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Automorphic properties of local height probabilities for integrable solid-on-solid models

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The local height probabilities of a class of integrable solid-on-solid models are obtained in terms of the modular forms describing irreducible decompositions of tensor products of $A_1^{(1)}$ modules. Their critical exponents agree with those of a coset construction of Goddard-Kent-Olive with c not necessarily less than 1.

The local height probabilities (LHP's) of two-dimensional integrable lattice models exhibit an automorphic property which enables us to compute the critical behavior of various order parameters.[1] Although Baxter's corner transfer-matrix method[1] gives one an effective way to compute the LHP's, the origin of the automorphic property is yet unknown. We report here a case such that the character formula[2] of an affine Lie algebra explains it.

In another paper,[3] we presented a class of integrable solid-on-solid (SOS) models which contains those studied by Andrews, Baxter, and Forrester (ABF).[4] As in the ABF case, there are four regimes to consider. In this Rapid Communication the LHP's in one of the regimes (regime III in ABF's notation) are computed. The remarkable fact is that they are expressed in terms of the modular forms describing irreducible decompositions of tensor products of $A_1^{(1)}$ modules.

It was pointed out by Huse[5] that the multicritical exponents of ABF in regime III coincide with those in the discrete series of unitary representations of the Virasoro algebra.[6] The significance of the latter is known in the conformal field theory (CFT) by Belavin, Polyakov, and Zamolodchikov.[7] This is the case $0<c<1$. The coset construction of Goddard, Kent, and Olive (GKO)[8] has shown its power in this case. It suggests further the existence of a variety of integrable CFT's with $c>1$. Fateev and Zamolodchikov[9,10] have constructed some of this sort. It is notable that the exponents of our lattice models agree with those of the coset pair $A_1^{(1)}\oplus A_1^{(1)}\supset A_1^{(1)}$.

Let us consider a two-dimensional square lattice. Fix two positive integers N,L such that $L\geq N+3$. With each site i, we associate a "height" l_i subject to the following conditions:

$$\begin{aligned}&l_i=1,2,\ldots,L-1,\ \ (l_i-l_j+N)/2=0,1,\ldots,N\ ,\\&(l_i+l_j-N)/2=1,2,\ldots,L-N-1\ ,\end{aligned}\tag{1}$$

where i and j are adjacent sites. The second condition implies in particular that if N is even the parity of l_i's are all the same, while if N is odd, it alternates from site to site. We consider an interaction-round-a-face model[1] by assigning a Boltzmann weight $W(a,b,c,d)$ to each height configuration a,b,c,d (ordered counterclockwise from the southwest corner) round a face. A model is known to be integrable if the Boltzmann weights satisfy the star-triangle relation.[1] In Ref. 3, we constructed a solution expressed in terms of the elliptic θ function,

$$\theta(u)=2p^{1/4}\sin\left(\frac{\pi u}{L}\right)\times\prod_{n=1}^{\infty}\left[1-2p^n\cos\left(\frac{2\pi u}{L}\right)+p^{2n}\right](1-p^{2n})^2\ .$$

The weights enjoy the following symmetries:

$$\begin{aligned}W(L-a,L-b,L-c,L-d\,|\,u)&=W(a,d,c,b\,|\,u)\\&=W(c,b,a,d\,|\,u)\\&=W(a,b,c,d\,|\,u)\ ,\end{aligned}$$

$$W(a,b,c,d\,|\,u)=(g_ag_c/g_bg_d)W(b,c,d,a\,|-1-u)\ ,$$

where $g_k=\varepsilon_k\sqrt{\theta(k)}$, $\varepsilon_k^2=1$, $\varepsilon_k\varepsilon_{k+1}=(-)^k$. The case $N=1$ was studied by ABF.[4]

In the rest of this paper we shall deal exclusively with the case $0<p<1$ and $-1<u<0$, which corresponds to regime III of ABF.[4] The model becomes critical as $p\to 0$. It is straightforward to compute the free energy per site f. We find that f is regular at the criticality except when N is odd and L is even, in which case the dominant singularity is $f_{\text{sing}}\sim p^{L/2}\ln p$. We have, thus,

$$2-\alpha=L/2\ .\tag{2}$$

A ground-state configuration has the same height, say b, on the center site 1 and those sites separated from 1 by even steps, while all other heights assume another fixed value, say c. By definition the LHP $P(a\,|\,b,c)$ is the probability of finding the height $l_1=a$ at the center site 1 under the condition that the boundary heights are fixed to those of the ground state specified by b and c, as above. Baxter's corner transfer-matrix method provides the following expression in the thermodynamic limit:

$$P(a\,|\,b,c)=x^{-\lambda_{a,L}}G_{a,L}(x)X(a\,|\,b,c)/M_{b,c}\ ,$$

$$M_{b,c}=\sum_{a=1}^{L-1}x^{-\lambda_{a,L}}G_{a,L}(x)X(a\,|\,b,c)\ .$$

Here $\lambda_{a,L}=(2a-L)^2/(8L)$,

$$G_{a,L}(x)=\sum_{v\in Z}(-)^v x^{(2Lv+2a-L)^2/(8L)},$$

and $X(a\,|\,b,c)$ is the one-dimensional partition sum

$$X(a\,|\,b,c)=\lim_{m\to\infty}\sum x^{v_m(l_1,\ldots,l_{m+2})},$$

$$v_m(l_1,\ldots,l_{m+2})=\sum_{j=1}^{m} j\,|l_j-l_{j+2}|/2\ . \tag{4}$$

In (4) we fix $l_1=a$, $l_{m+1}=b$, $l_{m+2}=c$ if m is even, and $l_1=a$, $l_{m+1}=c$, $l_{m+2}=b$ if m is odd. The sum is over $l_2,\ldots,l_m$ such that l_i and $l_{i+1}(i=1,\ldots,m)$ satisfy the condition (1). The parameter x is related to the nome p through $p=\exp(-\varepsilon)$, $x=\exp(-4\pi^2/\varepsilon L)$. It turns out that $P(a\,|\,b,c)$ can be neatly described in terms of the characters of the affine Lie algebra $A_1^{(1)}$ as shown below in (6) and (10).

Let $V_{k,m}(0\le k\le m)$ denote the irreducible highest weight $A_1^{(1)}$ module of level m and spin $k/2$, namely, the one generated from the "vacuum" vector v such that $e_iv=0(i=0,1)$, $h_0v=(m-k)v$, and $h_1v=kv$ (the notations here follow those of Ref. 2). Consider the coset pair $A_1^{(1)}\oplus A_1^{(1)}\supset\Delta(A_1^{(1)})$, where Δ signifies the diagonal embedding. Irreducible decomposition of tensor modules with respect to $\Delta(A_1^{(1)})$ (analogous to the Clebsch-Gordan rule) reads

$$V_{k_1,m_1}\otimes V_{k_2,m_2}=\bigoplus_{k=0}^{m} W_k,\ m=m_1+m_2\ . \tag{5}$$

Here W_k is isomorphic to a direct sum of copies of $V_{k,m}$, and vanishes if $k_1+k_2\not\equiv k$ mod2.

In order to relate the LHP $P(a\,|\,b,c)$ to this setting, we make the following identification:

$$m_1=N,\ k_1=d-1,\ m_2=L-N-2\ ,$$

$$k_2=e-1,\ m=L-2,\ k=a-1\ ,$$

where $d=1+(b-c+N)/2$, $e=(b+c-N)/2$. Note that the conditions (1) mean precisely $0\le k_i\le m_i$ for $i=0,1,2(m_0=m,\,k_0=k)$. Put

$$H=2(L_0^{(1)}+L_0^{(2)})-(h_1^{(1)}+h_1^{(2)})/2\ .$$

Here $\{L_n\}$ are the Virasoro operators in the Sugawara form,[8] and the superscripts refer to the components in $A_1^{(1)}\oplus A_1^{(1)}$. Our main result is

$$P(a\,|\,b,c)=\mathrm{tr}_{W_{a-1}}(x^H)/\sum_{k=0}^{m}\mathrm{tr}_{W_k}(x^H)\ . \tag{6}$$

Using the character formula of $A_1^{(1)}$, we rewrite the right-hand side of (6) in terms of modular forms.

Define the character of $V_{k,m}$ by

$$\chi_{k,m}(z,q)=q^{-\mu}\mathrm{tr}(q^{L_0}z^{-h_1/2}),$$

with $\mu=m/[8(m+2)]$. Explicitly, it is given by a ratio of θ functions

$$\chi_{k,m}(z,q)=F_{k+1,m+2}(z,q)/F_{1,2}(z,q)\ ,$$
$$F_{j,l}(z,q)=\sum_{\gamma\in Z+j/2l}q^{l\gamma^2}(z^{-l\gamma}-z^{l\gamma})\ . \tag{7}$$

Following GKO,[8] we introduce the Virasoro operators $K_n=L_n^{(1)}+L_n^{(2)}-\Delta(L_n)(n\in Z)$ commuting with $\Delta(A_1^{(1)})$. They act on the space of vacuum vectors

$$\Omega_k=\{v\in W_k\,|\,\Delta(e_i)v=0,\ i=0,1\}\ ,$$

with the central charge

$$\frac{3m_1}{m_1+2}+\frac{3m_2}{m_2+2}-\frac{3m}{m+2}\ .$$

We define the character of Ω_k by

$$B_{k_1,k_2;k}(q)=q^{-v}\mathrm{tr}_{\Omega_k}q^{K_0}=q^{-v+h_k}(1+\cdots)\ ,$$

$$v=m_1/[8(m_1+2)]+m_2/[8(m_2+2)]-m/[8(m+2)].$$

The spectrum of K_0 on Ω_k takes the form $\{h_k, h_k+1, h_k+2,\ldots\}$. The lowest eigenvalue h_k is given by

$$h_k=\frac{k_1(k_1+2)}{4(m_1+2)}+\frac{k_2(k_2+2)}{4(m_2+2)}-\frac{k(k+2)}{4(m+2)}+R_k\ , \tag{8}$$

with a nonnegative integer R_k defined as follows. In view of the symmetries (i) $k_1\leftrightarrow k_2$, $m_1\leftrightarrow m_2$, (ii) $k_1\leftrightarrow m_1-k_1$, $k_2\leftrightarrow m_2-k_2$, $k\leftrightarrow m-k$, we may assume that $k_1\le k_2$ and $k\le k_1+k_2$. We then have

$$\begin{aligned}R_k&=0\quad(k=k_1+k_2),\\&=t(k_2-k-tm_1)\quad(k_2-k_1-2tm_1\le k<k_1+k_2-2tm_1),\\&=t[k_2-k-(t+1)m_1]+(k_2-k_1-k)/2\quad(k_1+k_2-2(t+1)m_1\le k<k_2-k_1-2tm_1)\ ,\end{aligned}$$

where $t=0,1,2,\ldots$.

In terms of the characters, (5) implies the identity

$$\chi_{k_1,m_1}(z,q)\chi_{k_2,m_2}(z,q)=\sum_{k=0}^{m}B_{k_1,k_2;k}(q)\chi_{k,m}(z,q). \tag{9}$$

The coefficients $B_{k_1,k_2;k}(q)$ are uniquely characterized by this θ function identity. The significance of the B's is that by the substitution $\tau\to-1/\tau$ [$q=\exp(2\pi i\tau)$] they undergo a linear transformation. This is a direct consequence of (9) and the transformation properties of the characters (7).[2] Since

$$\mathrm{tr}(q^{L_0^{(1)}+L_0^{(2)}}z^{-(h_1^{(1)}+h_1^{(2)})/2})$$

reduces to $\mathrm{tr}(x^H)$ by the specialization $q=x^2$ and $z=x$, (6) is rephrased as

$$P(a\,|\,b,c)=\frac{G_{a,L}(x)G_{1,2}(x)}{G_{d,N+2}(x)G_{e,L-N}(x)}B_{d-1,e-1;a-1}(x^2), \tag{10}$$

where $G_{j,l}(x)=x^{l/8}F_{j,l}(x,x^2)$ is given by (3) and

$B_{k_1,k_2;k}(q)$ is the modular form defined by (7) and (9). The result agrees with the regime III result of ABF[4] ($N=1$) and with Ref. 11 ($N=2$, $L=7$).

Let us examine the critical behavior of the LHP's by using the formula (10). At $p=0$ the dependence on the boundary heights drops off altogether, giving $P_a^0=(4/L)\sin^2(a\pi/L)$. The deviation $P(a\,|\,b,c)-P_a^0$ is expressed as a sum of the series of the form $p^\beta(\text{const}+\dots)$. According to the scaling hypothesis,[1] the critical scale dimensions are given by $\eta=2\beta/(2-\alpha)=4\beta/L$, where we have used (2). We find that the exponents β of the LHP give precisely the values $\eta=2h_k$ in (8).

In conclusion, we have shown that the SOS models of Ref. 3 are the integrable lattice models corresponding to the coset pair $A_1^{(1)}\oplus A_1^{(1)}\supset A_1^{(1)}$. This means that the LHP's of the former are coded in the character formulas of the latter. We boldly propose the following principle of correspondence: (i) an integrable lattice model $\leftrightarrow$ a coset pair, an affine Lie algebra $\mathfrak{g}$ and its subalgebra $\mathfrak{h}$, (ii) ground states $\leftrightarrow$ irreducible representations of $\mathfrak{g}$, (iii) heights of the central site $\leftrightarrow$ irreducible representations of $\mathfrak{h}$, and (iv) the LHP's $\leftrightarrow$ the decomposition of the specialized characters.

Another realization of this principle is given in Ref. 12. The lattice model is an N-state Ising-type model[13] and the coset pair is $A_1^{(1)}$ and $\mathfrak{s}$($\mathfrak{s}$: the homogeneous Heisenberg subalgebra). Precisely speaking, the decomposition of the character of the $A_1^{(1)}$ module is carried out with respect to $W\times\mathfrak{s}$(W: the affine Weyl group). It is quite interesting to add further examples of this sort, especially ones with $\mathfrak{g}$ of higher rank.

The above principle is to be supplemented by the third ingredient to form a triplet consisting of a coset pair, an integrable lattice model, and an integrable CFT. The continuum limit of the second at criticality should lead to the third, and its conformal structure should be governed by the first. This point of view is important especially for the case $c>1$, where the unitarity argument alone does not single out integrable CFT's. The cases $N=1,2,4$ of our SOS model correspond to the known integrable CFT's: the minimal theory,[7] its supersymmetric extension,[14] and the Z_3-symmetric self-dual theory,[10] respectively. For the N-state Ising-type model mentioned in the last paragraph, the corresponding CFT has been found by Fateev and Zamolodchikov.[9]

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