{2d} - \frac{\Delta}{T^2}\right]F_2(k)\;. \tag{A16}$$

The second term gives the following contribution:

$$\epsilon_{zij}\epsilon_{zmn}\int dt_3 dt_4 dt_1 dt_5 \int^{<} dt_6 \frac{1}{N}\sum_{\mathbf{x},\mathbf{x}'} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\langle i\hat{S}_i(t_1)S_j(t_2)i\hat{S}_k(t_3)i\hat{S}_l(t_4)\rangle_{\mathrm{MF}}$$
$$\times[\langle S_k(x,t_3)S_l(x,t_4)i\hat{S}_m(x',t_5)i\hat{S}_n(x',t_6)\rangle] = -\tfrac{4}{5}q\Delta F_1(k)\;. \tag{A17}$$

Collecting terms together, we obtain

$$-\tfrac{4}{5}q\Delta F_1(k)+\left[T^2\frac{(ka)^2}{2d}-\frac{\Delta}{T^2}\right]F_2(k)=\tfrac{8}{5}T^2q\Delta\ . \quad \text{(A18)}$$

We now have a closed system of equations, which can be solved for F_1 and F_2. The determinant D is given by

$$D=\left[T^2\frac{(ka)^2}{2d}-\frac{\Delta}{T^2}\right]^2-\tfrac{4}{5}q^2\frac{\Delta}{T^2}\simeq-2\Delta\frac{(ka)^2}{2d}\ . \quad \text{(A19)}$$

This immediately yields

$$F_1(k)=\frac{2d}{(ka)^2}T^2\ . \quad \text{(A20)}$$

Note that this result is independent of the parameters of the quartic theory. This indicates that the result (A20) is valid beyond the quartic approximation to L, even though it has been derived only within the truncated model.

The final step in our calculation is the relation of F_1 to $\chi^\delta(k)$ via Eq. (4.10). We solve it for time ordering $t_1<t_5<t_6<t_2$ and all time separations $\ll\tau_\Delta$ and t_2-t_6 integrated over:

$$-\frac{\Delta}{T^2}4T\chi^\delta-\frac{q}{T^2}F_1(k)=0 \quad \text{(A21)}$$

or

$$\chi^\delta=\frac{Tdq}{2\Delta(ka)^2}\ . \quad \text{(A22)}$$

APPENDIX B: DIAGONALIZATION OF A HIERARCHICAL MATRIX IN THE $n\to 0$ LIMIT

In this appendix we diagonalize a matrix $Q_{\alpha\beta}$ having the Parisi hierarchical structure. The simplest examples of matrices of this kind are $q_{\alpha\beta}$, the Parisi order parameter matrix and integer powers thereof $Q_{\alpha\beta}=(q_{\alpha\beta})^p$. To set the notation we review the Parisi construction. The k stage of the hierarchical structure is defined by dividing the $m_0\times m_0$ ($m_0\equiv n$) matrix into $m_1\times m_1$ blocks; these blocks are in turn subdivided into $m_2\times m_2$ blocks. The procedure is iterated k times. There are (m_{l-1}/m_l) blocks of size m_l inside a block of size m_{l-1}. The row index α can be replaced by the sequence of hierarchical block numbers $\alpha=(i_1,i_2,\ldots,i_{k+1})\, i_1=1,\ldots,(n/m_1)$, $i_2=1,\ldots,(m_1/m_2)$, $\ldots$, $i_{k+1}=1,\ldots,(m_k/m_{k+1})$. We define $m_{k+1}\equiv 1$, and i_{k+1} labels replicas in the smallest block (of size m_k). For simplicity we consider here the matrix $q_{\alpha\beta}$.

The overlap of two replicas, $\alpha\cap\beta$, $\alpha=(i_1,\ldots,i_{k+1})$ and $\beta=(j_1,\ldots,j_{k+1})$, is the integer l such that $i_1=j_1,\ldots,i_l=j_l$ but $i_{l+1}=j_{l+1}$. (Pictorially, m_l is the smallest block size containing α and β). A hierarchical matrix $q_{\alpha\beta}$ depends on the replica indices α,β only via their overlap. That is, $q_{\alpha\beta}=q_i$ with $i=\alpha\cap\beta$.

The matrix q can be written as

$$q=q_0\,|\,m_0\,|+(q^1-q^0)\,|\,m_1\,|^{m_0/m_1}+\cdots+(q^k-q^{k-1})\,|\,m_k\,|^{m_0/m_k}-q^k\,|\,m_{k+1}\,|^{m_0}\ . \quad \text{(B1)}$$

$|\,m_i\,|$ denotes the $m_i\times m_i$ matrix having all its entries equal to 1. $|\,m_k\,|(m_0/m_k)$ is a $m_0\times m_0$ matrix obtained by juxtaposing (m_0/m_k) blocks $|\,m_k\,|$. The matrix $|\,m_j\,|$ has (m_j-1) linearly independent eigenvectors $e^\lambda[m_{j-1}]$ having zero eigenvalue, we label them by an index λ. Notice that $\sum_k e_k^\lambda[m_j]=0$.

The eigenvectors of $q_{\alpha\beta}$ can be written as

$$T_{l_1,\ldots,l_{k+1}}^{[\lambda,(i_1i_2,\ldots,i_j),m_j]}=\delta_{l_1i_1}\cdots\delta_{i_jl_j}e_{l_{j+1}}^\lambda[m_j/m_{j+1}]\ . \quad \text{(B2)}$$

Pictorially, this eigenvector is different from zero only inside a particular block of size m_j, labeled by $(i_1,\ldots,i_j)$. Inside this block it depends only on the sub-block index l_{j+1}, as $e_{l_{j+1}}^\lambda[(m_j/m_{j+1})]$. Notice that $|\,m_j\,|^{m_0/m_j}T^{[\,,m_k]}=0$ if $k\geq j$ and $|\,m_j\,|^{m_0/m_j}T^{[\,,m_k]}=m_jT^{[\,,m_k]}$ if $k<j$.

The decomposition (B1) gives

$$qT^{[\,,m_l]}=\epsilon[m_l]T^{[m_l]} \quad \text{(B3)}$$

with

$$\epsilon[m_l]=\sum_{j=l+1}^{k}(q^j-q^{j-1})m_j-q^km_{k+1}\ . \quad \text{(B4)}$$

The eigenvalues $\epsilon[m_l]$, only depend on the "band index" m_j, $\lambda(i_1,i_2,\ldots,i_j)$ serve as additional labels. The degeneracy of the m_j band is given by

$$\deg(m_j)=\frac{m_0}{m_j}\left[\frac{m_j}{m_{j+1}}-1\right]\ . \quad \text{(B5)}$$

Notice that $\sum_{j=0}^k\deg(m_j)=n-1$. This procedure gives all the eigenvectors obeying $\sum e_i=0$. Adding the additional eigenvector $(1,\ldots,1)$ we check that the total number of eigenvectors equals the dimension of the matrix q.

The continuum limit is taken by letting $n\to 0$, $m_i=i/(k+1)$, and $k\to\infty$; $\{m_i\}$ is a partition of $[m_0,m_{k+1}]=[0,1]$. The continuous variable x replaces the discrete index m_i. In this limit the eigenvalues all are given by

$$\epsilon(x)=-\int_x^1 q(y)dy-q(x)x \quad \text{(B6)}$$

and the number of modes parametrized by scales between x and $x+dx$ is given by the continuum version of Eq. (B5), i.e., $-(dx/x^2)n$.

In Sec. V we have used the eigenvalues of the matrix

$$\left[\sum_\sigma q_{\alpha\sigma}^2\delta_{\alpha\beta}-q_{\alpha\beta}^2\right]\ . \quad \text{(B7)}$$

The first term induces a constant shift of the spectrum,

$$\sum_\sigma q_{\alpha\sigma}^2=\sum_{j=0}^k q_j^2(m_j-m_{j+1})\ . \quad \text{(B8)}$$

Replacing q by q^2 in Eq. (B6) and combining it with Eq. (B8) we find $\epsilon_2[m_j]$, the eigenvalues of Eq. (B7).

$$\epsilon_2[m_j]=\sum_{j=0}^{l} q_j^2(m_j-m_{j+1})+q_l^2 m_{l+1} . \tag{B9}$$

The continuum limit $k\to\infty$ $n\to 0$ $m_i\sim x$, [Eq. (B9)], (B10) lead to Eq. (5.32)

$$\epsilon_2(x)=\left[q^2(x)x-\int_0^x q^2(y)dy\right] . \tag{B10}$$

APPENDIX C: SPIN-WAVE PROPAGATORS IN THE SHERRINGTON-KIRKPATRICK AND IN THE BROKEN REPLICA SYMMETRY THEORY

In this appendix we explicitly exhibit the relation between the eigenvectors of hierarchical matrices and the spin-wave propagators.

Restricting ourselves to fluctuations $\delta Q_{\alpha\beta}^{ij}=\epsilon_{ij}^{z}\delta Q_{\alpha\beta}$, the quadratic part of $\delta F(\delta Q)$ [see Eq. (5.35)] has the form

$$\delta L=\tfrac{1}{2}\sum_{\substack{\alpha<\beta\\ \gamma<\delta}} \delta Q_{\alpha\beta}L_{\alpha\beta;\gamma\delta}\delta Q_{\gamma\delta} .$$

Notice that because of the antisymmetry one can define $\delta Q_{\alpha\beta}=-\delta Q_{\beta\alpha}$ for $\alpha>\beta$.

To perform explicit calculations we restrict ourselves to the vicinity of T_c and keep terms up to cubic order in $q_{\alpha\beta}$ in $L_{\alpha\beta;\gamma\delta}$. The resulting L is given by

$$L=(r+k^2)I-6wR+8(2y_1+y_2)I , \tag{C1}$$

where

$$(\alpha\beta\,|\,I\,|\,\alpha'\beta')=q_{\alpha\beta}^2\delta_{\alpha\alpha'}\delta_{\beta\beta'} .$$

The association matrix R is defined by

$$\begin{aligned}(\alpha\beta\,|\,R\,|\,\alpha\gamma)&=q_{\beta\gamma},\quad \gamma\neq\alpha,\beta ,\\ (\alpha\beta\,|\,R\,|\,\gamma\alpha)&=-q_{\beta\gamma},\quad \gamma\neq\alpha,\beta ,\\ (\alpha\beta\,|\,R\,|\,\gamma\delta)&=0,\quad \gamma\neq\delta\neq\alpha\neq\beta .\end{aligned} \tag{C2}$$

The order parameter $q_{\alpha\beta}$ is determined from the equation of state

$$rq_{\alpha\beta}-6w\sum_{\delta=1}^{n} q_{\beta\delta}q_{\delta\alpha}+(2y_1+y_2)q_{\alpha\beta}^3=0 . \tag{C3}$$

From symmetry considerations we know that [see Eq. (5.26)]

$$Q_{\alpha\beta}^{(\lambda)}=q_{\alpha\beta}\frac{[U_\alpha(\lambda)-U_\beta(\lambda)]}{\sqrt{T}\,\epsilon(\lambda)},\quad \lambda=1,\ldots,n-1 \tag{C4}$$

are normalized eigenvectors $[\sum_{\alpha<\beta}(Q_{\alpha\beta})^2=1]$ of L with eigenvalue k^2. In this appendix we will be concerned with the relation between the eigenvectors (C4) and the propagators $\langle\delta Q_{\alpha\beta}\delta Q_{\alpha\beta}\rangle$.

In the Sherrington-Kirkpatrick solution $q_{\alpha\beta}=q$ for $\alpha\neq\beta$ and $q_{\beta\alpha}=0$. The equation of state (C3) reduces to

$$rq-6wq^2(n-2)+8(2y_1+y_2)q^3=0 \tag{C5}$$

and the matrix $(\alpha\beta\,|\,L\,|\,\gamma\delta)$ is easily diagonalized. The eigenvectors dictated by symmetry are

$$q[e_\alpha^\lambda(n)-e_\beta^\lambda(n)]=Q_{\alpha\beta}^{(\lambda)},\quad \lambda=1,\ldots,n-1 , \tag{C6}$$

and have eigenvalue k^2. The vectors $e_\alpha(n)$ were defined in the preceding appendix. L is a symmetric $(n-1)n/2\times n(n-1)/2$ matrix and therefore has $n(n-1)/2$ eigenvectors.

In addition to (C6) there are $(n-1)(n-2)/2$ eigenvectors of the form $E_{\alpha\beta}^{\lambda,\lambda'}=(e_\alpha^\lambda e_\beta^{\lambda'}-e_\beta^\lambda e_\alpha^{\lambda'})$, $\lambda<\lambda'$, $\lambda,\lambda'=1,\ldots,(n-1)$; the corresponding eigenvalue is $k^2+[n\,|r|\,/(2-n)]$ to lowest order in q.

The propagators $\langle\delta Q_{\alpha\beta}\delta Q_{\gamma\delta}\rangle$ are found by inverting the matrix L with the following *Ansätz:*

$$\langle\delta Q_{\alpha\beta}\delta Q_{\alpha\beta}\rangle\equiv G,\quad \alpha\neq\beta ,$$

$$\langle\delta Q_{\alpha\beta}\delta Q_{\alpha\gamma}\rangle\equiv X,\quad \alpha\neq\beta\neq\gamma ,$$

and

$$\langle\delta Q_{\alpha\beta}\delta Q_{\gamma\delta}\rangle\equiv 0,\quad \alpha\neq\beta\neq\gamma\neq\delta .$$

Alternatively, one combines Dyson equation

$$G=G_0+G_0 6w\mathrm{R}G , \tag{C7}$$

$$(\alpha\beta\,|\,G_0\,|\,\gamma\delta)=\frac{\delta_{\alpha\gamma}\delta_{\beta\delta}}{r+k^2+8(2y_1+y_2)q_{\alpha\beta}^2} \tag{C8}$$

and the equation of state

$$rq-6q^2(n-2)w+8(2y_1+y_2)q^3=0 \tag{C9}$$

to obtain, after some algebra,

$$G=\frac{2}{nk^2}+\left[1-\frac{2}{n}\right]\frac{1}{k^2+n\,|r|\,/(2-n)} , \tag{C10}$$

$$X=\frac{1}{n}\frac{1}{k^2}-\frac{1}{n}\left[\frac{1}{k^2+n\,|r|\,/(2-n)}\right] . \tag{C11}$$

The first terms in Eqs. (C10) and (C11) are the spin-wave contributions to G and X, respectively. That is,

$$\frac{2}{n}=\sum_\lambda Q_{\alpha\beta}^{(\lambda)}Q_{\alpha\beta}^{(\lambda)}$$

$$\frac{1}{n}=\sum_\lambda Q_{\alpha\beta}^{(\lambda)}Q_{\alpha\gamma}^{(\lambda)} .$$

This contribution combines with the contribution of the other modes [second term in Eqs. (C10) and (C11)] to give a finite limit as $n\to 0$,

$$G=\frac{1}{k^2}+\frac{|r|}{k^4} ,$$

$$X=\frac{|r|}{2k^4} .$$

The fact that X diverges faster than $(1/k^2)$ indicates that the spin-wave stiffness vanishes in the absence of replica symmetry breaking.

Consider now the first stage of replica symmetry breaking. The matrix L [see Eq. (C1)] has now a richer spectrum. Using the notation of Appendix B we list the eigenvalues and eigenvectors and their degeneracy in Table I.

The last two entries in the table are the spin-wave eigenvectors (C4), that arise from symmetry considerations alone. The eigenvectors listed in the table are not

TABLE I. Eigenvectors, eigenvalues, and degeneracy of the Hessian, at the first stage of replica symmetry breaking. The vector $C_i^{\lambda' l_1}$ is zero when $i=l_1$ and obeys $\sum_i C_i^{\lambda' l_1}=0$. $\lambda_1=(6w/q_1)[(q_1^2-q_0^2)m_1+m_0q_0^2]$. $\lambda_2=6w[nq_0+2(q_1-q_0)m_1]$. $\lambda_3=6wnq_0$. $\lambda_4=6w[m_0q_0+(q_1-q_0)m_1]$.

Unnormalized eigenvectors	Eigenvalue	Degeneracy
$(e^{\lambda}_{j_2}e^{\lambda'}_{i_2}-e^{\lambda}_{i_2}e^{\lambda'}_{j_2})\delta^{l_1}_{i_1}\delta^{l_1}_{j_1}$ $\lambda=1,\ldots,m_1-1;\ \lambda<\lambda'$ $\lambda'=1,\ldots,m_1-1;\ l_1=1,\ldots,(m_0/m_1)$	$k^2+\lambda_1$	$\frac{m_0}{2m_1}(m_1-1)(m_1-2)$
$e^{\lambda}_{i_2}e^{\lambda'}_{j_2}\delta^{l_1}_{i_1}\delta^{l_2}_{j_1}-e^{\lambda}_{j_2}e^{\lambda'}_{i_2}\delta^{l_2}_{i_1}\delta^{l_1}_{j_1}$ $l_1=1,\ldots,(m_0/m_1)$ $l_2=1,\ldots,(m_0/m_1)$ $l_2\le l_1$ $\lambda,\lambda'=1,\ldots,m_1-1,$	$k^2+\lambda_2$	$\frac{1}{2}\frac{m_0}{m_1}\left(\frac{m_0}{m_1}-1\right)(m_1-1)^2$
$q_0\delta^{l_1}_{i_1}\delta^{l_1}_{j_1}(e^{\lambda}_{i_2}-e^{\lambda}_{j_2})+\frac{q_1m_1}{(m_1-n)}[e^{\lambda}_{i_2}\delta^{l_1}_{i_1}(1-\delta^{l_1}_{j_1})$ $-e^{\lambda}_{j_2}\delta^{l_1}_{j_1}(1-\delta^{l_1}_{i_1})],\ l_1=1,\ldots,m_0/m_1,\ \lambda=1,\ldots,(m_1-1)$	$k^2+\lambda_1$	$\frac{m_0}{m_1}(m_1-1)$
$\delta^{l_1}_{i_1}e^{\lambda}_{i_2}C^{\lambda' l_1}_{j_1}(1-\delta^{l_1}_{j_1})-\delta^{l_1}_{j_1}e^{\lambda}_{j_2}C^{\lambda' l_1}_{i_1}(1-\delta^{l_1}_{i_1})$	$k^2+\lambda_4$	$\frac{m_0}{m_1}(m_1-1)\left(\frac{m_0}{m_1}-2\right)$
$e^{\lambda}_{i_1}e^{\lambda'}_{j_1}-e^{\lambda'}_{i_1}e^{\lambda}_{j_1},\ \lambda=1,\ldots,(m_0/m_1)-1$	$k^2+\lambda_3$	$\left(\frac{n}{m_1}-1\right)\left(\frac{n}{m_1}-2\right)\Big/2$
$e^{\lambda}_{i_1}-e^{\lambda}_{j_1},\ \lambda=1,\ldots,m_0/m_1-1$	k^2	$\frac{m_0}{m_1}-1$
$q_1(e^{\lambda}_{i_2}-e^{\lambda}_{j_2})\delta^{l_1}_{i_1}\delta^{l_1}_{j_1}+q_0[\delta^{l_1}_{i_1}e^{\lambda}_{i_2}(1-\delta^{l_1}_{j_1})$ $-\delta^{l_1}_{j_1}e^{\lambda}_{j_2}(1-\delta^{l_1}_{i_1})],\ \lambda=1,\ldots,m_1-1,\ l_1=1,\ldots,m_0/m_1$	k^2	$\frac{m_0}{m_1}(m_1-1)$

normalized to unity. The first ones are constructed from eigenvectors of a hierarchical matrix U^{λ} belonging to the first band which varies over the block scale m_0. The second one is built of eigenvectors varying over a scale m_1. The third entry from below indicates the existence of $(n/m_1-1)(n/m_1-2)/2$ modes that become massless in the limit $n\to 0$. This can be predicted on general grounds since the free energy has to be proportional to n, and therefore, the number of massless modes has to be proportional to n. The spin-wave modes only provide $(n-1)$ massless modes. To evaluate the propagators we use Dyson equation (C7) and the equation of state (C3) which gives,

$$\begin{aligned}0&=rq_1-6w[q_0^2(n-m_1)+q_1^2(m_1-2)]\\&\quad+8(2y_1+y_2)q_1^3\ ,\\0&=rq_0-6w[q_0^2(n-2m_1)+2q_0q_1(m_1-1)]\\&\quad+8(2y_1+y_2)q_0^3\ .\end{aligned}\tag{C12}$$

Inserting the *Ansätz:*

$$\begin{aligned}&\langle\delta Qi_1,i_2;j_1j_2\delta Ql_1l_2;m_1m_2\rangle=0,\quad i_1\neq j_1\neq l_1\neq m_1\\&\langle\delta Qi_1,i_2;i_1j_2\delta Qi_1i_2;i_1j_2\rangle\equiv g_4(=\,=)\ ,\\&\langle\delta Qi_1,i_2;i_1j_2\delta Qi_1i_2;i_1m_2\rangle\equiv g_4(=\,\neq)\ ,\\&\langle\delta Qi_1,i_2;i_1j_2\delta Qi_1i_2;m_1m_2\rangle\equiv g_5\ ,\\&\langle\delta Qi_1,i_2;j_1j_2\delta Qi_1i_2;m_1m_2\rangle\equiv g_7(=)\ ,\\&\langle\delta Qi_1,i_2;j_1j_2\delta Qi_1l_2;m_1m_2\rangle\equiv g_7(\neq)\ ,\\&\langle\delta Qi_1,i_2;j_1j_2\delta Qi_1i_2;j_1m_2\rangle\equiv g_6(=\,\neq)\ ,\\&\langle\delta Qi_1,i_2;j_1j_2\delta Qi_1l_2;j_1m_2\rangle\equiv g_6(\neq\,\neq)\ .\end{aligned}$$

In Dyson equations (C7) and (C8) we obtain after some algebra,

$$g^4(==)+(m_1-2)g^4(=\neq)=\frac{k^2+6wm_1q_1}{k^2(k^2+\lambda_1)},\quad g^4(==)-2g^4(=\neq)=\frac{1}{k^2+\lambda_1},\quad g_5(=)=\frac{\lambda_3}{nk^2(k^2+\lambda_1)}$$

$$m_1[g_7(=)-g_7(\neq)]=\frac{k^2+6w[(n-m_1)q_0/q_1-(n-2m_1)]}{k^2(k^2+\lambda_1)}-\frac{1}{k^2+\lambda_4},\quad g_7(=)+(m_1-1)g_7(\neq)=\frac{\lambda_3}{nk^2(k^2+\lambda_3)}, \tag{C13a}$$

$$\lambda_1=\frac{6w}{q_1}[(q_1^2-q_0^2)m_1+nq_0^2],\quad \lambda_3=6wnq_0,$$
$$\lambda_2=6w[nq_0+2(q_1-q_0)m_1],\ \lambda_4=6w[nq_0+(q_1-q_0)m_1]. \tag{C13b}$$

The short-time stiffness was defined in terms of the propagators $\langle \delta Q i_i i_2 i_1 j_2 \delta Q i_1 i_2 i_1 j_2 \rangle \equiv G^{\alpha\beta,\alpha\beta}$ with $\alpha n\beta=1$. [$g_4(==)$, in the present notation.] Solving (C13) for $g_4(==)$, we find

$$g^4(==)=\frac{2}{m_1}\left[\frac{k^2+6wm_1q_1}{k^2(k^2+\lambda_1)}-\frac{1}{k^2+\lambda_1}\right]+\frac{1}{k^2+\lambda_1},$$
$$g^4(=\neq)=\frac{1}{m_1}\left[\frac{k^2+6wm_1q_1}{k^2(k^2+\lambda_1)}-\frac{1}{k^2+\lambda_1}\right]. \tag{C14}$$

The spin-wave stiffness is defined in terms of the residue of the pole of $g^4(==)$ at $k^2=0$. From Eqs. (C14) and (C13b), we find

$$\lim_{k^2\to 0} g^4(==)k^2=\frac{q_1^2 2}{q_1^2 m_1+q_0^2(m_0-m_1)}. \tag{C15}$$

This equation can be examined in the Sommers limit ($m_1\to\infty$, $(q_1-q_0)m_1\to\Delta'$, $m_0\to 0$ and $q_1\to q$). We find,

$$g^4(==)\sim\frac{q}{\Delta' k^2} \tag{C16}$$

in agreement with the results of the dynamical calculation. Alternatively, (C15) can be computed using the spectral decomposition of $g^4(==)$ in terms of the eigenvectors listed in Table I. Only eigenvectors with k^2 eigenvalue (i.e., spin waves) contribute to the residue (C15). From the explicit expressions in Table I, it is immediate that eigenvectors belonging to the second band which are constant inside an m_1-sized block give zero contribution to the spectral decomposition of $g^4(==)$. Therefore, only spin waves varying over the shortest scale contribute to $g^4(==)$.

To illustrate this point we calculate the short-time stiffness from the eigenvalue decomposition. $E^{\lambda l_1}$ are the unnormalized eigenvectors from the last entry of Table I.

$$\lim_{k^2\to 0} k^2 g^4(==)=\sum_{\lambda,l_1}\frac{E^{\lambda l_1}_{i_1 i_2,i_1 j_2}E^{\lambda l_1}_{i_1 i_2,i_1 j_2}}{||E||^2} \tag{C17}$$

$$||E||^2=\tfrac{1}{2}\sum_{i_1,i_2,j_1,j_2}(E^{\lambda l}_{i_1 i_2 j_1 j_2})^2. \tag{C18}$$

The sums (C17) and (C18) are elementary and we find

$$k^2 g^4(==)=\frac{2q_1^2}{(q_1^2-q_0^2)m_1+nq_0^2} \tag{C19}$$

in agreement with the result of the direct inversion of L, Eq. (C15). This approach is easily generalized to the k stage of replica symmetry breaking.

The relevant eigenvectors are

$$E^{l_1,\ldots,l_k;\lambda}_{i_1,\ldots,i_{k+1};j_1,\ldots,j_{k+1}}=q_{i_1,\ldots,i_{k+1};j_1,\ldots,j_{k+1}}(e^{\lambda}_{i_{k+1}}\delta^{i_k}_{l_1}\cdots\delta^{i_k}_{l_k}-e^{\lambda}_{j_{k+1}}\delta^{j_1}_{l_1}\cdots\delta^{j_k}_{l_k}).$$

Equations (C18) and (C17) generalize to

$$\sum_{\substack{\lambda\\ l_1,\ldots,l_k}}(E^{\lambda;l_1,\ldots,l_k}_{i_1,\ldots,i_{k+1};i_1,\ldots,i_k j_{k+1}})^2=2q_k,$$

$$||E^{\lambda;l_1,\ldots,l_k}||^2=\sum_{j=0}^{k-1}q_j^2(m_j-m_{j+1})+q_k^2 m_k.$$

Equation (C19) generalizes to

$$\lim_{k^2\to 0}k^2 g^4(==)=\frac{2q_k^2}{\left[\sum_{j=0}^{k-1}q_j^2(m_j-m_{j+1})+q_k^2 m_k\right]}. \tag{C20}$$

The continuum limit of Eq. (C20) results in (5.39).

[1]S. F. Edwards and P. W. Anderson, J. Phys. F 5, 965 (1975).
[2]B. I. Halperin and W. M. Saslow, Phys. Rev. B **16**, 2154 (1977).
[3]K. H. Fischer, Phys. Status Solidi B **116**, 357 (1983); **130**, 13 (1985); K. Binder and A. P. Young, Rev. Mod. Phys. (to be published).
[4]K. H. Fischer, Z. Physik B **39**, 37 (1980).
[5]W. M. Saslow, Phys. Rev. B **22**, 1174 (1980); Physica **107B**, 769 (1981); C. L. Henley, H. Sompolinsky, and B. I. Halperin,

Phys. Rev. B **25**, 5849 (1982).
[6]L. R. Walker and R. E. Walstedt, Phys. Rev. B **22**, 3816 (1980).
[7]R. E. Walstedt, Phys. Rev. B **24**, 1524 (1981).
[8]R. B. Gronzka and M. A. Moore, J. Phys. C **16**, 1109 (1983).
[9]P. Reed, J. Phys. C **12**, L475 (1979).
[10]J. R. Banavar and M. Cieplak, Phys. Rev. B **26**, 2662 (1982).
[11]D. L. Huber, W. Y. Ching, and M. Fibich, J. Phys. C **12**, 3535 (1979).
[12]W. Y. Ching, D. L. Huber, and K. M. Leung, Phys. Rev. B **23**, 6126 (1981); U. Krey, Z. Phys. B **38**, 243 (1980); **42**, 231 (1981).
[13]H. Maletta, W. Zinn, H. Scheuer, and S. M. Shapiro, J. Appl. Phys. **52**, 1735 (1981); S. M. Shapiro, C. R. Fincher, Jr., A. C. Palumbo, and R. D. Parks, Phys. Rev. B **24**, 6661 (1981).
[14]A. P. Murani, Phys. Rev. B **28**, 432 (1983); G. Aeppli, S. M. Shapiro, R. J. Birgeneau, and H. S. Chen, Phys. Rev. B **29**, 2589 (1984); B. Hennion, M. Hennion, F. Hippert, and A. P. Murani, J. Phys. F **14**, 489 (1984).
[15]I. A. Campbell, Phys. Rev. Lett. **47**, 1473 (1981).
[16]H. Sompolinsky, G. Kotliar, and A. Zippelius, Phys. Rev. Lett. **52**, 392 (1984).
[17]G. Parisi, Phys. Rev. Lett. **43**, 1754 (1979); J. Phys. A **13**, L115 (1980); **13**, 1887 (1980).
[18]H. Sompolinsky, Phys. Rev. Lett. **47**, 935 (1981).
[19]G. Parisi, Phys. Rev. Lett. **50**, 1946 (1983).
[20]D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **35**, 1792 (1975); S. Kirkpatrick and D. Sherrington, Phys. Rev. B **17**, 4384 (1978).
[21]M. Mézard, G. Parisi, N. Sourlas, G. Toulouse, and M. Virasoro, Phys. Rev. Lett. **52**, 1156 (1984); J. Phys. (Paris) **45**, 843 (1984).
[22]C. De Dominicis, Phys. Rev. B **18**, 4913 (1978).
[23]H. Sompolinsky and A. Zippelius, Phys. Rev. B **25**, 6860 (1982).
[24]H. Sompolinsky and A. Zippelius, J. Phys. C **15**, L1059 (1982).
[25]J. R. L. de Almeida and D. J. Thouless, J. Phys. A **11**, 983 (1978).
[26]H. J. Sommers, Z. Phys. B **31**, 301 (1978); **32**, 173 (1979).
[27]C. De Dominicis and T. Garel, J. Phys. (Paris) Lett. **40**, L576 (1979).
[28]H. Sompolinsky and A. Zippelius, Phys. Rev. Lett. **50**, 1297 (1983).
[29]D. M. Cragg, D. Sherrington, and M. Gabay, Phys. Rev. Lett. **49**, 158 (1982); M. Gabay, T. Garel, and C. De Dominicis, J. Phys. C **15**, 7165 (1982).
[30]A. P. Young, A. J. Bray, and M. A. Moore, J. Phys. C **17**, L149 (1984).
[31]D. S. Fisher and H. Sompolinsky, Phys. Rev. Lett. **54**, 1063 (1985).
[32]R. G. Palmer and C. M. Pond, J. Phys. F **9**, 1451 (1979).
[33]A. J. Bray and M. A. Moore, J. Phys. C **14**, 2629 (1981).