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Spiral-lattice-site animals: An exact enumeration study

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Spiral-lattice-site animals are studied on a square lattice by exact enumeration methods. The number of animals g_s, their average perimeter $\langle t_s \rangle$, and the average radius of gyration R_s are obtained exactly for all s-sited animals up to $s=11$. We find as $s \rightarrow \infty$, $g_s \sim \lambda^s$ with $\lambda = 2.811 \pm 0.002$, $\langle t_s \rangle \sim As$ with $A=1.157 \pm 0.020$, and $R_s \sim s^{\rho}$ with $\rho = 0.499 \pm 0.030$. These results are discussed and comparisons with ordinary and directed lattice animals are made.

Lattice animals are clusters of connected sites or bonds embedded in a lattice. They are useful in explaining various clustering phenomena, right from the formation of organisms to the problem of branched polymers in solution in the dilute limit. Much work has been done on the statistics of these animals (see, e.g., Stauffer[1] and references therein) which is mostly concerned with finding the exponents characterizing the asymptotic behavior of animals. It is also well known that if global directional constraints are put on the growth process of animals, the asymptotic behavior changes. An example is the problem of directed lattice animals[2] which belongs to a universality class different from that of ordinary animals. The picture, however, is less clear and more complicated if one puts local constraints instead of global constraints on the growth of the clusters. An example of local constraints is the spiraling constraint for which the animal can grow either in the same direction as that in its previous stage of growth or clockwise with respect to it. In this Brief Report, we study spiral-lattice-site animals on a square lattice in order to understand how local constraints affect the asymptotic behavior of ordinary animals.

We define spiral-lattice-site animals to be a subset of ordinary site animals with the proviso that there is a site of the cluster, namely, the origin to which every other site of the cluster is attached, through at least one spiral path. Figure 1(a) shows one such animal of ten sites grown from the origin X. The arrows have been drawn for the convenience of recognizing the spiral paths. Figure 1(b) gives the example of an ordinary-lattice-site animal which is not spirally connected; none of the sites of the cluster can serve as an origin for a spiral lattice animal.

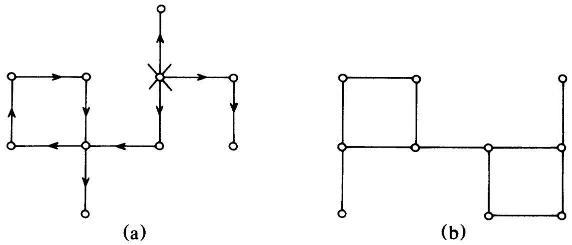

FIG. 1. (a) A spiral lattice animal of 10 sites on a square lattice grown from the origin X. (b) An ordinary lattice animal on a square lattice which is not spirally connected.

Li and Zhou[3] have recently studied labeled spiral bond animals without loops, i.e., spiral trees. These animals which are more numerous than ordinary animals, have no relevance in the present study. We perform series-expansion-type calculations in order to study various properties of the spiral animals we have defined. Exact enumeration of the number of spiral site animals on a square lattice has been performed on an Iris-80 computer for cluster size up to $s=11$, where s is the number of sites in the animal cluster. The computer algorithm that has been used consists of two parts: The first part uses Martin's algorithm[4] to generate ordinary animals. In the second part, the clusters are individually checked for spiral connection. Table I gives the number of spiral site animals for $s=1$ to 11. It also includes data for the average perimeter $\langle t_s \rangle$ of s-sited spiral animals and the corresponding average radius of gyration[5] $R_s = \langle (\sum_{i=1}^{s} r_i^2/s)^{1/2} \rangle$, where r_i is the distance of a cluster site i from the center of mass of the cluster.

TABLE I. Results of exact determination of spiral animal properties for animal size s up to 11. g_s is the number of lattice animals, $\langle t_s \rangle / s$ is the average perimeter-to-size ratio, and R_s is the average radius of gyration for s-sited clusters.

S	g_s	$\frac{\langle t_s \rangle}{s}$	R_s
1	1	4.0000	
2	2	3.0000	0.5000
3	6	2.4443	0.7166
4	17	2.1765	0.9103
5	49	2.0082	1.0780
6	140	1.8905	1.2318
7	396	1.8037	1.3743
8	1114	1.7344	1.5067
9	3131	1.6766	1.6300
10	8794	1.6276	1.7459
11	24733	1.5848	1.8546

The starting point for any lattice-animal theory is the generating function $G(x)$ defined as $G(x)=\sum_s g_s x^s$, where g_s is the number of lattice animals of size s and x^s is the weight associated with each of the clusters. $G(x)$ is expected to exhibit a power-law singularity of the form $G(x) \sim (x-x_c)^{\theta-1}$ as $x \rightarrow x_c^-$. This would imply that in the asymptotic limit $s \rightarrow \infty$, g_s has the form

$$g_s \sim \lambda^s s^{-\theta} , \tag{1}$$

where $\lambda = 1/x_c$. λ is known as the "growth parameter" because asymptotically $g_s/g_{s-1} \rightarrow \lambda$. For spiral lattice animals, the variation of this ratio g_s/g_{s-1} with $1/s$ is shown in Fig. 2. Extrapolation of the last few points to $s \rightarrow \infty$ gives $\lambda = 2.811 \pm 0.002$ and θ very close to zero. For ordinary (see Ref. 1) lattice animals $\lambda \simeq 4.06$, $\theta \simeq 1$, and for directed animals[2] $\lambda = 3$, $\theta = \frac{1}{2}$. We find that there is a systematic reduction in the values of λ and θ as one goes from ordinary to directed to spiral lattice animals. The value of θ is reminiscent of the growth processes (Eden process, diffusion-limited aggregation, etc.) for which it is exactly zero. θ is also exactly zero for one-dimensional lattice animals. We find that for both spiral and directed animals the generating function has a power-law type of singularity, in contrast to ordinary animals where the singularity is of logarithmic type.[6] However, for spiral animals, whether θ is exactly zero or has a very small value can be decided only with the help of a bigger series for animal numbers.

The average perimeter $\langle t_s \rangle$ for s-sited animals is $\langle t_s \rangle = \sum_t t g_{st}/g_s$, where g_{st} is the number of lattice animals with perimeter t and $g_s = \sum_t g_{st}$. The perimeter sites include both external and internal "hole" sites. Stauffer[7] has pointed out that with such a definition of the perimeter, the perimeter or surface in the case of sufficiently large clusters will always be proportional to the size s of the clusters. The limiting value, for large s, of the perimeter-to-size ratio can be regarded as a measure of the internal disorder of large clusters. For both ordinary[1] and directed[8] lattice animals, the average perimeter distribution law for large s is

$$\frac{\langle t_s \rangle}{s} = A + \frac{B}{s} . \tag{2}$$

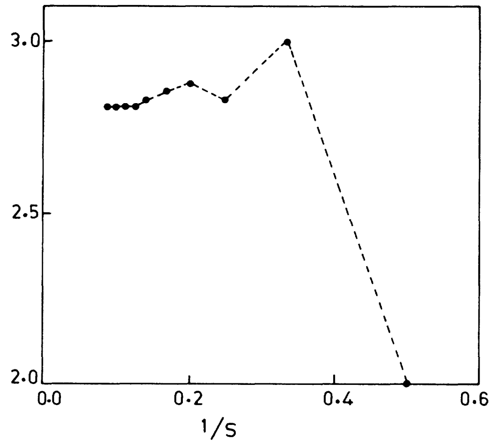

FIG. 2. Plot of the ratios g_s/g_{s-1} vs $1/s$ for spiral animals on a square lattice.

Spiral animals also obey this distribution law, as shown in Fig. 3. The average perimeter is linear in size with a Bethe-type first-order correction. "A," the measure of "internal disorder," has values 1.157 ± 0.020 for spiral animals and 1.20, 0.75 for ordinary[1] and directed[8] animals, respectively. From the data for perimeter-to-size ratios, one can also have an alternative estimate of the growth parameter λ. In fact, as in the case of ordinary animals,[1,5,9] the leading s dependence of g_{st} for large s can be expressed as $g_{st} \sim \Lambda^s e^{sg}$, where

$$\Lambda = \Lambda(A) = \lim_{s \rightarrow \infty} \frac{(a+1)^{a+1}}{a^a} ,$$

$a = \langle t_s \rangle / s$ for any s, and

$$g = g(A) = \lim_{s \rightarrow \infty} \left[\frac{1}{s} \ln (g_{st} \Lambda^{-s}) \right]$$

is the exponent for the leading exponential variation of the ratio g_{st}/Λ^s with size s. As $s \rightarrow \infty$, $a \rightarrow A$, we get for the growth parameter

$$\lambda = \Lambda(A) e^{g(A)} . \tag{3}$$

Figure 4 shows the variation of $g(a)$ with "a" from which we obtain $g(a=A) = -0.48$ and using Eq. (3), λ comes out to be 2.8. This value of λ is in complete agreement with the value 2.811 obtained earlier from Eq. (1). This provides a good consistency check of the relevant animal theory in case of spiral lattice animals.

Finally, we study the variation of the average radius of gyration R_s with s. The values of R_s for different s are listed in Table I. We find that the spiral animals obey the same scaling law $R_s \sim s^\rho$ as do the ordinary[10] and directed[2] animals. From successive values of animal radii R_{s-1} and

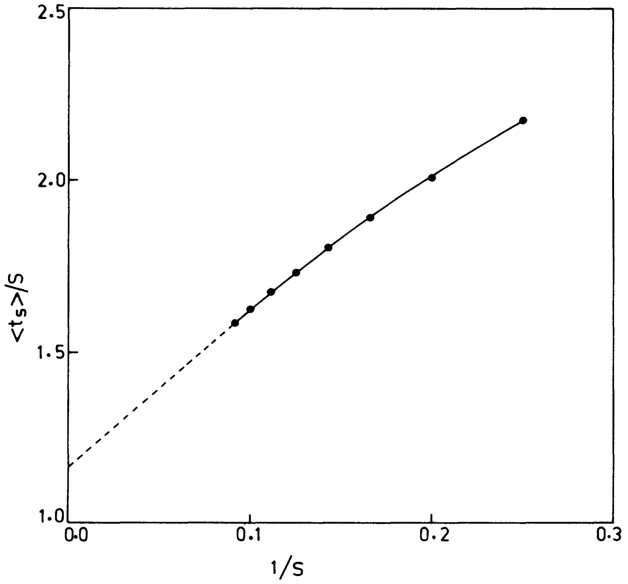

FIG. 3. Variation of average perimeter-to-size ratio $\langle t_s \rangle / s$ with $1/s$ for spiral animals on a square lattice. Extrapolation to $s \rightarrow \infty$ gives $\langle t_s \rangle / s = 1.157 \pm 0.020$.

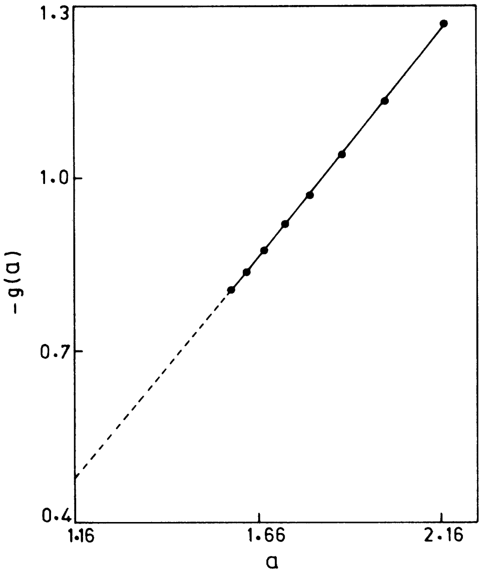

FIG. 4. Variation of $g(a)$ with a. The extrapolation to $a = A$ gives $g(A) = -0.48$ for spiral lattice animals.

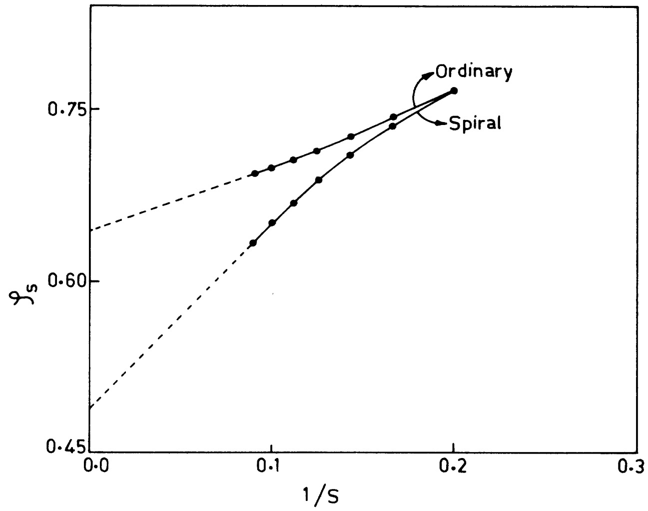

FIG. 5. ρ_s vs $1/s$ for both ordinary (using the results of Ref. 10) and spiral animals. Extrapolation to $s \rightarrow \infty$ gives ρ(ordinary) $\simeq 0.635$ and ρ(spiral) $\simeq 0.499 \pm 0.030$.

R_s, an estimate of ρ_s for the exponent ρ is obtained from

$$\rho_s = \frac{[\log_{10}(R_s) - \log_{10}(R_{s-1})]}{[\log_{10}(s) - \log_{10}(s-1)]} . \tag{4}$$

In Fig. 5, we plot ρ_s vs $1/s$ and extrapolate the plot to $1/s = 0$. The variation of the last few points is quite linear but with a superimposed small-amplitude oscillation. Numerical estimate of $\rho = \lim_{s \rightarrow \infty} \rho_s$ is obtained as the average of the linear intercepts from the last few points and comes out to be $\simeq 0.499 \pm 0.030$. In the same figure the corresponding plot for the ordinary animals (obtained from the results given in Ref. 10) is also exhibited for the sake of comparison. The value of ρ for ordinary animals is $\simeq 0.635$. This radius exponent ρ describes the geometry of animals in terms of an effective dimensionality[1] $d_f = 1/\rho$, the cluster mass M varying as $M \propto (R_s)^{d_f}$. If $d_f < d$, the spatial dimension of the lattice, the animals are said to be ramified, and if $d_f = d$, the animals are compact. For spiral animals we find that d_f is very close to d, i.e., spiral clusters are almost perfectly compact. We argue that this is due to the nature of the constraint itself. Spiraling constraint is effective only in the presence of rotational force fields. A cluster growing in such a field will tend to have a compact structure because of the centripetal force acting on it. Other lattice models which compact clusters are a characteristic feature of include some growth models like the Eden model.[11]

To sum up, we find that lattice animals with spiraling local constraint belong to a new universality class. The various distribution laws and scaling forms, however, remain unaltered. This is true for global directional constraints also. The spiral animals have many interesting features: The animal number exponent θ is very close to zero and the fractal dimensionality d_f is almost equal to the Euclidean dimensionality d. Granting that enumeration of animal properties for larger values of s is essential for rigorous verification of these interesting estimates, one does not, however, fail to notice the close agreement of these results with the corresponding exact results for the Eden growth model. The lattice-animal problem is a static problem with clusters of a given size being assigned equal weight, whereas the growth model describes a nonequilibrium situation where the probability of obtaining a particular configuration depends on the history of growth. It is an open problem to relate both these models by finding growth rules that would generate lattice animals with the correct statistical weight. The similarity of exponents for the spiral-lattice-site animal problem and the Eden growth model gives added justification for further studies in this direction. Finally, we would like to point out the dominant role played by loops in the formation of spiral animals. For ordinary lattice animals, it is known[12] that loops are irrelevant for the study of asymptotic properties. Havlin, Trus, and Stanley[13] have considered a growth model for which the clusters are chemically linear. Such clusters exhibit the same asymptotic properties as do geometrically linear chains; the loops are insignificant in this case also. But, for spiral lattice animals, loops are absolutely essential for cluster connectivity and have an important contribution to the cluster statistics. A detailed analysis of the role played by loops in the geometry of spiral lattice animals will be reported elsewhere.

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