Quantum Hall effect with realistic boundary conditions

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Earlier theories of the quantum Hall effect depend on boundary conditions (cylindrical, toroidal, etc.) which are very different from that of the experimental devices (essentially striplike). To remove this discrepancy, we show that the Hall conductivity has an exponential locality property whenever the Fermi energy lies between the levels of the bulk extended states, and that this is true in spite of the edge states at the Fermi energy. We describe in detail how this locality property can be used to adapt the gauge-symmetry argument of Laughlin and the topological-invariant approach of Niu, Thouless, and Wu to conditions that are much closer to real experimental ones. The resulting conclusion is that the boundary correction to the quantization of the Hall conductance is exponentially small when the system size is large compared with a microscopic length (typically the magnetic length).

I. INTRODUCTION

For the most part the theory of the integer quantum Hall effect[1] (QHE) is well understood, and there is no mystery about why the Hall conductance is a multiple of e^2/h with such high accuracy. On the other hand there seems to be a gap between the theoretical derivations of the QHE and the experimental measurements, in that the theories seem, for the most part, to rely on special forms of boundary conditions which do not correspond to the actual conditions used in experiments. In this paper we examine these boundary conditions more closely and show how the gaps between theory and experiment can be closed.

Figure 1 illustrates a typical device used to make a four-terminal measurement of the Hall voltage for an inversion layer in a strong magnetic field. A strip of inversion layer [essentially a two-dimensional (2D) electron layer] is connected to a source and a drain (three dimensional) at the two ends. A strong and nearly uniform magnetic field is applied in the direction normal to the inversion layer. A current I is passed through the strip. A pair of voltage probes on opposite sides of the strip, also part of the inversion layer, allow the voltage V_H across the strip to be measured with a voltmeter attached to the probes. A second pair of probes further along the side enables the longitudinal component of the resistance to be measured simultaneously. The Hall conductance is

$$\sigma_H = I/V_H \ . \tag{1}$$

This is found to be quantized in multiples (integer for the integer QHE, exact fractions with quite small denominators for the fractional QHE) of e^2/h whenever the longitudinal component of the resistance vanishes. Energy dissipation does occur near the ends of the strip, and, for the plateaus of the QHE, the voltage drop measured in a two-terminal measurement is very close to V_H, as can be seen from the arguments of Kawaji.[2] It is, however, an experimental fact that the disturbance of the two ends has little influence on the high precision of the measured quantization of σ_H, so long as the strip is long enough.[3] This immediately suggests that the Hall conductivity has a locality property.

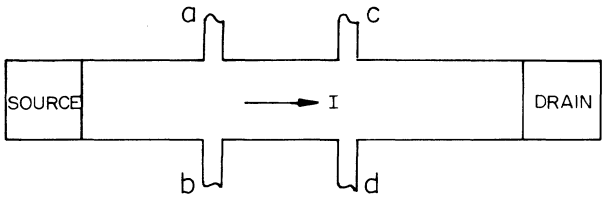

FIG. 1. Illustration of a typical device (strip geometry) used to make a four-terminal measurement of the QHE. The magnetic field is in the normal direction. A current I is passed through the strip. Between a and b the Hall voltage is measured and between a and c the longitudinal voltage is measured.

Theoretical analyses are all agreed that there should be a precise quantization of the Hall conductance at sufficiently low temperatures whenever the Fermi energy lies in a gap of the density of states, or in an energy region where there are electron states, but these are all localized. Such an energy gap or mobility gap will be signaled by the vanishing of the longitudinal component of resistance (or, equivalently, of the longitudinal component of conductance). Derivations of this quantization are, however, based on geometries very different from typical experimental situations. For example, the very important argument of Laughlin[4] is based on a cylindrical geometry for the device, with the current flowing round the cylinder. Niu *et al.*[5] assumed that the device was a torus. Halpe-

rin[6] gave a modified version of Laughlin's argument in which an annulus was used, and particular attention was paid to the currents flowing round the circles bounding the annulus. There is also a set of arguments based on perturbation theory[7–9] which show that in the bulk two-dimensional system the two-dimensional conductivity, the ratio of the current per unit length to the transverse electric field, is quantized, and is undisturbed by perturbations due to disorder or interactions; integration of this relation leads to the conclusion that the Hall conductance is indeed quantized away from the edges. All these arguments use boundary conditions that are very different from those used in real systems.

The key observation is that, when the Fermi energy is in a mobility gap, the conductivity, which is purely transverse, depends only on the local environment, up to exponentially small terms. The locality property of the transverse conductivity σ_{xy} can best be analyzed through the Kubo formula expressed in terms of Green functions.[5,9] The conductivity is expressed as an integral over the energy parameter of the Green function taken round a contour in the complex plane which cuts the real axis only at the Fermi energy and at some energy below the spectrum of the system. If the Fermi energy is in a mobility gap then the Green function is an exponentially decreasing function of distance for all values of the energy on the contour of integration. We have used this property in earlier work both to show that there are no perturbative corrections to the Hall conductance to any power of the electric field,[10] and to justify averaging over generalized periodic boundary conditions.[5]

On a strip geometry, there are states which damp exponentially away from the edges, but are extended along the edges. The energies of these edge states fill the gap or mobility gap between the levels of the extended states in the bulk. We are thus forced to consider the effects of the edge states on the long-distance behavior of the Green function. Our main result is that the Green function behaves in a semilocal manner when its two position variables are confined in a narrow region along an edge, that is, it is only extended in one direction along the edge, but is exponentially localized in the other direction. Since the Kubo formula involves a product of two or three (depending on the specific expression) Green functions, a full exponential locality of the Hall conductivity is then ensured.

As we will show in the text, the locality of the Hall conductivity, together with a current conservation law, leads to an important conclusion about the finite-size correction to the quantization of the Hall conductance. Under quite general conditions, the correction will be exponentially small when the linear size of the system is large compared with a microscopic length which is typically the magnetic length. This estimate gives a much smaller bound to the correction than those given earlier.[11]

In Sec. II we present a detailed calculation of the Green function for a free-electron system in a long strip bounded by two edges. In Sec. III we derive a version of the Kubo formula which is particularly suitable for the locality analysis. At the end of this section, a primary argument to justify the use of the cylinder geometry is also given. In Sec. IV we use an argument of adiabatic charge transport to justify the use of the torus geometry. In Sec. V, we describe in detail how Laughlin's argument can be adapted to conditions that are much closer to real experimental conditions. In Sec. VI we examine the expression for the Hall conductance as a topological invariant[5,12,13] and show how this is also applicable under conditions more realistic than those used in earlier discussions. For the most part we restrict the discussion to the integer QHE, but where the argument can be extended to cover the fractional QHE we mention the fact.

II. THE SEMILOCALITY OF THE GREEN FUNCTION

Consider a 2D free-electron gas on a strip under a uniform magnetic field $\mathbf{B}=B\hat{\mathbf{z}}$ in the normal direction of the surface. The strip is assumed to be infinitely long in the y direction and to have edges along $x=0$ and $x=L_x$. It is convenient to use the Landau gauge $\mathbf{A}=(0,Bx,0)$ so that the Schrödinger equation has the separable form

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{1}{2m}\left[-i\hbar\frac{\partial}{\partial y}+eBx\right]^2\right]\psi(x,y) = E\psi(x,y)\ , \quad (2)$$

which is to be solved under the boundary conditions

$$\psi(x=0)=\psi(x=L_x)=0\ . \quad (3)$$

The wave functions can be written in the product form $e^{-iky}\psi_n(x,k)$, where $\psi_n(x,k)$ is an eigenstate of the one-dimensional (1D) problem

$$\left[\frac{d^2}{dx^2}+\nu+\tfrac{1}{2}-\tfrac{1}{4}(k-x)^2\right]\psi(x,k)=0\ . \quad (4)$$

In the last equation we have used $\sqrt{\hbar/2eB}$ as the length scale in the x direction and $\sqrt{2\hbar/eB}$ in the y direction, and have expressed E as $(\nu+\frac{1}{2})(\hbar eB/m)$.

The Green function in the spectral representation is

$$G(\mathbf{r},\mathbf{r}';z)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ik(y'-y)}\sum_{n=0}^{\infty}\frac{\psi_n(x,k)\psi_n(x',k)}{z-E_n(k)} = \int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ik(y'-y)}g(x,x';\nu,k)\ , \quad (5)$$

where g is the Green function of the homogeneous equation (4). In the rest of this section we will concentrate our attention on the edge along $x=0$, and forget about the other by formally taking $L_x\to+\infty$. The reduced Green function g then satisfies the following boundary conditions:

$$g(0,x';\nu,k)=0,\quad g(+\infty,x';\nu,k)=0 \ \text{ for } x'>0\ . \quad (6)$$

As a 1D problem, we can write for $\nu\neq 0,1,2,3\ldots$ (away from the bulk Landau levels)

$$g(x,x';\nu,k)=\frac{1}{W}[D_\nu(x-k)f(x',k)\theta(x-x') + f(x,k)D_\nu(x'-k)\theta(x'-x)]\ , \quad (7)$$

where $D_\nu(x)$ is the solution of Eq. (4) (with $k=0$) that vanishes as $x\to+\infty$. The function $\theta(x)$ in Eq. (7) is just the unit step function, while the other functions are given in terms of $D_\nu(x)$ by

$$f(x,k)\equiv D_\nu(k)D_\nu(x-k)-D_\nu(-k)D_\nu(k-x)\ ,$$
$$W=-2D_\nu(-k)D_\nu(0)D'_\nu(0)\ . \tag{8}$$

The function $D_\nu(x)$ is called the parabolic cylinder function in the literature of mathematical physics, and some of its important properties are listed in the Appendix.

To be explicit, let us assume $x>x'>0$, then we have

$$g(x,x';\nu,k)=\frac{D_\nu(x-k)D_\nu(k-x')}{2D_\nu(0)D'_\nu(0)}-\frac{1}{2D_\nu(0)D'_\nu(0)}\frac{D_\nu(x-k)D_\nu(x'-k)D_\nu(k)}{D_\nu(-k)}\ . \tag{9}$$

The first term on the right-hand side of the above equation is the reduced Green function in the absence of the boundary, and when inserted in the expression (5), produces a Gaussian-like localized behavior of $G(\mathbf{r},\mathbf{r}';z)$ in the separation $(\mathbf{r}-\mathbf{r}')$ in all directions. Thus, if there is an extended behavior of $G(\mathbf{r},\mathbf{r}';t)$ it must come from the contribution of the second term which is induced by the boundary condition at $x=0$. It is therefore sufficient to consider the following object:

$$\Delta(\mathbf{r},\mathbf{r}';\nu)\equiv\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ik(y'-y)}\frac{D_\nu(x-k)D_\nu(x'-k)D_\nu(k)}{D_\nu(-k)}\ . \tag{10}$$

First, consider the case that ν is real negative or ν is away from the real axis of the energy plane. Then $D_\nu(-k)$ has no zero in a finite strip, I_k, parallel to the real axis in the complex k plane. Since $D_\nu(k)$ is analytic in the k plane (excluding ∞), the integrand in Eq. (10) is analytic in the strip I_k. Within I_k, the function $D_\nu(k)$ has the following asymptotic behavior:

$$k^\nu e^{-(1/4)k^2}\ \text{as}\ \mathrm{Re}(k)\to+\infty\ ,$$
$$k^{-\nu-1}e^{(1/4)k^2}\ \text{as}\ \mathrm{Re}(k)\to-\infty\ , \tag{11}$$

so that the integrand in (10) behaves (in absolute value) as

$$\frac{1}{|k|}e^{-(1/2)|k|(x+x')},\quad |k|\to\infty \tag{12}$$

for fixed x and x' (which are positive). Thus, we can replace the contour of integration of (10) by a line below or above the real axis to make $\Delta(\mathbf{r},\mathbf{r}';\nu)$ an apparent localized function of $y-y'$. The localization of Δ in $(y-y')$ is therefore exponential.

Now suppose ν is real positive and is between two bulk Landau levels n and $n+1$. In this case, the function $D_\nu(-k)$ has $n+1$ zeros on the real axis of the k plane, corresponding to the $n+1$ edge states at energy ν. We denote these zeros as k_j, $j=0,1,2,\ldots,n$. If we add to ν an infinitesimal imaginary part $i\epsilon$, then these zeros become

$$k_j+\frac{\partial k_j}{\partial\nu}i\epsilon\ , \tag{13}$$

Since the energy levels decrease as k increases [see Eq. (4)], the derivative is real negative. Thus, when ϵ is positive the zeros move just below the real axis. From the analytic property of $D_\nu(-k)$ there is no other zero in a finite strip I_k parallel to the real axis. The asymptotic behavior of $\Delta(\mathbf{r},\mathbf{r}';\nu+i\epsilon)$ as a function of $(y-y')$ can be readily found as follows.

Let us fix $\epsilon>0$. For $y-y'<0$ we replace the contour of integration in (10) by a line which is a finite distance above the real axis (but still within I_k). The new integral equals the original one because there are no poles of the integrand in the region between the new and the old contour, and because the asymptotic behavior of the integrand still holds as in Eq. (12). It is therefore clear that $\Delta(\mathbf{r},\mathbf{r}';\nu+i\epsilon)$ is exponentially small for $y\ll y'$, for k has a finite positive imaginary part along the new contour. In the other case, $y-y'>0$, we replace the contour of integration in (10) by a line below the real axis. The integral along the new contour gives an exponentially small contribution to $\Delta(\mathbf{r},\mathbf{r}'\nu+i\epsilon)$ for $y\gg y'$. In addition to this, we must consider now the contributions from the poles of the integrand [the zeros of $D_\nu(-k)$]. The contributions from the poles are plane waves in $(y-y')$, which make $\Delta(\mathbf{r},\mathbf{r}';\nu+i\epsilon)$ extended for $y\gg y'$.

Thus, the retarded Green function $G^+(\mathbf{r},\mathbf{r}';z)$ for z between two bulk Landau levels behaves quite differently for $y\ll y'$ and $y\gg y'$. In the former case it is exponentially small, while in the latter case it is plane-wave-like. This particular behavior of the Green function can in fact be intuitively understood from the following classical picture. A classical electron circles around anticlockwise in the $x-y$ plane under a magnetic field in the z direction. When its orbit hits the wall at $x=0$, the electron bounces and travels to the positive y direction along the edge. It is this one way traveling behavior of the electron that makes the Green function a semilocalized function of $y-y'$.

The advanced Green function G^- also has the semilocality property but in the opposite sense. It is localized for $y\gg y'$ and extended for $y\ll y'$. This is not surprising, for changing ϵ from positive to negative effectively reverses the time.

So far we have only considered the behavior of the Green function in the y direction (along the edge). The behavior in the x direction can also be studied through the integral in Eq. (10), with the properties of the parabolic cylinder function $D_\nu(k)$. The results are quite simple. For z between two bulk Landau levels, the Green function (either retarded or advanced) is an exponentially localized function of the separation $|x-x'|$.

Before closing this section, we would like to make some remarks about the results obtained so far. First, the localization lengths in the behavior of the Green functions should be of order unity in the dimensionless variables, if the energy z is not very close to the bulk Landau levels. When we put the length scales back into the equations, we should find that the localization lengths are of order of the magnetic length $\sqrt{\hbar/eB}$. Second, we expect that the general features of the Green functions should hold in the

presence of a disordered potential. One primary reason for this is that the nature of the spectrum for the extended states including the extended edge states are quite the same with or without disorder.[6,14] With disorder, the localization length may however be different from the magnetic length.

III. LOCALITY OF THE HALL CONDUCTIVITY

The Kubo formula for the Hall conductivity has been the basis for many theories of the quantum Hall effect. In the following we give a brief derivation of this formula in a form which is particularly suitable for the analysis of its locality. The arguments used in the derivation can be largely found from a paper of Streda and Smrcka.[15]

In the limit of linear response, a uniform electric field E in the x direction produces a perturbation ρ_1 to the density matrix, given by

$$\rho_1 = -ieE \int_0^\infty dt\, e^{-\epsilon t} e^{-iHt}[x,\rho_0]e^{iHt} , \tag{14}$$

where ρ_0 is the unperturbed density matrix (namely the Fermi-Dirac distribution operator), and ϵ is a positive infinitesimal. The y component of the electric current density at $\mathbf{r}=\mathbf{r}_0$ is then given by

$$J_y(\mathbf{r}_0) = ie^2 E \int_0^\infty dt\, e^{-\epsilon t} \mathrm{Tr}(j_y e^{-iHt}[x,\rho_0]e^{iHt}) , \tag{15}$$

where j_y is the current density operator given in terms of the velocity operator V_y by

$$j_y = \tfrac{1}{2}[V_y \delta(\mathbf{r}-\mathbf{r}_0) + \delta(\mathbf{r}-\mathbf{r}_0)V_y] . \tag{16}$$

The Hall conductivity at $\mathbf{r}_0$ is simply

$$\sigma_H(\mathbf{r}_0) = ie^2 \int_0^\infty dt\, e^{-\epsilon t} \mathrm{Tr}(j_y e^{-iHt}[x,\rho_0]e^{iHt}) . \tag{17}$$

Now we can expand the trace operation in the energy eigenspace and carry out the time integration to yield

$$\begin{aligned}\sigma_H(\mathbf{r}_0) &= e^2 \sum_{m,n} \rho_0(E_m) \left[\frac{(j_y)_{mn} X_{nm}}{E_m - E_n + i\epsilon} + \text{c.c.} \right] \\ &= e^2 \int_{-\infty}^{\infty} d\eta\, \rho_0(\eta) \\ &\quad \times \sum_{m,n} \delta(\eta - E_m) \left[\frac{(j_y)_{mn} X_{nm}}{\eta + i\epsilon - E_n} + \text{c.c.} \right] ,\end{aligned} \tag{18}$$

where $\rho_0(\eta)$ is now the Fermi-Dirac distribution function. Using the definition of the Green functions

$$G^{\pm}(\eta) = \frac{1}{\eta \pm i\epsilon - H} \tag{19}$$

and the formal relation

$$\delta(\eta - H) = \frac{i}{2\pi}(G^+ - G^-) , \tag{20}$$

we can rewrite Eq. (18) as

$$\begin{aligned}\sigma_H(\mathbf{r}_0) &= \frac{ie^2}{2\pi} \int_{-\infty}^{\infty} d\eta\, \rho_0(\eta)\, \mathrm{Tr}[(G^+ - G^-)(j_y G^+ x + x G^- j_y)] \\ &= \frac{ie^2}{2\pi} \int_{-\infty}^{\infty} d\eta\, \rho_0(\eta)\, \mathrm{Tr}(G^+ j_y G^+ x - G^- x G^- j_y) \\ &= \frac{-e^2}{2\pi} \int_{-\infty}^{\infty} d\eta\, \rho_0'(\eta) \int_{-\infty}^{\eta} d\eta'\, \mathrm{Tr}[G^+(\eta') j_y G^+(\eta') x - G^-(\eta') j_y G^-(\eta') x] .\end{aligned} \tag{21}$$

The integral over η' in the last expression can be turned into a contour integral in the complex energy plane with the result

$$\begin{aligned}\sigma_H(\mathbf{r}_0) &= \frac{ie^2}{2\pi} \int_{-\infty}^{\infty} d\eta\, \rho_0'(\eta) \int_{C(\eta)} dz\, \mathrm{Tr}[G(z) j_y G(z) x] \\ &= \frac{ie^2}{2\pi} \int_{-\infty}^{\infty} d\eta\, \rho_0'(\eta) \int_{C(\eta)} dz \int\int d\mathbf{r}\, d\mathbf{r}'\, G(\mathbf{r},\mathbf{r}',z) j_y G(\mathbf{r}',\mathbf{r};z) x ,\end{aligned} \tag{22}$$

where in the last step the double coordinate integration represents the trace operation and the current density operator j_y acts on the $\mathbf{r}'$ variable of the second Green function. The contour $C(\eta)$ in the energy integration of (22) is now in the complex energy plane surrounding the energy spectrum below η, with the infinitesimal segment $(\eta - i\epsilon, \eta + i\epsilon)$ being omitted.

Equation (22) is the Kubo formula that has the desired form for the locality analysis. At low enough temperature ($k_B T \ll \hbar\omega_c$), the quantity $\rho_0'(\eta)$ is highly peaked about the chemical potential which we assume to lie between two bulk Landau levels or the levels of extended states in the bulk. In the absence of boundaries, the contour $C(\eta)$ is away from the spectrum of extended states, so that the Green functions are exponentially localized in the separation $|\mathbf{r}-\mathbf{r}'|$. Because of the δ function $\delta(\mathbf{r}'-\mathbf{r}_0)$ contained in the operator j_y, the contribution to $\sigma_H(\mathbf{r}_0)$ only comes from a neighborhood of $\mathbf{r}_0$. This neighborhood has a typical linear scale of a magnetic length, corresponding to the localization length of the Green functions. Now, suppose our system is in a strip bounded by two parallel edges. In this case, the contour $C(\eta)$ can come close to the spectrum of the edge states near the Fermi energy. These edge states are extended along the edges, which make the Green functions also extended along the edges. According to the analysis of the previous sections, the extendedness of the Green functions is, however, only in one way near each edge. Together with the fact that the

Green functions are still localized in the direction perpendicular to the edges, it is now easy to see that the product $G(\mathbf{r},\mathbf{r}',z)j_yG(\mathbf{r}',\mathbf{r},z)$ is in fact localized in all directions in the separation $(\mathbf{r}-\mathbf{r}')$. Again, because of the δ function contained in the operator j_y, the contribution to $\sigma_H(\mathbf{r}_0)$ only comes from a small neighborhood of $\mathbf{r}_0$. This completes our locality analysis of the Hall conductivity.

In the usual experiments of the quantum Hall effect, the Hall voltage is measured across a section (say, at $y=y_0$) in the middle of the strip (see Fig. 1). The Hall conductance can thus be expressed by

$$\sigma_H(y_0)=\frac{1}{L_x}\int_0^{L_x}dx_0\sigma_H(\mathbf{r}_0)\ , \tag{23}$$

where L_x is the width of the strip at y_0. All the complications due to the two ends, where the strip is connected to the source and the drain, die out exponentially at y_0 if the ends are sufficiently far away from the voltage probes. This explains why the Hall conductance can be so accurately determined in spite of the shorting effect at the two ends.

Owing to the exponential locality of the Hall conductivity, we can use whatever boundary conditions at the two ends which is convenient for a theoretical analysis. In particular, we can use a periodic boundary condition which joins the two ends making the strip into a cylinder. If this is done, the Hall conductance can be related to the coherent response of the extended states with respect to the change of a magnetic flux through the center of the cylinder as was done by Laughlin.[4] The gauge symmetry of the system then leads to the quantization of the Hall conductance averaged over a flux quantum. At this step, one can see very clearly the important role played by the locality of the Hall conductivity in establishing the relevance of Laughlin's theory to the experiments.

The average procedure over the flux quantum was not justified in Laughlin's paper. In a paper of Niu *et al.*,[5] an argument is given that explains why the Hall conductance can be replaced by its average. The theoretical basis of this argument is again the locality of the Hall conductivity, although the effect of the edge states (which is now shown to be harmless) was not taken into account by us.

In Sec. V we will give a more detailed analysis of Laughlin's theory.

IV. THE TORUS GEOMETRY

It is convenient to start with the cylinder geometry, the use of which has primarily been justified in Sec. III. Let us continue to use (x,y) as the coordinates for the cylinder surface, where x is confined into the interval $[0,L_x]$ while y and $y+L_y$ are identified as the same. Suppose we have a magnetic flux through the center of the cylinder. We denote $C(x_0,\phi)$ as the induced charge transport through the circle at $x=x_0$ when this flux is increased adiabatically from zero to ϕ. Following Laughlin's idea we equate the Hall conductance with the flux derivative of the adiabatic charge transport,

$$\sigma_H(y_0)=\tilde{C}(x_0)\equiv\left.\frac{\partial C(x_0,\phi)}{\partial\phi}\right|_{\phi=0}\ , \tag{24}$$

where $\phi_H(y_0)$ is defined in Eq. (23), and x_0 may be taken as $L_x/2$. Later we will give an expression for $\tilde{C}(x_0)$, from which we can show that $\tilde{C}(x_0)$ has exponentially small contributions from the region $|x-x_0|\gg l$, where l is the magnetic length. Thus, if $L_x\gg l$, then we can change the boundary conditions at $x=0$ and $x=L_x$ with exponentially small error introduced into $\tilde{C}(x_0)$ and hence into $\sigma_H(y_0)$ by Eq. (24). In particular, we can use a periodic boundary condition which joins the edges of the cylinder together to form a torus. Thus, if we can prove Eq. (24) and establish the exponential locality of $\tilde{C}(x_0)$, then we can justify the use of the torus geometry.

Let us first derive a formula for $\tilde{C}(x_0)$ to show its locality. Through the dependence on the flux, the Hamiltonian varies adiabatically in time. To the first order in $\dot{\phi}$, the perturbation ρ_1 to the density matrix is determined by the equation of motion

$$i\hbar\dot{\rho}_0+[\rho_1,H]+i\epsilon\rho_1=0\ , \tag{25}$$

where $\dot{\rho}_0$ is the time derivative of the instantaneous density matrix corresponding to the Fermi-Dirac distribution at fixed flux, and ϵ is an positive infinitesimal. Taking the matrix element of Eq. (25) between two orthonormal instantaneous eigenstates of the Hamiltonian, we obtain

$$(\rho_1)_{mn}=i\hbar\frac{\langle\dot{m}\mid n\rangle\rho_0(E_n)+\langle m\mid\dot{n}\rangle\rho_0(E_m)}{E_n-E_m+i\epsilon}\ , \tag{26}$$

where we have used the identity

$$\langle m\mid\dot{\rho}_0\mid n\rangle=-\langle\dot{m}\mid n\rangle\rho_0(E_n)-\langle m\mid\dot{n}\rangle\rho_0(E_m)\ , \tag{27}$$

because ρ_0 is diagonal in the basis of the eigenstates. The induced current through the circle at $x=x_0$ is given by the trace of the product of ρ_1 and the current operator $i_x\equiv(-e/2)[V_x\delta(x-x_0)+\delta(x-x_0)V_x]$, that is

$$\begin{aligned}I_x(x_0)&=\sum_{m,n}\frac{i\hbar[\langle\dot{m}\mid n\rangle\rho_0(E_n)+\langle m\mid\dot{n}\rangle\rho_0(E_m)]}{E_n-E_m+i\epsilon}(i_x)_{nm}\\&=i\hbar\int d\eta\,\rho_0(\eta)\sum_n\delta(\eta-E_n)(\langle\dot{n}\mid G^-i_x\mid n\rangle\\&\qquad-\langle n\mid i_xG^+\mid\dot{n}\rangle)\ .\end{aligned} \tag{28}$$

The adiabatic time derivative of an eigenstate can be written as

$$\begin{aligned}P_n\mid\dot{n}\rangle&=\frac{1}{E_n-H}P_n\dot{H}\mid n\rangle\\&=\tfrac{1}{2}[G^+(E_n)+G^-(E_n)]\dot{H}\mid n\rangle\ ,\end{aligned} \tag{29}$$

where P_n is the operator that projects off the state $\mid n\rangle$. Using the above result and the formal relation (20), Eq. (28) can be rewritten as

$$I_x(x_0)=\frac{\hbar}{4\pi}\int d\eta\,\rho_0(\eta)\,\mathrm{Tr}[i_x G^+(G^++G^-)\dot{H}(G^+-G^-)+\text{c.c.}]$$
$$=\frac{\hbar}{4\pi}\int d\eta\,\rho_0(\eta)\,\mathrm{Tr}\left[i_x\left[G^+G^+\dot{H}G^++G^-\dot{H}G^-G^-+\frac{\partial}{\partial\eta}(G^+\dot{H}G^-)\right]\right], \tag{30}$$

where in the last step we have used the following identities:

$$[G^+,G^-]=0,\quad G\dot{H}G=\dot{G},\quad GG=-\frac{\partial}{\partial\eta}G\ . \tag{31}$$

The Hamiltonian depends on time through the y component of the canonical momentum which contains an additional term of $e\phi/L_y$. As a result of this, we have $\dot{H}=e\dot{\phi}V_y/L_y$. The charge transport $C(x_0,\phi)$ is just the time integral of $I_x(x_0)$, so that the quantity $\tilde{C}(x_0)$ as defined in Eq. (24) is given by

$$\tilde{C}(x_0)=\frac{\hbar e}{4\pi L_y}\int d\eta\,\rho_0(\eta)\,\mathrm{Tr}\left[i_x\left[G^+G^+V_yG^++G^-V_yG^-G^-+\frac{\partial}{\partial\eta}(G^+V_yG^-)\right]\right], \tag{32}$$

where everything is now evaluated at $\phi=0$. To reveal the locality of $\tilde{C}(x_0)$, we integrate the right-hand side of the above equation by parts to get the result

$$\tilde{C}(x_0)=\frac{-\hbar e}{4\pi L_y}\int d\eta\,\rho_0'(\eta)\left[\mathrm{Tr}(i_xG^+V_yG^-)+\int_{-\infty}^{\eta}d\eta'\,\mathrm{Tr}[i_x(G^+G^+V_yG^++G^-V_yG^-G^-)]\right]. \tag{33}$$

At low enough temperature ($k_BT\ll\hbar\omega_c$), $\rho'(\eta)$ is highly peaked about the chemical potential which lies between the levels of the extended states in the bulk. For each term in the integral over η', the contour of integration can be deformed into the complex energy plane, so that the only place that η' is close to the spectrum is at the Fermi energy. The Green functions are then exponentially local functions of the separation in the x directions as were shown in Sec. II. As a result of the locality of the Green functions and the δ functions $\delta(x-x_0)$ contained in the operator i_x, the quantity $\tilde{C}(x_0)$ has exponentially small contributions from the region $|x-x_0|\gg l$, where l is the magnetic length.

Having shown the locality of $\tilde{C}(x_0)$, we now proceed to establish the identity

$$\sigma_H(y_0)=\tilde{C}(x_0)\ , \tag{34}$$

which was stated in Eq. (24) and is given here for convenience. Since the two sides of the above equation depend on different variables, we must show that $\tilde{C}(x_0)$ is independent of x_0 and that $\sigma_H(y_0)$ is independent of y_0. The reason for the constancy of $\tilde{C}(x_0)$ can be found from the current conservation law. It is important to notice that we only need the adiabatic current $I_x(x_0)$ in a vanishingly small neighborhood of the flux to evaluate the quantity $\tilde{C}(x_0)$. In this neighborhood the effect of charge accumulation or decumulation due to the edges can be neglected. The reason for the constancy of $\sigma_H(y_0)$ can be found in a similar way. We are thus left to show the identity (34) when both sides are averaged over the position variables.

From a simple electrodynamic argument, the adiabatic current $I_x(x_0)$ is in fact the Hall current under the electromotive force generated by the changing flux. The averaged quantity $\langle\tilde{C}\rangle\equiv(1/L_x)\int dx_0\tilde{C}(x_0)$ is just the Hall conductivity σ_{xy} (with a minus sign) over the whole surface. On the other hand, $\langle\sigma_H\rangle\equiv(1/L_y)\int dy_0\sigma_H(y_0)$ is the average of the Hall conductivity σ_{yx} over the surface. The Onsager relation $\sigma_{xy}=-\sigma_{yx}$ immediately leads to the identity $\langle\sigma_H\rangle=\langle\tilde{C}\rangle$, which is what we wanted to prove.

To summarize, we started with a cylinder geometry for the Hall system, and reexpressed the Hall conductance as the flux derivative of the adiabatic charge transport from one edge to the other. The new expression can be justified by an electrodynamic argument originally used by Laughlin, or by mathematical manipulations on the Kubo formula. Since the charge transport can be evaluated from a circle in the middle of the cylinder, we can use the locality property of the Green function to show its insensitivity to the boundary conditions at the two edges. The flux derivative of the charge transport is calculated in a vanishing neighborhood of the flux, so it is a property of the ground state of the system at zero flux. Then we can join the two edges to turn the cylinder into a torus without worrying about the inability of using a flux in the new geometry. On the torus we can restore the new expression for the Hall conductance back to the Kubo formula, from which a topological invariant expression can be derived.

In Sec. VI we will study a Hall system which has a topology of a torus with a hole in it. As will be pointed out there, such a system could be realized in real experiments. For such a system, a topological invariant expression for the Hall conductance will be derived to show its quantization.

V. LOCALITY AND LAUGHLIN'S ARGUMENT

Laughlin's[4] argument involves consideration of a device with the geometry of a finite cylinder, although it can equally well be applied to an annulus or any other figure with the same topology. A uniform magnetic field is applied normal to the cylindrical surface, and there is a solenoid along the axis of the cylinder through which a magnetic flux Φ passes. It is supposed that this flux Φ can be varied without changing the magnetic field which acts on the surface. This arrangement is shown in Fig. 2. It is supposed, in the simplest form of this argument, that the electrons are noninteracting and at zero temperature,

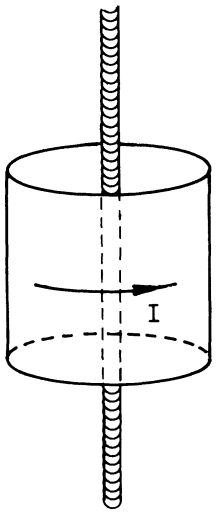

FIG. 2. Cylinder geometry used in Laughlin's theory. Through the center of the cylinder a magnetic flux is passed. The flux derivative of the induced adiabatic charge transport from one edge to the other determines the ratio of the Hall current I flowing around the cylinder to the voltage drop between the two edges.

and that the Fermi energy is such that there are no bulk extended states near that energy; there may be localized states at that energy, of a size much less than the circumference of the system, and there will in general be extended states at the edges of the cylinder which are confined to a region much smaller than the height of the cylinder.[6]

The effect of changing the flux Φ passing through the solenoid by one quantum unit h/e is now considered. For the localized states this merely maps each one into the equivalent state related by a gauge transformation. For each extended state in flux Φ there is an equivalent gauge-transformed state for flux $\Phi+h/e$, but the continuous change of flux can map the set of occupied extended states into a different set of states. Occupied bulk states must remain occupied, since their energies are well away from the Fermi energy, but it is possible for an integer number n of electrons to be transferred from one edge to the other. By Faraday's law the changing flux produces an electromotive force (emf) round the cylinder, and so the ratio of the integral over time of the current from one edge to the other to the time integral of the emf in ne^2/h. If this relation for the integrated current produced by a discrete change in the flux can be changed to a differential relation, the quantization of the ratio between the voltage down the cylinder and the current round it follows. This has the same topology as is usual in the Corbino disk arrangement.

There are two very important conclusions that can be drawn from this argument. The first is that it is only the existence of a mobility gap between the regions of bulk extended states that is necessary to produce the integer QHE, and the gap in which the Fermi energy lies does not have to be obtained by perturbation from an ideal system of electrons in a uniform potential. The second as pointed out by Tao and Wu,[16] is that a fractional QHE, with a conductance pe^2/qh, where p and q are integers with no common factors, implies that in flux Φ there are q equivalent ground states, and the system is mapped through all q of them when the flux is changed by q quanta.

There are a number of points in this argument which have been criticized. The first is that the replacement of the ratio of a charge transfer to a flux change by the conductance, which is its limiting value for an infinitesimal change in flux, is not justified. The second is that a special sort of geometry is assumed, and this is not obviously related to the usual geometry of experimental devices. We show how the locality properties of the Green functions can be used to overcome both of these objections. There is a further criticism of the theory, which is that it is assumed that the rate of change of flux is so slow, or the electric field so small, that adiabatic theory can be used for the charge transfer. We have addressed this problem in an earlier paper,[10] but we had to use perturbative arguments rather than the Laughlin argument to show that the restriction to vanishingly weak electric fields is not essential.

Now it is possible to ask how the Laughlin argument[4] could be applied to the kind of experimental arrangement sketched in Fig. 1. The quantity in which we are interested is the current flowing across a line such as the one shown in the figure going from the end of one voltage probe to the end of the opposite probe divided by the voltage difference between the two ends of the line. The Kubo formula allows the current density at a point to be expressed in terms of the integral of the two-particle Green function of the many-electron system multiplied by the electric field. In a mobility gap the relevant Green function falls off exponentially when the separation between the two points (the point where the current density is calculated, and the point in the integral where the electric field is measured) is increased. This is true despite the existence of extended edge states, as is shown in Sec. II. This exponential localization of the Green function is the key property that allows us to estimate the effect of modifying the boundary conditions. We can expect the localization length to be the order of magnitude of the magnetic length, which is about 10 nm under typical conditions.

Two changes can be made to bring the experimental arrangement shown in Fig. 1 into the correspondence with the geometry of Laughlin's argument shown in Fig. 2, and the locality of the Green function allows us to argue that each of these changes will have a negligible effect on the result. The first change is to replace the current driven round a circuit connected to the source and drain by two passive electron reservoirs at the source and drain. The same current will still flow, but it is to be driven by an emf applied to the voltage probes. Conditions are exactly the same except in the neighborhood of the source and drain, and these are very far from the line in which we are interested, so only exponentially small changes in the ratio of current to voltage occur.

The other change is to replace the emf supplied by the

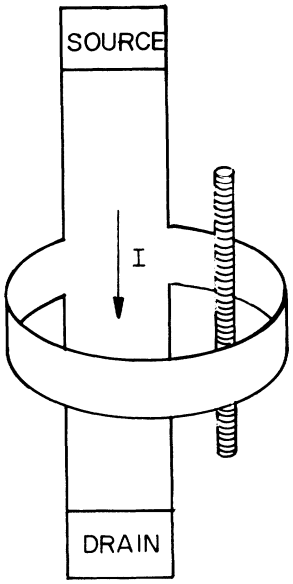

FIG. 3. A pair of opposite voltage leads in Fig. 1 is connected by a ribbon. Through the hole between the ribbon and the main part of the device passes a changing magnetic flux which drives the Hall current.

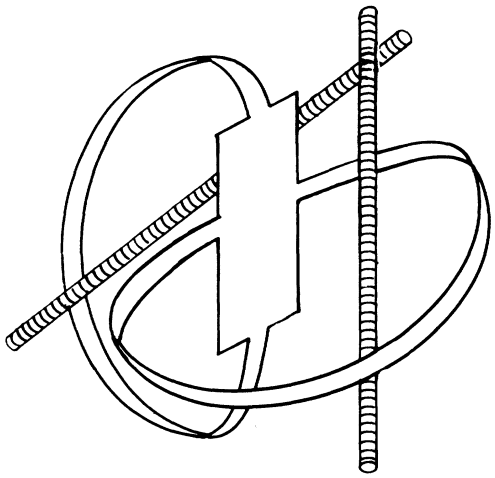

FIG. 4. A rectangle with its two ends connected by a strip (the current lead) and its two sides by another strip (the voltage lead). Through the holes are the magnetic flux solenoids.

voltmeter by a voltage source similar to that assumed in Laughlin's argument. The two ends of the pair of opposed voltage probes are connected to one another by a strip of two-dimensional material, and a solenoid is inserted between this strip and the main part of the device as shown in Fig. 3. The emf is produced by a uniform rate of change of the flux through the solenoid. The nature of the emf should be irrelevant to the physical effects, and any changes in the local conditions in the voltage probes due to replacement of the voltmeter leads by more two-dimensional material can be compensated by imposing local irrotational electrostatic fields. The Laughlin argument can be applied directly to his device shown in Fig. 3. Change of flux by one quantum must drive an integer number of electrons from one side of the system to the other, unless there is a ground state with q-fold degeneracy, in which case the fractional QHE is obtained.

VI. QUANTUM HALL CONDUCTANCE AS A TOPOLOGICAL INVARIANT

Niu *et al.*[5] showed that the quantum Hall conductance of a torus could be written as a topological invariant which is stable against the presence of disorder in the substrate, interactions between the electrons, etc. Although a torus geometry can be mathematically transformed from a strip geometry up to exponentially small errors in the Hall conductance, such a geometry is nevertheless impossible to be realized without using a magnetic monopole source for a uniform magnetic field, so we are now constructing a more realistic version of the theory.

We consider the system shown in Fig. 4 which consists of a rectangle, with its two ends connected by a strip which we call the "current lead," and its two sides connected by a strip which we call the "voltage lead." All of this, the rectangle and the two sets of leads, are made of the same two-dimensional material and are in the same approximately uniform magnetic field. This is evidently a system that could be realized in practice. It has the topology of a torus with a simple hole in it, and its edge is a single simply connected curve. We also suppose that there are two solenoids with flux ϕ_I and ϕ_V which pass through each of the two sets of leads. The boundary conditions round each set of leads are periodic functions (up to a gauge transformation) of ϕ_I and ϕ_V, while an emf can be induced around either set of leads by making a uniform change in the appropriate flux.

Now we use the Kubo formula[17,5,9] to determine the total current I which flows round the current leads in response to a weak electric field $\mathbf{E}_V$ induced by a steady change of the flux ϕ_V. We write the emf induced round the voltage leads as V_V. It is convenient to introduce a fictional electric field $\mathbf{E}_I$ which is locally irrotational, has equipotentials that go all the way round the voltage leads, and has a line integral V_I round the current leads; such a field could be induced by a steady change of ϕ_I if the solenoid were appropriately placed in relation to the voltage leads to make the equipotentials go all round the leads. The current can be written as

$$I=\sum_{\alpha(\neq 0)}\frac{i\hbar}{V_I(\varepsilon_0-\varepsilon_\alpha)^2}\left[\left\langle\Psi_0\left|\int \mathbf{j}\cdot\mathbf{E}_V\right|\Psi_\alpha\right\rangle\left\langle\Psi_\alpha\left|\int \mathbf{j}\cdot\mathbf{E}_I\right|\Psi_0\right\rangle-\left\langle\Psi_0\left|\int \mathbf{j}\cdot\mathbf{E}_I\right|\Psi_\alpha\right\rangle\left\langle\Psi_\alpha\left|\int \mathbf{j}\cdot\mathbf{E}_V\right|\Psi_0\right\rangle\right], \tag{35}$$

where Ψ_0 is the many-body ground state of energy ε_0, the Ψ_α are excited states of energy ε_α. The operator $\mathbf{j}$ is the sum over all electrons of

$$\mathbf{j}=(e/m)(-i\hbar\nabla-e\mathbf{A}) . \tag{36}$$

The integrals involving $\mathbf{j}\cdot\mathbf{E}_I$, which give the current that flows in response to the perturbation $\mathbf{j}\cdot\mathbf{E}_V$, are written as integrals over the entire space and divided by V_I, but, because of the continuity of current flow, they could equally well be written as integrals between two close equipotentials of $\mathbf{E}_I$ and divided by the difference in potential δ. If these equipotentials go round the system inside the voltage leads it can be argued that the Green functions implied by Eq. (35) are insensitive to the boundary conditions applied round the current leads, and so the resulting current I is independent of the flux ϕ_I. This insensitivity to boundary conditions will also be used at a later stage in the argument.

The operators in Eq. (35) can be expressed in terms of the partial derivatives of the Hamiltonian H with respect to the flux through the solenoids. We have

$$\int \mathbf{j}\cdot\mathbf{E}_I=-V_I\frac{\partial H}{\partial\phi_I}=-\frac{V_I e}{\hbar}\frac{\partial H}{\partial\eta_I} , \tag{37}$$

and a similar equation for $\mathbf{j}\cdot\mathbf{E}_V$. This enables us to rewrite Eq. (35) as

$$I=\sum_{\alpha(\neq 0)}\frac{ie^2V_V}{\hbar(\varepsilon_0-\varepsilon_\alpha)^2}\left[\left\langle\Psi_0\left|\frac{\partial H}{\partial\eta_V}\right|\Psi_\alpha\right\rangle\left\langle\Psi_\alpha\left|\frac{\partial H}{\partial\eta_I}\right|\Psi_0\right\rangle-\left\langle\Psi_0\left|\frac{\partial H}{\partial\eta_I}\right|\Psi_\alpha\right\rangle\left\langle\Psi_\alpha\left|\frac{\partial H}{\partial\eta_V}\right|\Psi_0\right\rangle\right]$$
$$=\frac{ie^2V_V}{\hbar}\left[\left\langle\frac{\partial\Psi_0}{\partial\eta_V}\left|\frac{\partial\Psi_0}{\partial\eta_I}\right\rangle-\left\langle\frac{\partial\Psi_0}{\partial\eta_I}\left|\frac{\partial\Psi_0}{\partial\eta_V}\right\rangle\right]\right. . \tag{38}$$

Now it can be argued that the phase η_V is averaged over because its steady change with time provides the emf V_V, and we have already argued that the current is independent of η_I, so we can certainly average over that to get the Hall conductance as

$$\frac{I}{V_V}=\frac{ie^2}{2\pi h}\int_0^{2\pi}d\eta_V\int_0^{2\pi}d\eta_I\left[\left\langle\frac{\partial\Psi_0}{\partial\eta_V}\left|\frac{\partial\Psi_0}{\partial\eta_I}\right\rangle-\left\langle\frac{\partial\Psi_0}{\partial\eta_I}\left|\frac{\partial\Psi_0}{\partial\eta_V}\right\rangle\right]\right. . \tag{39}$$

The double integral in this equation now has the form of the topological invariant that defines the first Chern class of the mapping of the torus onto the complex projective space of many-particle wave functions.[12,13] Provided that continuous changes in η_V,η_I by multiples of 2π map the ground state $|\Psi_0\rangle$ into itself, apart from phase factors, this integral is an integer multiple of 2π, and we get the integer QHE. Edge states are unimportant in this argument, as there is a single edge in the system, and so no current can result from transfer of electrons from one edge to the other. The detailed geometrical shape of the sample, substrate disorder, slight inhomogeneity of the magnetic field, and electron-electron interactions are unimportant, provided that the ground state remains isolated from other states, so that there is not a continuum of other states into which the continuous gauge transformations can map the ground state.

The fractional QHE will occur if there is a discrete set of q equivalent states into which $|\Psi_0\rangle$ can be mapped. In such a case η_V must be changed by $2\pi q$ before the state returns to its initial form, so that the integral on the right side of Eq. (39) must be replaced by an integral equal to $1/q$ times an integer multiple of 2π.

The relation between this system shown in Fig. 4 and the experimental situation shown in Fig. 1 is very similar to the relation described for the Laughlin geometry in Sec. V. We argued in Sec. V that the voltmeter leads can be replaced by a loop of two-dimensional material with an emf applied around it, so we have already justified the special form of the voltage leads in Fig. 4. The fact that the current density depends only on the local environment of the line across which it is measured allows us to replace the source and drain by the current leads of Fig. 4, provided that we also apply some local electric fields (with zero circulation) to maintain the pattern of current flow in the system.

ACKNOWLEDGMENT

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APPENDIX: PROPERTIES OF $D_\nu(k)$

In this Appendix we list some of the important properties of the function $D_\nu(k)$, following the reference book by Bateman.[18] The solutions of the equation

$$\frac{d^2f}{dk^2}+(\nu+\tfrac{1}{2}-\tfrac{1}{4}k^2)f=0 ,$$

are called parabolic cylinder functions. For any value of ν there is a solution $D_\nu(k)$, which damps to zero as $k\to+\infty$. From the reflectional symmetry of the equation, $D_\nu(-k)$ is also a valid solution. If ν is zero or a positive integer corresponding to the harmonic oscillator levels, the two solutions $D_\nu(k)$ and $D_\nu(-k)$ become linearly dependent, and

$$D_\nu(k)=2^{-(1/2)\nu}e^{-k^2/4}H_\nu(k/\sqrt{2})$$

under an appropriate normalization for $D_\nu(0)$, where H_ν is the Hermite polynomial of degree ν. In the following,

we are interested in the case with energies off the bulk Landau levels, so we will only consider those values of ν which are off the oscillator levels mentioned above.

(i) The functions $D_\nu(k)$ and $D_\nu(-k)$ are linearly independent of each other, and are analytical functions of k in the whole complex plane excluding ∞.

(ii) The function $D_\nu(k)$ has the following asymptotic expansions when $|k| \to +\infty$:

$$k^\nu e^{-(1/4)k^2} \quad \text{for } -\tfrac{3}{4}\pi < \arg k < \tfrac{3}{4}\pi ,$$

$$-\frac{(2\pi)^{1/2}}{\Gamma(-\nu)} e^{\nu\pi i} k^{-\nu-1} e^{(1/4)k^2} \quad \text{for } \tfrac{3}{4}\pi < \arg k < \tfrac{5}{4}\pi ,$$

$$-\frac{(2n)^{1/2}}{\Gamma(-\nu)} e^{-\nu\pi i} k^{-\nu-1} e^{(1/4)k^2} \quad \text{for } -\tfrac{5}{4}\pi < \arg k < -\tfrac{3}{4}\pi .$$

(iii) For ν real and positive, $D_\nu(k)$ has $[\nu+1]$ real zeros, where $[\nu+1]$ denotes the largest integer less than $\nu+1$. For ν real and negative, there are no real zeros. In the above cases, there may be zeros off the real axis, but the smallest distance of these zeros to the real axis should be at least of order 1. The complex zeros with large magnitudes should lie about the lines of $|\arg k| = \frac{3}{4}\pi$, following from the asymptotic expansions of $D_\nu(k)$. For nonreal values of ν, there are no zeros on the real axis. In any case, the positions of the zeros should depend continuously on ν (except at $\nu=0,1,2,\ldots$), and the complex zeros are not close to the real axis if ν is not close to positive portion of the real axis of the energy plane.

(iv) For $\nu \neq 0,1,2,\ldots$, $D_\nu(k)$ has infinite number of zeros on the complex plane. This statement can be proved easily by using the growth theory of entire functions.[19] Suppose $D_\nu(k)$ has n zeros ($n < \infty$), then $D_\nu(k)$ can be written as $P_n(k)Q(k)$, where $P_n(k)$ is an nth degree polynomial having the zeros of $D_\nu(k)$, and $Q(k)$ is an entire function without zeros. From the asymptotic behavior of $D_\nu(k)$ we know that $D_\nu(k)$ and hence $Q(k)$ has growth of order 2. A simple theorem says that $Q(k)$ must be of the from $\exp(ak^2+bk+c)$, where a, b, and c are constants. But this form of $Q(k)$ cannot give the correct asymptotic behavior of $D_\nu(k)$ in all directions, unless ν is zero or an positive integer. This contradiction proves our statement.

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