Transport properties of continuum systems near the percolation threshold

Shechao Feng
Schlumberger-Doll Research, Old Quarry Road, Ridgefield, Connecticut 06877-4108

B. I. Halperin
Department of Physics, Harvard University, Cambridge, Massachusetts 02138

P. N. Sen
Schlumberger-Doll Research, Old Quarry Road, Ridgefield, Connecticut 06877-4108
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We present the results of studies of transport properties of several classes of *continuum* disordered systems near the percolation threshold. We find that the problem can be mapped onto a percolation network with randomly occupied bonds, whose strengths of transport have a wide distribution. We show, using both the variational method and the scaling approaches, that these wide distributions of bond strengths can considerably increase the values of the various transport percolation exponents. We are able to place bounds, which are rather narrow, to these new transport percolation exponents.

I. INTRODUCTION

The purpose of this paper is to detail and extend a study by the same authors reported recently in Physical Review Letters.[1] In that note, it was shown that the critical exponents governing the behaviors of electrical conductivity, elastic constants, and viscous-fluid permeability near the percolation threshold of random-void *continuum* systems specialized to spherical voids ("Swiss-cheese" model) can differ considerably from their counterparts in the conventional discrete-lattice models. This is a sharp contrast to the critical exponents for the geometrical percolation properties, such as the correlation-length exponent ν, which have been shown to be the same for both continuum and discrete-lattice models.[2,3] Two other classes of continuum models, the "inverted Swiss-cheese" and "potential models," which were discussed by the authors in a second communication,[4] will also be discussed more fully below.

The transport properties of percolation systems have been the subject of many recent theoretical and experimental studies. These studies, apart from their theoretical interest, may serve as guidance for understanding the transport properties of many types of inhomogeneous materials, such as polymer gels near the sol-gel transition[5] and low-porosity sedimentary rocks.[6]

Let us first review the percolation transport properties of the "standard" discrete lattice percolation model, in which the bonds (or sites) of the lattice are randomly occupied with a probability p, and all occupied bonds have the *same* strength. If the occupied bonds in the network are taken to be electrical conductors of unit strength, we can define the percolation conductivity exponent t as $\Sigma \sim (p - p_c)^t$, for $p \gtrsim p_c$, where Σ is the network conductivity, and p_c is the percolation threshold. The values of t are relatively well established, e.g., $t \approx 1.3$ in two, $t \approx 1.9$ in three, and $t = 3$ in six dimensions.[7]

When the occupied bonds in the network represent elastic interactions among the neighboring sites (masses), we can define the percolation elasticity exponent f as $Y \sim (p - p_c)^f$, where Y is the Young's modulus of the system. The value of f had long been thought to be the same as t, based on the similarities of the elasticity problem to that of the electrical conductivity.[5] Recently however, it has been demonstrated through both theoretical[8-14] and experimental[15,16] means, that because of the higher tensorial order of the elastic problem as compared with the electrical conduction one, f can be actually quite different from t. Below we will summarize three classes of elastic network models that have been studied in the past which seem to introduce different universality classes.

(i) The *non-rotationally-invariant* Born model. If the change in energy is given as a sum over the occupied bonds of the squares of the relative displacements in a random network one arrives at the Born model. It is easy to show that this model falls into the same class as the conductivity problem, thus $f = t$ in this case. This model, however, violates the physical requirement of rotational invariance.[8] Alexander[5] has argued that for networks under internal (or external) stress which may be relevant to systems like rubbers and gels, there are terms in the Hamiltonian similar to those in the Born model. Because these systems differ from the Born model in other important ways, however, such as the presence of nonlinear terms in their Hamiltonians and the occurrence of negative as well as positive Born coefficients, it is not clear that the result $f = t$ should apply to such systems.

(ii) The *rotationally invariant* bond-bending class of elastic percolation networks. In the bond-bending model, one considers a percolation network where the occupied bonds are elastic upon stretching the bond lengths and upon changing the angles of given pairs of bonds. Another model that is in the same class as the bond-bending model is the granular model, where sites on a network

represent approximately rigid grains which have both translational degrees of freedom of the centers of mass and angles of rotation of the grains, and the randomly occupied bonds represent nearest neighbor elastic forces.[14] For models of the above type, one has the property that $p_e = p_c$, where p_e denotes the threshold for the vanishing of elastic moduli. This model is a natural description of disordered elastic systems which do obey rotational invariance, and is relevant to, for example, the sintered materials studied by Deptuck *et al.*[16] Recent works[9–12,14–17] suggest strongly that for these models $f - t \approx 2\nu$, where ν is the exponent for the percolation correlation length ξ, defined through $\xi \sim (p - p_c)^{-\nu}$. Numerically,[17] $f \approx 4$ in all dimensions larger than two. There is even indication that $f = t + 2\nu$ may be an exact relation.[17,18]

(iii) The central-force model. Suppose in a random network we only retain the nearest neighbor central forces, i.e., bonds are taken as pure Hooke springs.[8,13] Here one finds that p_e is much higher than p_c. For instance, for a two-dimensional triangular lattice, $p_e \approx 0.6$, whereas $p_c \approx 0.34$. This is because many connected clusters near percolation contain quasi-one-dimensional chains which do not resist deformation through changes in bond angles on the chains. It appears that the elastic "backbone" for this model differs very much from that of electrical conduction, and it appears that there exist distinct exponents such as ν_{cen} and β_{cen} which describe the geometry of this central force backbone.[13] Because of this peculiarity, the central force model is said to be underconstrained. Here, one can again define an elasticity percolation exponent f_{cen} through $Y \sim (p - p_e)^{f_{\text{cen}}}$. In Ref. 13, Tremblay and co-workers have found that $f_{\text{cen}} \approx 1.4 \pm 0.2$, which is different from the value for f in the bond-bending-type models, but close to the value of t. (The value of f_{cen} is smaller than the estimate given earlier in Ref. 8 by Feng and Sen.) In any case, since the geometrical exponents of the central-force model are different from the usual models, we believe that the percolation elasticity models in which $p_e \neq p_c$ belong to a universality class that is distinct from both the bond-bending type models and that of the electrical conduction (or Born model).

If the occupied bonds on a random network are interpreted as pipes containing a viscous fluid, then one can define the network permeability κ as the ratio between the macroscopic fluid current density and the applied pressure gradient. A permeability critical exponent e is defined through the relation $\kappa \sim (p - p_c)^e$. Since in the standard discrete model all the occupied bonds are identical, one can map this problem exactly onto that of the electrical conduction problem, with the electrical current density replacing the fluid current density, and electrical field replacing the fluid pressure gradient. Thus one has $e = t$. We will see later that this trivial relation no longer holds for many continuum systems.

Now let us consider the continuum percolation systems and their transport properties. As an illustration, let us consider the so-called "Swiss-cheese" model[1] in three dimensions, where uniformly-sized spherical holes are placed at random in a uniform transport medium. The spherical holes are allowed to overlap with one another. It is quite obvious that there exists a critical hole-volume fraction q_c such that when $q > q_c$ the system ceases to support any transport. For $q \lesssim q_c$, one can again define all the transport percolation exponents through: $\Sigma \sim (q_c - q)^{\bar{t}}$, $Y \sim (q_c - q)^{\bar{f}}$, and $\kappa \sim (q_c - q)^{\bar{e}}$.

Near the percolation threshold q_c, the transport properties of the Swiss-cheese model are limited by many narrow "necks," each of which is bounded by three interpenetrating holes. A pictorial illustration of the shape of a neck is shown in Fig. 1. These narrow necks are just barely connected to support transport through the system. Thus it is plausible that the Swiss-cheese model can be mapped onto a random network problem, with the narrow necks playing the role of occupied bonds. Unlike in the standard discrete percolation models where all the occupied bonds are identical, however, the necks in the Swiss-cheese model have a wide distribution in widths. We shall see below, that if one denotes the neck width to be δ, then the probability density function of neck width $P(\delta)$ is finite in the limit $\delta \to 0$. Because the transport "strength" (e.g., conductance) of a given neck of width δ has a power-law dependence on δ, this distribution of neck widths results in a singular distribution of the transport strengths of the necks.

It has been known for some time that percolation random networks with a singular distribution of transport strengths can have a transport percolation exponent that differs significantly from that of the standard network models in which all occupied bonds are identical.[19–22] Thus combining the above reasoning, one concludes that the critical exponents governing the transport properties of continuum percolation systems can differ significantly from their counterparts in the standard discrete models.

It should be emphasized that, as noted by Elam *et al.*,[3] the geometrical percolation exponents such as ν for the Swiss-cheese-type systems do *not* differ from their counterparts in the discrete models. This is because the geometrical exponents depend only on the connectivity of the necks, but not on the widths of individual necks; thus all the narrow necks in the Swiss-cheese system can be re-

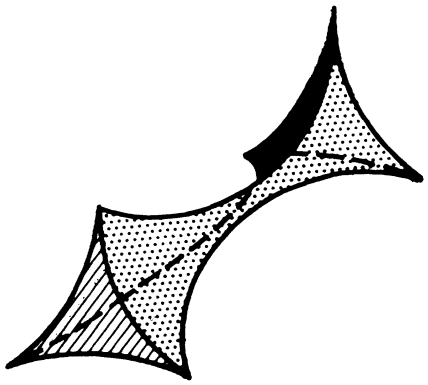

FIG. 1. Narrow portion of a bond (neck), passing between three overlapping spherical holes, in the three-dimensional Swiss-cheese model.

garded identical in the context, and the problem is essentially equivalent to that of the connectivity properties of a standard discrete percolating network model.

The organization of the rest of the paper is as follows. In Sec. II, we will analyze in detail the transport exponents of lattice percolation models where the bond strengths follow a singular power-law distribution. In particular, we will derive both upper and lower bounds of these exponents. We also consider the problem of a mixture of superconducting and normal bonds, where the normal bonds have a singular distribution near zero resistance. In Sec. III we consider the mapping of a few different continuum models onto equivalent discrete networks and compute the transport bond-strength distributions. From these distributions we then determine the values of transport exponents. Concluding remarks are contained in Sec. IV.

II. TRANSPORT PROPERTIES OF RANDOM NETWORKS WITH A DISTRIBUTION OF BOND STRENGTHS

In this section we discuss the percolation-transport critical exponents of random networks in which fraction $1-p$ of the bonds are unoccupied, and fraction p of the bonds are occupied. The occupied bonds are *not* of the same kind, but rather follow a singular distribution of transport strengths.[19–23] Let g be the generic transport strength of a given bond, which is the conductance of the bond when one is addressing the electrical conduction problem, or is the bond-bending force constant for the elasticity problem, or is the viscous fluid flow conductance for the permeability problem. The distribution of transport strengths for the occupied bonds is assumed to have the following limiting behavior for $g \to 0$:

$$p(g) \sim g^{-\alpha}, \quad \text{as } g \to 0 , \tag{1}$$

where α is understood to be in the physically accessible range $0 < \alpha < 1$. For most of this section we assume that there are no correlations among the occupation probabilities and the bond strengths on different sites and that the underlying lattice is regular. We shall discuss in Sec. II D the situation when these restrictions are removed.

To facilitate contact with the continuum models later, let us introduce another representation of the bond-strength distribution. Let δ be a "width" parameter such that g depends on δ in the form

$$g = \delta^{y+1} , \tag{2}$$

where y is related to α in Eq. (1) by the relation

$$y = \frac{\alpha}{1-\alpha}, \quad \text{or } \alpha = \frac{y}{y+1} . \tag{3}$$

Then the probability distribution for δ has a finite limit for $\delta \to 0$, i.e.,

$$P(\delta) \to \text{const} \equiv \frac{1}{\delta_0} \quad \text{as } \delta \to 0 . \tag{4}$$

[This follows immediately from the condition $P(\delta)d\delta = p(g)dg$.]

We will also assume that the range of g is bounded above, i.e., there is a maximum value of bond transport strength $g_{\max}$. This in turn implies that there exists a maximum value of δ, which we denote by $\delta_{\max}$.

We define the transport percolation exponents $\bar{t}$, $\bar{f}$, and $\bar{e}$ as $\Sigma \sim (p-p_c)^{\bar{t}}$, $Y \sim (p-p_c)^{\bar{f}}$, and $\kappa \sim (p-p_c)^{\bar{e}}$, for $p \gtrsim p_c$. It turns out that we are able to obtain nonrigorous lower bounds on the values of these exponents based on the linear chain analysis in the "nodes-links-blobs" picture of the conducting backbone[24–25] as well as a set of rigorous lower and upper bounds based on a variational principle. The combined results can be summarized in the following form.

(i) When $y \le 0$ or $\alpha \le 0$, the distribution of bond strengths does not result in any change in transport exponents, i.e., $t = t$, $\bar{f} = f$, and $\bar{e} = e = t$.

(ii) When $y \ge 0$ or $0 < \alpha \le 1$, the transport exponents are bounded by

$$\begin{aligned} &\max(t_1 + y, t) \le \bar{t} \le t + y , \\ &\max(f_1 + y, f) \le \bar{f} \le f + y , \\ &\max(t_1 + y, t) \le \bar{e} \le t + y \end{aligned} \tag{5}$$

where t and f are the exact values of the standard lattice percolation electrical conductivity and elasticity exponents, respectively, and $t_1 \equiv 1 + (d-2)\nu \le t$, $f_1 \equiv 1 + d\nu \le f$ are the lower bounds of these exponents derived within the linear chain analysis. Since the bounds in Eq. (5) are quite close to each other, one is able to obtain good estimates of these transport exponents. It is to be noted that the inequalities involving t are rigorous, but those involving t_1 and f_1 are nonrigorous. The inequalities involving f are rigorous for one variant of the elastic percolation problem, but the upper bound is not rigorous in the case of greatest interest to us here. (See discussion at the end of subsection II B, below).

The conductivity problem described above was studied originally by Kogut and Straley in 1979.[19] They employed an effective medium theory and a Cayley tree model, as well as the variational method discussed in Sec. II B, below. Their proposed form of the exponent $\bar{t}$ coincided with the upper bound $t + y$, for $y > 0$. Subsequently Ben-Mizrahi and Bergman[20] applied an approximate renormalization group method to this problem, in two dimensions. This analysis led them to propose a form $\bar{t} = At + By$, for $y > 0$, where A and B are constants slightly different from unity.

A later paper by Straley[21] reexamined the conductivity problem by means of a "nodes-links-blobs" picture analysis, which was then combined with the renormalization group[22] and the results of the previous analyses. Straley concluded that the correct value for $\bar{t}$ is given by the lower bound in Eq. (5), i.e., $\bar{t} = \max(t, t_1 + y)$. Since t is slightly larger than t_1, this implies that the necessary condition for $\bar{t}$ to differ from t is then $y > (t - t_1)$, rather than simply $y > 0$. We refer the reader to Straley's paper for details,[19,21,22] which will not be repeated here.

Since permeability and conductivity are mathematically equivalent in the lattice model under consideration, Straley's analysis implies that $\bar{e} = \max(t, t_1 + y)$. An extension of Straley's analysis would also suggest that the

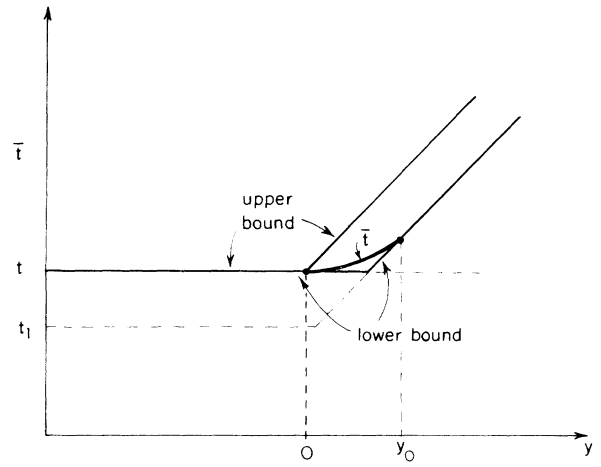

FIG. 2. Behavior of exponent $\bar{t}$ as a function of the distribution parameter y. The straight solid lines indicate the lower and upper bounds to $\bar{t}$ discussed in the text, while the curved heavy solid line shows the conjectured deviation of $\bar{t}$ from its lower bound. [For $y<0$ and $y>y_0$, $\bar{t}$ is given by its lower bound (see Ref. 23).]

correct value for $\bar{f}$ is the lower bound given in Eq. (5), $\bar{f}=\max(f,f_1+y)$.

Recently, Lubensky and Tremblay[23] have studied this problem in $6-\epsilon$ dimensions, and have found results which agree with Straley's for most, but not all values of y. Specifically they find that for a given spatial dimension d, there exists a critical value y_0, such that $\bar{t}=t$ for $y<0$ and $\bar{t}=t_1+y$ for $y>y_0$. In the range $0<y<y_0$, however, the value of $\bar{t}$ is larger than both lower bounds; t and t_1+y. We illustrate this result in Fig. 2, where both the upper and lower bounds of $\bar{t}$ and the conjectured behavior of the exponent itself are plotted as a function of the bond-strength distribution parameter y.

Our own discussion of the lattice models with varying bond-strengths will be given in two subsections. In subsection II A, we use the linear chain analysis to derive the non-rigorous lower bounds to these transport exponents. In subsection II B, we use a variational approach to derive the rigorous lower and upper bounds of these exponents. In subsection II C we generalize our analysis to the superconducting and "superelastic" networks with bond strength distribution. In subsection II D we discuss the influence of correlation of disorder on our results.

A. Nonrigorous lower bounds on $\bar{t}$, $\bar{f}$, and $\bar{e}$

Nonrigorous lower bounds to the transport critical percolation exponents of random networks with a wide distribution of bond strengths can be obtained by applying a scaling analysis based on the nodes-links-blobs picture of the percolation clusters. Since the argument is based on a scaling assumption, these lower bounds are not rigorous. We will first discuss the electrical conduction property, for purpose of clarity. A similar argument for this case was previously given by Straley, in Ref. 21.

In the nodes-links-blobs picture[24] of the percolation cluster near p_c, the "backbone" which supports electrical conductivity through the system is imagined to consist of quasi-one-dimensional string segments ("links"), tying together a set of "nodes" whose typical linear separation is the percolation correlation length ξ. Each string is supposed to consist of several sequences of singly-connected bonds, in series with thicker regions, "blobs," where there are two or more conducting bonds in parallel.

A lower bound to the conductivity exponent is obtained if one ignores the resistance of the blobs i.e., the blobs are taken to be perfectly conducting. It is apparent that this assumption leads to an overestimate of the true conductivity of the system. To be specific, the conductance of a string (link) is given by

$$G^{-1}\geq\sum_{\alpha=1}^{L_1} g_\alpha^{-1}\ , \tag{6}$$

where g_α denotes the conductance of a given bond i, and the sum is restricted to the L_1 singly-connected bonds on the string. It has been shown rigorously[25] that the typical value of L_1 is proportional to $(p-p_c)^{-1}$.

In the standard discrete lattice case where all occupied bonds have unit resistance, the conductance of a string G will be bounded from above by L_1^{-1}, and the conductivity Σ of the entire network, which is related to G by the geometrical relation $\Sigma\sim\xi^{2-d}G$, will therefore be bounded from above by $\xi^{2-d}L_1^{-1}$, where d is the spatial dimension. Thus one obtains a lower bound to the conductivity exponent t of the standard lattice percolation problem: $t>t_1\equiv 1+(d-2)\nu$. The value of t_1 is slightly smaller than the best computer simulation estimates[7] of the exponent t; numerically, $t_1=1$, $t\approx 1.29$ in $d=2$, $t_1\approx 1.85$, $t\approx 1.9$ in $d=3$. It is believed that $t_1=t=3$ in $d=6$, while for $d=6-\epsilon$, one has $t=(d-2)\nu+\zeta$ with $\zeta=1+\epsilon/42+O(\epsilon^2)$.[26]

In the present case, the occupied bonds on the string have a wide distribution of conductances. We can apply statistical results due to Levy and others[27] on sums of random variables, to Eq. (6). The probability density of this sum of random variables cannot be evaluated by the central limit theorem if $y>0$. However, Levy, Feller, and others have worked out some limiting formulas[27] which can be applied here. To use these results, we write $R\equiv G^{-1}$, $r_\alpha=g_\alpha^{-1}$. From Eq. (2.18) of Weiss and Rubin,[27] one finds that when g_α is distributed as in Eq. (1) or equivalently when the probability density of bond resistances is given as $\hat{\mathbf{p}}(r)\sim r^{-(y+2)/(y+1)}$ for large r, the probability density of R, the sum of L_1 resistances, behaves asymptotically for large L_1 as

$$P_{L_1}(R)\sim\frac{L_1}{R^{(y+2)/(y+1)}}\quad\text{for } y>0\ . \tag{7}$$

Then the median is given by

$$\frac{L_1}{R^{1/(y+1)}}\sim\frac{1}{2}\ .$$

Thus a typical value of the sum scales, roughly speaking,

as

$$R \sim L_1^{y+1} \quad \text{for } y > 0 .$$

For $y \leq 0$, one obtains

$$R \sim L_1 \quad \text{for } y \leq 0 .$$

Thus the macroscopic conductivity of the entire network $\Sigma \approx \xi^{2-d} G$, will therefore be given as $\xi^{2-d} L_1^{-1}$, for $y < 0$ and $\xi^{2-d} L_1^{-(y+1)}$, for $y > 0$, leading to

$$\begin{aligned} &\bar{t} \geq 1 + (d-2)\nu \equiv t_1, \quad \text{if } y < 0 \\ &\bar{t} \geq 1 + (d-2)\nu + y \equiv t_1 + y, \quad \text{if } y > 0 . \end{aligned} \tag{8}$$

A more intuitive derivation of these results can be obtained as follows. We can estimate the sum in Eq. (6) by an integral over the probability distribution $P(\delta)$, provided that, first, L_1 is very large (i.e., we are sufficiently close to the percolation threshold), and second, we properly control the contribution of the weakest bond on the string. In particular, we have

$$G^{-1} \gtrsim \frac{1}{\delta_{\min}^{y+1}} + L_1 \int_{\delta_{\min}}^{\infty} \frac{P(\delta)}{\delta^{y+1}} d\delta , \tag{9}$$

where $\delta_{\min}$ is the minimum value of δ_α along the L_1 singly connected bonds on the string. The probability distribution for $\delta_{\min}$ may itself be computed by considering the probability that *none* of the L_1 singly connected bonds has width smaller than a specified value ϵ. One thus finds

$$\text{Prob}(\delta_{\min} \geq \epsilon) = \left[1 - \int_0^{\epsilon} P(\delta) d\delta \right]^{L_1} \approx e^{-\epsilon L_1 / \delta_0} . \tag{10}$$

Thus we see that the typical value of $\delta_{\min}$ is of order δ_0 / L_1. (See also the discussion at the end of this subsection.)

In the case where $y < 0$, the integral in Eq. (9) converges in the limit $\delta_{\min} \to 0$. Moreover, the term $1/\delta_{\min}^{y+1}$ is typically of order L_1^{y+1} which is negligible compared to the second term. Thus, we find for $y < 0$,

$$G^{-1} \gtrsim L_1 \langle g^{-1} \rangle , \tag{11a}$$

where $\langle \ \rangle$ denotes an average over the distribution $P(\delta)$. In this case, the upper bound of the resistance of the string depends on the *mean resistance* of a bond, in the same manner as if all the bonds on the string had equal strengths. By contrast, for $y > 0$, the bound on G^{-1} given by Eq. (9) is typically determined by $\delta_{\min}$. Using either of the terms of Eq. (9), one now finds that for almost all strings,

$$G^{-1} \gtrsim L_1^{y+1} / \delta_0^{y+1} . \tag{11b}$$

We now use Eq. (11) in the estimate $\Sigma \approx \xi^{2-d} G$ of the conductivity of the network. We thus obtain a lower bound to the percolation conductivity exponent $\bar{t}$, as given in Eq. (8) above.

The above derivation can be easily generalized to treat the elasticity and permeability problems on percolation networks with wide distribution of bond strengths. For the permeability problem, the above analysis goes through in exactly the same way, and one obtains

$$\begin{aligned} &\bar{e} \geq 1 + (d-2), \nu \equiv t_1, \quad \text{if } y < 0 , \\ &\bar{e} \geq 1 + (d-2)\nu + y \equiv t_1 + y, \quad \text{if } y > 0 . \end{aligned} \tag{12}$$

For the elasticity problem, however, one needs to consider the "bending moments" associated with the string displacements. In particular, let us define a force constant K for a string such that $\frac{1}{2} K u^2$ is the energy cost to displace one end of the string by a small distance u, when the last bond at the other end of the string is clamped in a fixed position and orientation. For a long chain of N bonds, one can easily derive an expression for the chain force constant K, by a generalization of the analysis of Kantor and Webman:[9]

$$K^{-1} = \sum_i^N \frac{\zeta_\alpha^2}{\gamma_\alpha} , \tag{13}$$

where γ_α is the bond-bending (or bond-twisting) force constant of bond i, and ζ_α is the moment arm of the ith bond. From this result, it is then easy to write down an upper bound to the force constant of a given string, by taking into account again only the single-connected bonds on the string,

$$K^{-1} \geq \sum_{i=1}^{L_1} \frac{\zeta_\alpha^2}{\gamma_\alpha} \tag{14}$$

where ζ_α can be assumed to be a length of the order of the correlation length ξ. If all the bonds have the same bond-bending constant, as is the case for the standard discrete percolation model, one finds $K \leq \gamma / (L_1 \xi^2)$. Since again the overall elastic moduli are proportional to $\xi^{2-d} K$, the above inequality implies the relation $f \geq f_1 \equiv d\nu + 1$, a result first obtained by Kantor and Webman.[9]

If the bond bending strengths γ_α have a singular distribution, of the form in Eq. (1) [or Eqs. (2) and (3)], we can estimate the sum in Eq. (14) by an integral, similar to Eq. (9). In particular, one finds that for $y < 0$, $K^{-1} \leq L_1 \xi^2$; and for $y \geq 0$, $K^{-1} \leq L_1^{y+1}$. Thus we obtain the lower bounds to $\bar{f}$ as

$$\begin{aligned} &\bar{f} \geq d\nu + 1 \equiv f_1, \quad \text{if } y < 0 \\ &\bar{f} \geq d\nu + 1 + y \equiv f_1 + y, \quad \text{if } y > 0 . \end{aligned} \tag{15}$$

Our use of the "typical value" of K or G to obtain the network transport coefficient may be made more precise and justifiable by means of a percolation argument, employed by Ambegaokar *et al.*[28] and Pollak[29] in their rederivations of Mott's law of hopping conductivity. Here we focus attention on the "super-network" of strings, which forms the percolating backbone at the length scale ξ. As in the hopping conduction problem, the conductances G (or the elastic force constants K) of strings in our problem assume a wide distribution of values. One finds then that the conductivity Σ (or the elastic modulus Y) of the entire system, which is connected by a quasiregular network of strings, can be approximated by that of a new network in which all of the strings less conductive than a "critical value" G_c are cut out. The value of G_c is defined such that the fraction of remaining strings is infinitesimally more than $\tilde{p}_c$, the percolation threshold of the "super" network of strings. The

reasoning behind this argument is that since the distribution of string conductances is wide, the currents are carried mostly in those strings that are not overly resistive. Thus it does not make much difference if one cuts out the most resistive strings. But, of course, the current carrying strings must still form a connected network, so that the fraction of bonds that are cut out must not exceed $1-\tilde{p}_c$. The electrical properties of the system are thus dominated by the most resistive remaining strings, which have conductance $\approx G_c$. The conductivity of the original network should thus be roughly the same as a regular network in which all strings have conductance G_c.

Now let us define a critical width parameter δ_c, by requiring that the fraction of strings with $\delta_{\min}<\delta_c$ be just equal to $1-\tilde{p}_c$. From the first term in Eq. (9), we have $G_c \le \delta_c^{y+1}$, and for $y>0$, we take our upper bound approximation to Σ as the conductivity of a regular network with string conductance δ_c^{y+1}. From Eq. (10), however, we see that

$$\delta_c = \frac{\delta_0}{L_1}\ln[1/(1-\tilde{p}_c)] \tag{16}$$

It is an essential assumption of the nodes-links-blobs picture that the percolation threshold $\tilde{p}_c$ of the quasiregular supernetwork is not especially close to either 0 or 1. Thus the logarithm in Eq. (16) is a number of order unity, and the critical value δ_c is indeed in the range of "typical values" of $\delta_{\min}$, of order δ_0/L_1, as claimed.

B. Rigorous upper and lower bounds of $\bar{t}$, $\bar{f}$, and $\bar{e}$

In this subsection, we derive the rigorous upper and lower bounds to the transport percolation exponents for a network with wide distribution of bond strengths. The treatment parallels that of Kogut and Straley,[19] and utilizes the variational principle. With a few simple theorems, we have made the argument rigorous in the elasticity case as well. We start by considering the electrical conduction problem. We shall utilize the following variational theorems.

Theorem 1. If the resistance of any bond or set of bonds in a network is increased, the resistance (or resistivity) of the entire sample must either increase or remain the same.

Proof. This theorem makes sense intuitively. The proof[30,31] is reproduced here for convenience of the reader. Consider a large system of size L. Its resistivity ρ is given by $\rho = RL^{d-2}$, where R is the resistance of the sample, measured from two opposite ends of the sample. Since R measures the overall energy dissipation of the system, we have

$$R = \sum_\alpha r_\alpha i_\alpha^2 \tag{17}$$

where i_α is the current on bond α when the total current injected into the system at one end and taken out from the other is held at unity. From the dissipation function formulation of the electrical conduction problem,[31] $\{i_\alpha\}$ are determined by minimizing the function R subjecting to the conditions that currents flowing into any internal node have to sum up to zero (Kirchhoff's first law) and the total external current $I=1$. So we can regard the resistance R as a functional of all the resistances in the network, i.e., $R=R(\{r_\alpha\})$. Now let us introduce a new network which has resistances $\{r'_\alpha\}$ such that $r_\alpha \le r'_\alpha$ for each α. We want to prove that $R(\{r_\alpha\}) \le R(\{r'_\alpha\})$. Suppose $\{i_\alpha\}$ are currents on each bond corresponding to the resistances $\{r_\alpha\}$, and $\{r'_\alpha\}$ are currents corresponding to $\{r'_\alpha\}$. Since we know that $\{i_\alpha\}$ minimizes $R=\sum_\alpha r_\alpha i_\alpha^2$, substituting $\{i'_\alpha\}$ in the above for $\{i_\alpha\}$ can not decrease the value of R, i.e., $R(\{r_\alpha\}) \le \sum_\alpha r_\alpha (i'_\alpha)^2$. Now if we substitute $\{r'_\alpha\}$ in the above for $\{r_\alpha\}$, we increase or maintain its value again, by definition. So $R(\{r_\alpha\}) \le \sum_\alpha r'_\alpha (i'_\alpha)^2 \equiv R(\{r'_\alpha\})$. This is what we wanted to prove. It follows immediately that if any bond(s) in a resistor network decreases its value of resistance, the resistance of the sample must either decrease or remain the same.

Theorem 2. If each resistance in a random network is replaced by its expectation value, the network resistance will be larger than the expectation value of the resistance of the original network.

Proof. Introduce a new network whose present resistances are all set to be the expectation value of the original network. Let us denote the currents on this new network by $\{i'_\alpha\}$ and its resistance R'. If we replace $\{i_\alpha\}$ in the functional $R=\sum_\alpha r_\alpha i_\alpha^2$ by $\{i'_\alpha\}$, from the variational principle, we obtain an upper bound to R; i.e., we have

$$R = \sum_\alpha r_\alpha i_\alpha^2 \le \sum_\alpha r_\alpha (i'_\alpha)^2 . \tag{18}$$

Now let us average Eq. (18) over the distribution of $\{r_\alpha\}$. We then arrive at the inequality $\langle R\rangle \le \sum_\alpha \langle r_\alpha\rangle (i'_\alpha)^2 \equiv R'$. The "reciprocal" of this theorem is also true.

Theorem 3. If the *conductances* in a random network are replaced by their expectation values, the new *conductivity* will be larger than the expectation value of the conductivity of the original network.

Proof. This theorem can be proved by using the formula for the conductance of the system $G=\sum_\alpha [(\Delta V_\alpha)^2/r_\alpha]$, where ΔV_α is the voltage drop across resistance α and a unit voltage drop is kept across the two opposite sides of the macroscopic sample. Once again we have the variational principle that the proper solution $\{\Delta V_\alpha\}$ are such that functional G is minimized, subject to the second law of Kirchhoff, i.e., the sum of the voltage drops along any closed loop inside the network is zero.[31] (The constraint is satisfied automatically if we write $\Delta V_\alpha = V_i - V_j$, where V_i and V_j are the voltages at the two sites connected by bond α. Minimization with respect to V_i then leads to Kirchhoff's first law: $\sum^{(i)} i_\alpha = 0$, where the sum is over all bonds connected to site i and $i_\alpha \equiv \Delta V_\alpha / r_\alpha$.) The remainder of the proof is similar to that of Theorem 2.

Theorems 2 and 3, in a continuum form, are the basis for the well known bounds of Hashin and Strickman,[30] on the conductivity, dielectric constant, and magnetic permeability of a composite medium. With these theorems, the rigorous upper and lower bounds for $\bar{t}$ can be derived very easily. The random resistor network under consideration contains fraction $1-p$ missing bonds and fraction p occupied ones; and the occupied bonds follow a

power-law distribution which can be formulated in terms of a random width parameter δ_α, such that $r_\alpha = \delta_\alpha^{-y-1}$ and $P(\delta) \to 1/\delta_0$ for $\delta \to 0$. Let us denote the resistivity of this network by $\rho = 1/\Sigma$, and denote this network by O. We now prepare a new variational system (network N) by removing all resistances in network O that are higher than $r_v \equiv \delta_v^{-y-1}$, where $\delta_v/\delta_0 \ll 1$, and the precise value of δ_v will be chosen later. Note that the fraction of occupied bonds in this new system is $p(1-\delta_v/\delta_0)$. We know from Theorem 1 that the resistivity of N is an upper bound to that of network O. Now using Theorem 2, we know that an upper bound to the resistivity of network N is the resistivity of still another network N', obtained from network N by replacing all its nonvanishing resistances by their ensemble average value, which is given by

$$\bar{r}_v = \frac{\int_{\delta_v}^{\infty} \frac{P(\delta)}{\delta^{y+1}} d\delta}{(1-\delta_v/\delta_0)} . \tag{19}$$

For small values of δ_v, the above equation may be evaluated simply to give

$$\bar{r}_v = \begin{cases} \text{const} & \text{for } y < 0 \\ \text{const}/\delta_v^y & \text{for } y > 0 . \end{cases} \tag{20}$$

It is apparent that the last network N' is a standard lattice percolation model with occupation probability $p(1-\delta_v/\delta_0)$ and resistance $\bar{r}_v$. Near percolation, the resistivity of network N' is given by $\rho' = \text{const} \times \bar{r}_v (p - p_c - p\delta_v/\delta_0)^{-t}$, where t is the exact percolation conductivity exponent. Thus we have the inequality

$$\langle \rho \rangle \leq \text{const} \times \bar{r}_v (p - p_c - p\delta_v/\delta_0)^{-t} . \tag{21}$$

Now we can vary δ_v to obtain the closest upper bound to the resistivity of the original network O. Since $p \gtrsim p_c$, the optimal value δ_v is small, thus the scaling form of $\bar{r}_v$ in Eq. (20) may be used. For $y < 0$, the optimal value of $\bar{\delta}_v$ is zero, while for $y > 0$, the optimal $\bar{\delta}_v$ is given by $\bar{\delta}_v/\delta_0 \approx y(p-p_c)/p_c(t+y)$. As given in the work of Kogut and Straley,[19] the result is

$$\langle \rho \rangle \leq \text{const} \times \begin{cases} (p-p_c)^{-t} & \text{for } y < 0 \\ (p-p_c)^{-(t+y)} & \text{for } y < 0 . \end{cases} \tag{22}$$

Assuming the power-law behavior $\Sigma \sim (p-p_c)^{\bar{t}}$ or $\rho \sim (p-p_c)^{-\bar{t}}$, we arrive at an upper bound to the exponent $\bar{t}$, namely $\bar{t} \leq t$ for $y < 0$ and $\bar{t} \leq t + y$ for $y > 0$. (Alternatively, one could avoid the assumption of power-law behavior by defining the exponents t and $\bar{t}$ in terms of $\limsup[\ln\langle\rho\rangle/\ln(p-p_c)]$ or $\liminf[\ln\langle\rho\rangle/\ln(p-p_c)]$, as $p \to p_c$.)

One can now use Theorem 1 or Theorem 3 to obtain the *rigorous* variational lower bound $\bar{t} \geq t$ for all the physically relevant range of $y > -1$. If there exists a maximum value for the bond conductance $g_{\max}$, then it is easy to obtain this lower bound by changing all the occupied bonds to have the highest possible conductance $g_{\max}$, and using Theorem 1.

The rigorous lower bound $\bar{t} \geq t$ is valid even if there is no maximum value of the bond conductance g, as long as there exists an average conductance $\bar{g}$, defined as $\bar{g} = \int_0^\infty g p(g) dg$. In this case, the bound is derived by a direct application of Theorem 3.

The rigorous lower bound, when combined with the rigorous upper bound, gives the exact equality $\bar{t} = t$ for $y < 0$. For $y > 0$, the rigorous lower bound can be combined with the (nonrigorous) linear chain lower bound and rigorous upper bound to give $\max(t, t_1 + y) \leq t \leq t + y$.

The above analysis can be easily generalized to obtain the rigorous upper and lower bounds to the exponents $\bar{e}$ and $\bar{f}$. The case of $\bar{e}$ parallels precisely the above analysis, since all we need to do is to recognize the complete analogy in the two problems between the electrical conductivity and fluid permeability, and between bond electrical conductance and the bond fluid flow conductance, defined as the amount of viscous fluid flowing across a bond per unit time under a unit pressure difference across the bond.

The elasticity case warrants some more explanation. The elastic Young's modulus Y, which is analogous to Σ in the electrical case, is related to the force constant K of a macroscopic sample by $Y = KL^{2-d}$, where L is the linear size of the sample and K is the force applied to the opposite ends of the sample when a unit displacement is achieved. By virtue of the linearity of our problem, K equals twice the elastic potential energy stored in the system when the applied end-to-end displacement is equal to unity.

The form of the potential energy function is obviously model dependent. In the present work, we specify our network model to be that of a "granular elastic network model," which resembles the actual geometries of the continuum percolation systems we will consider later. This model was first introduced by Schwartz, Johnson, and Feng[32] to model sandstone rocks, and its percolation properties have been studied in some detail in Ref. 14. The main conclusion of Ref. 14 is that the percolation critical exponent f for the granular network model should be the same as its counterpart in the more conventional bond-bending network model studied earlier, and this equality is supported by numerical simulations.

Let us now describe briefly the granular network model. In this model, the sites are spherical rigid grains, whose infinitesimal motions are described by their center of mass displacements $\mathbf{u}_i$, and their rotational angles $\boldsymbol{\theta}_i$. The sites are taken to lie on a regular lattice and the bonds are taken to be randomly occupied. Three types of elastic couplings exist between two nearest-neighbor grains if the bond connecting them is occupied, and the potential function for the system can be expressed as

$$U(\{\mathbf{u}_i\},\{\boldsymbol{\theta}_i\}) = \tfrac{1}{2} \sum_{\langle ij \rangle} g_{ij} (\alpha_{ij} [(\mathbf{u}_i - \mathbf{u}_j)\cdot\hat{\mathbf{r}}_{ij}]^2 + \beta_{ij} \{\hat{\mathbf{r}}_{ij} \times [\hat{\mathbf{r}}_{ij} \times (\mathbf{u}_i - \mathbf{u}_j)] + R(\boldsymbol{\theta}_i + \boldsymbol{\theta}_j) \times \hat{\mathbf{r}}_{ij}\}^2 + \gamma_{ij} (\boldsymbol{\theta}_i - \boldsymbol{\theta}_j)^2) , \tag{23}$$

where $\hat{\mathbf{r}}_{ij}[\equiv(\mathbf{R}_j-\mathbf{R}_i)/2R]$ is the unit vector which points in the direction of the bond ij, R is the radius of the spherical grains, g_{ij} is the random parameter which assumes the value unity with probability p and zero with probability $1-p$. The force constants α_{ij}, β_{ij}, and γ_{ij} describe the elastic restoring forces for stretching, sliding and counter-rotational motions of the two neighboring grains. In Ref. 14, it has been shown that the counter-rotation of our present model corresponds to the bond-bending model in the conventional bond-bending motion; therefore one can take the force constant γ_{ij} to be the bond-bending force constant if one wishes.

Now that we have defined our model, we can state the variational principle for the elastic problem: The force constant K can be obtained by minimizing its corresponding functional $K(\{\mathbf{u}_i\},\{\boldsymbol{\theta}_i\})=2U(\{\mathbf{u}_i\},\{\boldsymbol{\theta}_i\})$ where U is the potential functional with the boundary conditions that grains on the two opposite sides of the sample are held fixed with a unit relative displacement and with no relative rotation. By comparing with the analysis we just gave on the electrical case, we can see that the generalized displacements $\{\mathbf{u}_i\}$ and $\{\boldsymbol{\theta}_i\}$ in the elastic case play the corresponding role of the site voltages V_i, forces and torques exerted by nearest neighbor grains upon each other correspond to the node-to-node currents, and α_{ij}, β_{ij}, and γ_{ij} play the role of the conductance g in the electrical case. It is then a relatively easy matter to prove theorems that are analogous to Theorem 1 and Theorem 3.

Theorem 1E. If any of the elastic constants (α, β, or γ) of any bond (grain to grain contact) is decreased, the force constant K (or the elastic modulus Y) must either decrease or remain the same. Also, if any of the elastic constants of any bond is increased, the force constant K must either increase or remain the same.

Theorem 3E. If each elastic constant of each bond in a random network is replaced by its expectation value, the new elastic modulus Y will be larger than the expectation value of the elastic modulus Y of the original network.

To prove a theorem for our elasticity problem that is analogous to Theorem 2, we need to formulate a variational principle for the elasticity problem which corresponds to the resistivity formulation in the electrical case. Let us define the force transmitted by bond ij as

$$\mathbf{f}_{ij}=-\mathbf{f}_{ji}=\alpha_{ij}\hat{\mathbf{r}}_{ij}[(\mathbf{u}_i-\mathbf{u}_j)\cdot\hat{\mathbf{r}}_{ij}] +\beta_{ij}\hat{\mathbf{r}}_{ij}\times[\hat{\mathbf{r}}_{ij}\times(\mathbf{u}_i-\mathbf{u}_j)+R(\boldsymbol{\theta}_i+\boldsymbol{\theta}_j)] , \tag{24a}$$

and let us define the additional torque due to counter-rotation of the bond by

$$\mathbf{t}_{ij}=-\mathbf{t}_{ji}=\gamma_{ij}(\boldsymbol{\theta}_i-\boldsymbol{\theta}_j) . \tag{24b}$$

For any site i in the interior of the sample, the equilibrium condition is

$$\mathbf{F}_i\equiv\sum_j \mathbf{f}_{ij}=0 ,$$

$$\mathbf{T}_i\equiv\sum_j (\mathbf{t}_{ij}+R\hat{\mathbf{r}}_{ij}\times\mathbf{f}_{ij})=0 ,$$

where the sums are over all neighbors to i, so that $\mathbf{F}_i$ and $\mathbf{T}_i$ are the total force and torque on site i.

We next define a function

$$W\{\mathbf{f}_{ij},\mathbf{t}_{ij}\}\equiv\tfrac{1}{2}\sum{}'\left[\frac{(\hat{r}_{ij}\cdot\mathbf{f}_{ij})^2}{\alpha_{ij}}+\frac{(\hat{\mathbf{r}}_{ij}\times\mathbf{f}_{ij})^2}{\beta_{ij}}+\frac{\mathbf{t}_{ij}^2}{\gamma_{ij}}\right] , \tag{24c}$$

where the sum is over all bonds in the "infinite" cluster, i.e., the cluster that connects the two ends of the sample at $x=L/2$ and $x=-L/2$. We shall now consider $\mathbf{f}_{ij}$ and $\mathbf{t}_{ij}$ to be arbitrary vectors with $\mathbf{f}_{ij}\equiv-\mathbf{f}_{ji}$, $\mathbf{t}_{ij}\equiv\mathbf{t}_{ji}$, and we shall then seek to minimize W, subject to the constraints that $\mathbf{F}_i=\mathbf{T}_i=0$ for all sites on the infinite cluster, except for sites at the ends $x=\pm L/2$, where we require $\mathbf{F}_i=\pm\hat{\mathbf{x}}$, $\mathbf{T}_i=0$. In order to perform the minimization we may introduce Lagrange multipliers $\mathbf{u}_i$ and $\boldsymbol{\theta}_i$, and minimize the function $W'=W-\sum_i(\mathbf{u}_i\cdot\mathbf{F}_i+\boldsymbol{\theta}_i\cdot\mathbf{T}_i)$, without regard to the constraints. The equations $\partial W'/\partial\mathbf{f}_{ij}=0$ and $\partial W'/\partial\mathbf{t}_{ij}=0$ lead directly to Eqs. (24a) and (24b), and we may therefore identify $\mathbf{u}_i$ and $\boldsymbol{\theta}_i$ with the displacement and the rotation angle of grain i.

We see that minimization of W with the required constraints gives the correct equilibrium solution for $\mathbf{f}_{ij}$ and $\mathbf{t}_{ij}$. It is also clear, that W is equal to the elastic energy stored in the system, and that W Is inversely proportional to the macroscopic Young's modulus Y of the system. Using this version of the variational principle, one easily proves the required theorem.

Theorem 2E. If the inverse elastic constants $1/\alpha_{ij}$, $1/\beta_{ij}$, and $1/\gamma_{ij}$ of each bond in a random network are replaced by their expectation values, the inverse elastic modulus $1/Y$ must be larger than or equal to the expectation value of the elastic modulus $1/Y$ of the original network.

Now with these theorems for the elastic problem, we can derive the rigorous upper and lower bounds to the exponent $\bar{f}$. Having in mind the physical origin of continuum systems to our network model with variable bond strengths, we assume the distribution of the various bond elastic constants to have the form $\alpha_{ij}=a\delta_{ij}^{q+1}$, $\beta_{ij}=b\delta_{ij}^{r+1}$, and $\gamma_{ij}=c\delta_{ij}^{y+1}$, where a, b, c are constants, δ_{ij} is the "width" random variable introduced earlier in the analysis of the electrical problem. It does not actually matter for our analysis whether there is a single random variable δ_{ij}, for each bond, or whether there are independent variables δ_{ij} for the three elastic constants, as long as $p(\delta_{ij})$ is finite in the limit $\delta_{ij}\to 0$ in each case. Now if $q=r=y$, then by following steps parallel to that in the electrical case, we derive the rigorous bounds $f\le\bar{f}\le f+y$ if $y>0$, and $\bar{f}=f$ if $y\le 0$, where f is the exact elasticity exponent for the standard lattice percolation model with bond force constants a, b, and c. (In proving the lower bound, we have assumed that the microscopic elastic constants α_{ij}, β_{ij}, and γ_{ij} have finite mean values.) It is straightforward to show that the exponent f is independent of the values a, b, and c, provided that all three are finite, since we obtain rigorous bounds to the Young's modulus when we replace a, b, and c by the minimum or the maximum of the three values.

The situation is more complicated when the exponents

q, r, and y are not equal. For our later applications to continuum models, we are most interested in the case where $y>q$ and $y>r$, so that the ratio of the bond stretching to counter-rotation (or bond-bending) force constants is infinite in the limit of small bond width δ_{ij}. Then the rigorous upper bound to $\bar{f}$ is replaced by the inequalities

$$\bar{f}<f+\max(q,r,0)\ , \tag{25a}$$

$$\bar{f}<f_\infty+\max(y,0)\ , \tag{25b}$$

where the quantity f_∞ is the exponent for the vanishing of the Young's modulus for $p\to p_c$ on a lattice where $\gamma_{ij}=1$ and $\alpha_{ij}=\beta_{ij}=\infty$, for all occupied bonds. According to the analysis of singly-connected bonds, only the bond-bending force constant is important near the percolation threshold, and therefore $f=f_\infty$ in that analysis. Rigorously, we only can prove that $f_\infty\le f$, but it seems most likely to us that the difference $(f-f_\infty)$ must be very small, if it is not actually zero, so that Eq. (25b) is the more useful upper bound in the cases of interest. For the sake of simplicity, we shall assume that $f_\infty=f$ in our discussion of the continuum models in Sec. III below.

We note that the rigorous lower bound $\bar{f}>f$, as well as the nonrigorous lower bound $\bar{f}>f_1+y\equiv d\nu+1+y$, for $y>0$, both remain valid in the case $y>r,p$. In the next subsection, we generalize the above results to treat the transport properties of random superconducting networks and random "superelastic" networks with singular distribution of bond strengths, below the percolation threshold.

C. Superconducting and superelastic networks with bond distributions

As a generalization of the analysis presented in the above two subsections, we consider the problem of superconductive and superelastic networks with wide distribution of bond strengths. Let us begin with the electrical case. We consider a random network in which fraction p of the bonds are "superconducting" (i.e., $g=\infty$), and the rest of the bonds are normal conductors. Unlike in the standard superconducting random network, these normal conductors are assumed to have a singular distribution of resistances (or conductances). In particular, we assume that the distribution of the bond resistances follow a power-law distribution for small resistances $r\,(r\equiv 1/g)$,

$$p_s(r)\sim r^{-\alpha},\quad \text{for } r\to 0\ . \tag{26}$$

Another representation of this distribution is given by introducing the length parameter δ, such that $r=\delta^{(y+1)}$; with $y\equiv\alpha/(1-\alpha)$. The distribution for δ has a finite limit as $\delta\to 0$, i.e., $P_s(\delta)\to\text{const}\equiv 1/\delta_0$ as $\delta\to 0$. We assume the conductivity of the system diverges as $\Sigma\sim(p_c-p)^{-\bar{s}}$ as $p\to p_c^-$.

First let us give the scaling analysis leading to the nonrigorous lower bound to $\bar{s}$, by studying the equivalent of the "nodes-links-blobs" picture below the percolation threshold.[33] Just below p_c, the superconducting regions are clusters of typical size ξ, which are the equivalent of the nodes, and are separated by a "wall" of normal conducting bonds, which are the equivalent of the links. If one denotes the conductance of a typical wall to be G', then once again we have for the conductivity of the system as $\Sigma\sim\xi^{2-d}G'$. If one looks closely at a wall, one finds that it consists of a parallel array of single bonds and bonds that form series. It has recently been shown[34,35] that the number of the single bonds between two superconducting clusters L_1' scales with $p-p_c$ the same way as L_1 above p_c, namely, $L_1'\sim(p_c-p)^{-1}$. If one only takes into account these single bonds in describing the conductivity of the system, and assumes the series bonds to be infinitely resistive, one arrives at a lower bound to the overall conductivity estimate.

Let us first consider the standard lattice model where each single bond has unit conductance. In this case a lower bound to G' is L_1' which leads to $\Sigma\ge\xi^{2-d}\xi^{1/\nu}$, thus giving a lower bound of the experiment s, i.e., $s\ge s_1\equiv(2-d)\nu+1$. For the more interesting case of bonds with wide distribution of strengths, the wall conductance has the lower bound

$$G'\ge\sum_\alpha^{L_1} g_\alpha\approx\frac{1}{\delta_{\min}^{(y+1)}}+L_1\int_{\delta_{\min}}\frac{P_s(\delta)}{\delta^{(y+1)}}d\delta\ . \tag{27}$$

Using an analysis that parallels exactly that in subsection II A, we see that $\bar{s}\ge s_1$ for $y<0$, and $\bar{s}\ge s_1+y$ for $y>0$. Note that the above analysis relies on the scaling picture of the percolation clusters below p_c, it therefore is not rigorous.

Rigorous upper and lower variational bounds to $\bar{s}$ can be obtained by use of the theorems which we proved in subsection II B. By replacing all bonds that have $\delta<\delta_v$ (or $g\ge\delta_v^{-(y+1)}$) in the original network network O by bonds of zero resistance, we obtain a network N whose conductivity is an upper bound to the conductivity of the network O, by virtue of theorem 1. By replacing in this network N all the finite conductances by their expectation value

$$\bar{g}_v=\frac{\int_{\delta_v}^\infty\frac{P_s(\delta)}{\delta^{y+1}}d\delta}{(1-\delta_v/\delta_0)}\ ,$$

one again obtains another network N' whose conductivity is an upper bound to that of network N, by virtue of theorem 3. This network N' is a standard random lattice superconducting-percolation network, and whose conductivity scales $\Sigma'\sim(p_c-p)^{-s}$. By varying the parameter δ_v, one obtains the rigorous upper bound to the exponent $\bar{s}$, i.e., $\bar{s}\le s$, for $y<0$ and $\bar{s}\le s+y$ for $y>0$.

By letting all the bond resistances in the network assume their maximum value and applying theorem 1, we find the rigorous lower bound to $\bar{s}$, i.e., $\bar{s}\ge s$. The above results can be condensed into the following inequality:

$$\max(s,s_1+y)\le\bar{s}\le s+y\ . \tag{28}$$

In two dimensions, $s_1=1$, and $s\approx 1.3$; in three dimensions, $s_1\approx 0.15$ and $s\approx 0.8$. Since the values of s_1 and s are rather far apart in 3D, the inequality (28) can be of only limited usefulness in determining the value of the exponent $\bar{s}$.

The above analysis can be easily extended to elasticity problems. Here one considers a network of perfectly rigid

bonds mixed with nonrigid ones that have finite elastic constants. For a standard lattice model, we define the exponent c such that $Y \sim (p_c - p)^{-c}$. Simple nodes-links-type scaling arguments[14] suggest that $c \approx s$. Numerical simulations of Ref. 14 indicate that c is slightly smaller than s; at least in two dimensions. A more recent numerical study[36] has suggested, however, that $c = s$.

Next we consider a model in which the nonrigid bonds have a singular distribution of elastic constants. Suppose that α_{ij}, β_{ij}, and γ_{ij} are all proportional to δ_{ij}^{-1-y}, where $p(\delta_{ij})$ is finite for $\delta_{ij} \to 0$, with $y > -1$. We define the superelasticity exponent $\bar{c}$ for this system such that $Y \sim (p_c - p)^{-\bar{c}}$. By generalizing the analysis for the electrical problem sketched above, we can show that $\max(c, s_1 + y) \leq \bar{c} \leq c + y$, for $y > 0$, and $\bar{c} = c$ for $y \leq 0$.

D. Correlated disorder

In Sec. III below, we shall map various continuum problems onto equivalent discrete networks, with a distribution of bond strengths. Unlike the networks considered above, however, the constructed networks will have some correlations among the occupation probabilities and the strengths of nearby bonds, and the underlying lattice is not regular. We must therefore argue that these modifications do not affect the critical exponents.

Let us first consider the effect of correlations in the occupation probabilities, on a regular lattice, with all bond strengths equal. There have been several studies of the effects of such correlations on the geometrical percolation exponents.[37,38] The conclusions of numerical studies, scaling arguments and renormalization-group analyses are that short-range correlations should have no effect on the critical exponents. (Of course, one must exclude pathologically strong correlations, such as those where the percolation threshold is shifted to $p_c = 0$ or $p_c = 1$.) Also, long-range correlations, which fall off as a power law of the separation at large distances, *can* change the critical exponents, if the fall-off is sufficiently slow.[38] If the underlying lattice is disordered, rather than periodic, this might be regarded as an additional source of short-range-correlated fluctuations in the bond occupations. This should have no effect on the geometrical critical exponents, therefore.

There is also good reason to believe that the transport exponents t or f are unaffected by short-range correlations in the bond occupations. This result follows for example, from the field-theoretic renormalization-group analysis in $6-\epsilon$ dimensions.[26] It is also easy to see that within the linear-chain approximation in the nodes-links-blobs picture, described in Sec. II A above, short-range correlations in the bond *occupations* will have no effect on t or f.

If correlations exist in the *strength* of bonds on neighboring sites, we can again estimate the effects on transport exponents using the linear-chain approximation of Sec. II A. (The bond strengths have no effect on the geometrical exponents, of course.) It appears that except for cases of pathologically strong correlations, the exponents $\bar{t}$, $\bar{e}$, or $\bar{f}$ will be unchanged in this approximation. For example, a sufficient condition for the analysis to hold is that the conditional probability density for the strength of bond i should vary like $g_\alpha^{-\alpha}$, for $g_\alpha \to 0$, with a fixed exponent α, regardless of the values of the conductances of the other singly-connected bonds on the same chain. It seems clear that this condition is fulfilled for the continuum examples considered in Sec. III, below.

The derivations of the rigorous bounds of Sec. II B are not applicable in the case where the strength distribution of the various occupied bonds are statistically correlated. If the bond strengths are statistically independent, however, the derivations remain valid even in the presence of correlations in the occupation probabilities or of disorder in the underlying lattice, provided that the exponents t and f, for the case where the occupied bonds have equal strengths, are unchanged from their standard lattice values.

This concludes our discussion of the bounds of transport percolation exponents for discrete random networks with power-law bond-strength distribution. In the next section, we will discuss the relevance of the above network models, i.e., we will see how the various continuum percolation transport problems can be mapped onto such networks.

III. BOND STRENGTHS IN CONTINUUM MODELS

In this section we discuss the mapping of a few continuum percolation models onto discrete random networks with wide distributions of bond strengths, and thereafter determine the critical transport percolation exponents for these models. We will consider the following continuum models individually: the "Swiss-cheese" model, the "inverted Swiss-cheese" model, and the "potential" model.

A. "Swiss-cheese" model

By the "Swiss-cheese" model or the random-spherical-void model, we mean a disordered continuum system where spherical holes are randomly placed in a uniform transport medium. A sketch of the model in two dimensions is given in Fig. 3, where the white area is the region where transport takes place, and the shaded area does not permit transport. It is apparent that there exists a critical percolation threshold q_c of the volume fraction of the punched holes, such that when $q > q_c$, the entire system ceases to support transport. If the transport medium is taken to be an electrical conductor, we can naturally define a continuum percolation conductivity exponent $\bar{t}$ as $\Sigma \sim (q_c - q)^{\bar{t}}$, where Σ is the conductivity of a Swiss-cheese system, for $q \lesssim q_c$. If the transport medium is an elastic solid, we can define an exponent $\bar{f}$ for the elastic moduli as $Y \sim (q_c - q)^{\bar{f}}$, where Y is the Young's modulus of the system. If the transport medium is taken as empty space where viscous fluid can flow through, and the spheres are taken as solid grains blocking the fluid, as is the case in a sedimentary rock, we can define $\bar{e}$ as the critical exponent describing the behavior of fluid permeability κ near q_c, by $\kappa \sim (q_c - q)^{\bar{e}}$. The permeability κ is defined as the amount of viscous fluid flow through a porous medium per unit time and unit area when a unit macroscopic pressure gra-

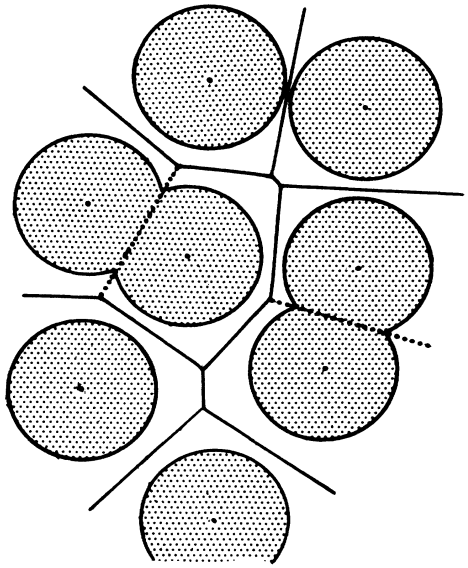

FIG. 3. Swiss-cheese model in two dimensions. Straight lines show the bonds of the superimposed discrete network; dotted lines are the missing bonds.

dient is applied to the system.

Now we proceed to discuss the mapping of the Swiss-cheese model onto a discrete random network with bond strength distributions, following the work of Kerstein[39] and Elam *et al.*[3] The discrete network is indicated by the straight line segments in Fig. 3, for the two-dimensional case. In three dimensions, the bonds are located on the edges of the Voronoi polyhedra constructed about the centers of the spherical holes. A bond is "present" or "occupied" if and only if it lies entirely within the conducting medium. (The Voronoi polyhedron, or Wigner-Seitz cell, associated with a hole centered at point $\mathbf{R}_\lambda$ is defined as the set of points closer to $\mathbf{R}_\lambda$ than to all other hole-centers in the system.)

We shall see below that the discrete network retains several important properties of the continuum problem. As was shown rigorously by Kerstein,[39] two distant regions of space are connected by the continuum conductor if and only if there is a path of open bonds on the discrete network connecting the two regions. Furthermore, if a current is flowing in the continuous medium, we may map each flow line onto the discrete network, in such a way as to determine a current flow on the network.

If the continuum system has a linear conductivity, the current through any bond in the network will be a linear function of the differences between the voltages at node points in the immediate vicinity of the bond; but the current is not in general determined precisely by the voltage difference between the two ends of the bond. Thus, strictly speaking one *cannot* assign a unique electrical resistance to each bond in the network that would precisely duplicate the current flow for all possible voltages applied to the outer-most nodes of the system. Nevertheless, we argue that it is possible to assign a unique resistance to any bond containing a very narrow neck, and that the probability distribution of such resistances should be sufficient to determine the percolation transport exponent $\bar{t}$.

For the cases of fluid flow or elastic response, we again find that for the case of a bond with a narrow neck, one can define uniquely a fluid flow resistance, or force constants that correspond to bending, twisting and stretching of the bond, and that these coefficients determine the critical transport properties at the percolation threshold. The mapping discussed below is *not* sufficiently accurate to give a reliable description of corrections to the asymptotic percolation behavior, and it may be quantitatively quite inaccurate if used outside of a narrow critical region near threshold.

1. Mapping

In his proof of the equivalence between the percolation properties in the Swiss-cheese continuum and on the lattice of edges of the Voronoi polyhedra, Kerstein[39] described a mapping of the interior of each unit cell onto its edges, which we shall also employ below. In two dimensions, Kerstein's mapping moves each point in the cell radially outward from the hole center $\mathbf{R}_\lambda$, until the point strikes an edge of the Wigner-Seitz cell. In three dimensions, the mapping proceeds in two steps. The interior point $\mathbf{r}$ is first carried in a straight line away from $\mathbf{R}_\lambda$, Until it hits a face of the Voronoi polyhedron. The point is then moved to the perimeter of the face, in a direction directly away from the point $\mathbf{X}$, which is the projection of the point $\mathbf{R}_\lambda$ onto the plane of the face. Kerstein's map has the important property that any continuous path embedded in the Swiss-cheese-model conductor is mapped into a continuous path on the edges of the polyhedra, which is also entirely inside the conducting medium.

The above construction can also be used to map a current flow in the continuum into an "equivalent" flow along the bonds. We use Kerstein's procedure to map each streamline in the conducting medium onto the edges of the Voronoi polyhedra, and we correspondingly map the motion of any particle along the streamline into a motion along the edges. The constructed edge current will clearly have zero divergence, if the continuum flow was already divergenceless. In the elasticity problem we use Kerstein's procedure similarly to convert the stress tensor distribution in the Swiss-cheese medium to a set of forces transmitted by bonds. The analogy to current flow is clear if we regard the stress component $\sigma_{\lambda\mu}$ as the current, in direction λ, of the momentum component μ, and then construct "streamlines" for the three values of μ. The equilibrium condition

$$\sum_{\lambda=1}^{3} \frac{\partial \sigma_{\lambda\mu}(\mathbf{r})}{\partial r_\lambda} = 0 \tag{29}$$

assures that there is vanishing force on each node, in the "equivalent" discrete network.

2. Description of bonds

We now consider a fixed set of hole centers $\{\mathbf{R}_\lambda\}$, and

examine the process by which various bonds are removed from the conducting network, as the sphere radius a is allowed to increase. In three dimensions, the edge of a Voronoi polyhedron is the set of points which are equidistant from three adjacent hole centers $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$, while further away from all other hole centers in the system. We shall classify the possible bonds into four types, represented by various combinations of the end and side views in Figs. 4 and 5.

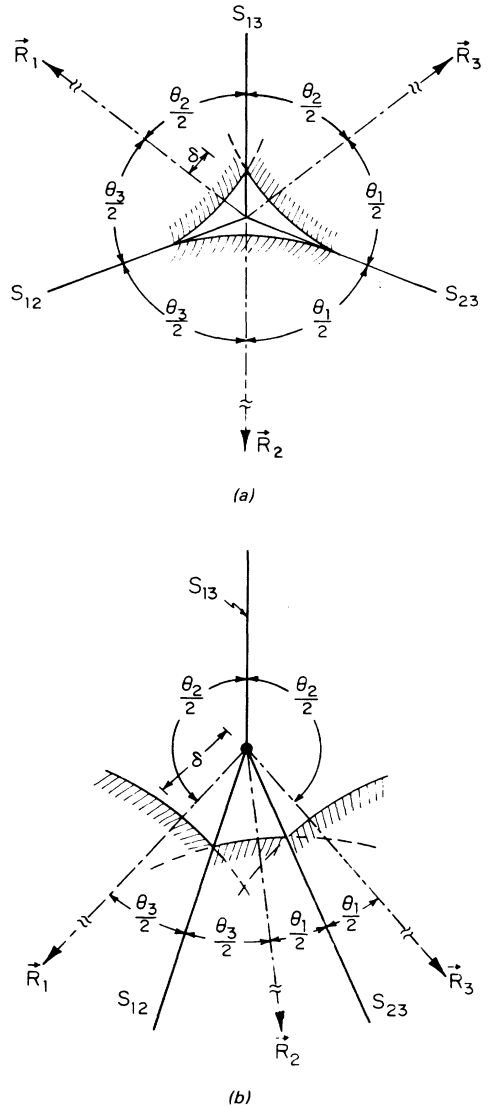

FIG. 4. End views of bonds in the Swiss-cheese model. Arrows point to hole centers $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$, separated by angles θ_1, θ_2, and θ_3. Solid lines are the projections of faces of the three Voronoi polyhedra, which meet at the bond. Shaded curves are the boundaries of the spherical holes centered at $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$, and $\delta=(s-a)$ is the distance from those boundaries to the projection of the bond. In part (a), the three angles θ_1, θ_3, and θ_3 are all less than 180°, as in bond types I and III; and in part (b) angle θ_2 is greater than 180°, as in bond types II and IV.

We shall first discuss bond types I and II, in which the bond in question passes through the phase containing the three adjacent centers. (The bond is clearly perpendicular to this plane.) We shall characterize the bond by the distance s to the three adjacent hole centers, and the three angles θ_1, θ_2, and θ_3 between the lines from the bond to the hole centers.

End views of bond types I and II are illustrated, respectively, in Figs. 5(a) and 5(b). For the type-I bond, the three angles θ_1, θ_2, and θ_3 are each less than 180°. For the type-II bond, one of the angles is greater than 180°.

Consider first the type-I bond in Fig. 5(a). (A perspective view of this bond is given in Fig. 1.) The arrows in Fig. 4 point in the directions of the adjacent hole centers $\mathbf{R}_1$, $\mathbf{R}_3$, and $\mathbf{R}_3$, while the solid lines bisecting the angles θ_1, θ_2, and θ_3 represents the faces S_{23}, S_{13}, and S_{12} of the three Voronoi polyhedra which meet at the edge. The dotted curves are the boundaries of the spherical holes, where they intersect the plane through the centers $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$. The triangular region between the circles is the smallest cross-section of the bond, and it is easy to see that in the limit $a \rightarrow s$, the cross-section area is given by

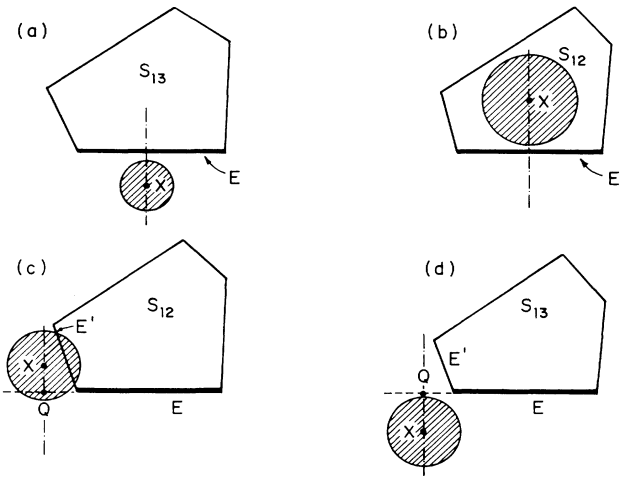

FIG. 5. Side views of bonds in the Swiss-cheese model. Plane of each part coincides with one face of a Voronoi polyhedron. The heavy line E represents the edge which is viewed end on in Fig. 4(a) or 4(b); the dot-dashed line represents the intersection of the plane of the figure with the plane containing the hole centers $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$; the points Q in parts (c) and (d) are the intersection of this plane with the linear extension of bond E. The points X are the projections of the "centers" of the two polyhedra which share the face in the plane of the figure, while the shaded curves are the intersections of the associated spherical holes with this plane. Cases (a) and (d) are two possibilities corresponding to the face S_{13} in Fig. 4, while cases (b) and (c) correspond to the faces S_{12} and S_{23} in Fig. 4(b), or to *any* of the three faces shown in the end view of Fig. 4(a). Thus (a) and (d) refer to bond types II and IV, respectively; view (b) may represent bond type I or II; view (c) may represent type III or IV.

$$A=\delta^2\left[\tan\left[\frac{\theta_1}{2}\right]+\tan\left[\frac{\theta_2}{2}\right]+\tan\left[\frac{\theta_3}{2}\right]\right], \tag{30}$$

where $\delta\equiv s-a$. The electrical resistance of the bond may be computed in the limit $\delta\ll a$, as

$$\frac{1}{g}\approx\frac{1}{A\sigma_0}\int_{-\infty}^{\infty}dz\frac{\delta^2}{\left[\delta+\frac{z^2}{2a}\right]^2}=\frac{\pi(a\delta)^{1/2}}{2^{1/2}A\sigma_0}, \tag{31}$$

where σ_0 is the conductivity of the Swiss-cheese material, z is the distance along the bond, and we assume the ends of the bond are much farther than $(\delta a)^{1/2}$ from the plane of the smallest cross section. The integrand in Eq. (31) is just the ratio of the smallest cross-sectional area to that at z, provided that $(z/a)\ll 1$. We see that as $\delta\to 0^+$, the bond conductance vanishes as $\delta^{3/2}$.

Next, consider the bond illustrated in Fig. 4(b), where one of the angles, θ_2, is greater than 180°. In this case, as $(s-a)\to 0$, the face S_{13} remains entirely outside the spheres, and continues to conduct when the bond is pinched off. A side view of the type-II bond, from a direction normal to S_{13} is illustrated schematically in Fig. 5(a). One can see from this figure that any current path which is mapped onto the surface S_{13}, in the first stage of the Kerstein construction, will be necessarily assigned to bonds on the perimeter other than the type-II bond (labeled E) and all of these bonds remain open when E is first pinched off.

A view of the type-II bond normal to the surface S_{12} or S_{23} is illustrated in Fig. 5(b), which may also be used to illustrate the type-I bond, from any of its faces. We see that the current paths on the surfaces S_{12} and S_{23} are necessarily pinched off at the critical hole radius $(a=s)$ for this bond. Thus in the Kerstein construction, the current assigned to the type-II bond must also vanish as $(s-a)\to 0$. It does not really matter how small a conductance we assign to the bond for small values of $(s-a)$, however, because *the type-II bond will be shorted out by the other conductances on the perimeter of* S_{13}. (Because of this, a type-II bond can never be one of the singly connected bonds in the nodes-links-blobs description of the percolation backbone.)

We now turn to bond types III and IV, where the plane containing $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$ intersects the line of the bond at a point Q which is off one end of the bond. In this case, the bond will be pinched off from the end, rather than the middle. We again define the angles θ_1, θ_2, and θ_3 between the vectors from Q to the hole centers.

In the type-III case, all three angles are less than 180°, and the end view is illustrated in Fig. 4(a), while a side view, projected onto the plane of *any* of the three faces S_{23}, S_{13}, or S_{12}, will look schematically as indicated in Fig. 5(c). The type-IV bond has one of the angles, θ_2, greater than 180°, and its end view is given by Fig. 4(b). Now, a side view, projected onto the plane of S_{13}, resembles the sketch in Fig. 5(d). In order to discuss the contribution of a bond of type III or IV to the network resistance, it is necessary to consider simultaneously the geometry of the three other bonds connected to the critical end of the bond. One such connecting bond is shown in each of the Figs. 5(c) and 5(d), and labeled E'. For the case shown in 5(c), it is seen that bond E' is actually pinched off before the end of bond E is reached by the growing sphere. For the case shown in 5(d), bond E' is removed from the system at the same time as bond E. In this case, the remaining bonds on the perimeter of the illustrated face remain open, as E' and E are removed, and these bonds carry any current that might have previously flowed from E' to E.

We find that all bonds connected to the critical end of a bond of type III or IV will behave like the bond E' in one or the other of Fig. 5(c) or 5(d). Thus there is a very small current through bond E, just prior to its removal from the system, and the resistance of this bond is of no consequence for the critical behavior of the network near percolation.

3. *Distribution of conductances*

Finally, we must compute the probability density $p(g)$, that a given bond has conductance g, in the limit as $g\to 0$. For the reasons stated above, we need only consider type-I bonds, which pass through the plane of the adjacent hole center and have θ_1, θ_2, and $\theta_3<180°$.

Let $p_0(s,\theta_1,\theta_2)$ be the probability density for finding a bond with given values of s, θ_1 and θ_2. (The remaining angle is determined by $\theta_3=2\pi-\theta_1-\theta_2$). It is clear that p is a smooth function of its variables. This probability distribution is independent of the choice of a, in the model we are considering, so there is nothing special about the value $s=a$ or $\delta=0$. The function $f(\theta_1,\theta_2)\equiv p_0(a,\theta_1,\theta_2)$ will be finite and nonzero for θ_1,θ_2 in the range of interest. We find, using Eq. (30) and Eq. (31), that

$$p(g)=\int_a^{\infty}ds\int d\theta_1 d\theta_2 p_0(s,\theta_1,\theta_2)\delta\left[g-\frac{2^{1/2}\sigma(s-a)^{3/2}\left[\tan\left[\frac{\theta_1}{2}\right]+\tan\left[\frac{\theta_2}{2}\right]+\tan\left[\frac{\theta_3}{2}\right]\right]}{\pi a^{1/2}}\right]$$

$$\approx\frac{2^{2/3}\pi^{2/3}a^{1/3}}{3g^{1/3}\sigma_0^{2/3}}\int d\theta_1 d\theta_2\frac{f(\theta_1,\theta_2)}{\left[\tan\left[\frac{\theta_1}{2}\right]+\tan\left[\frac{\theta_2}{2}\right]+\tan\left[\frac{\theta_3}{2}\right]\right]^{2/3}}. \tag{32}$$

The angular integrations in this equation are restricted to the region where θ_1, θ_2 are in the range $(0,\pi)$ while $\theta_1+\theta_2$ is in the range $(\pi, 2\pi)$; there is no problem with convergence of the integral.

The result $p(g) \propto g^{-1/3}$ coincides with the form obtained in Sec. II for the case where g is a function of a single variable δ, when $g \propto \delta^{3/2}$ and $P(\delta=0^+)$ is finite. In two dimensions, the equivalent result is that $g \propto \delta^{1/2}$; this is essentially the conductance of a rectangle of width δ and length $(\delta a)^{1/2}$.

The difference between the permeability and conductivity problems in the Swiss-cheese model arises from the different behaviors of the bond strengths, for small δ. In three dimensions, the fluid flow conductance of a viscous fluid through a narrow channel like that in Fig. 1 is proportional to $\delta^4/l \propto \delta^{7/2}$, while in a two-dimensional version of the model, the fluid flow conductance varies as $\delta^3/l \propto \delta^{5/2}$. These results can be easily derived from standard Poiseuille theory for viscous fluid flow in fluid dynamics.

In the elasticity problem, we need to find the bond-bending force constant γ that describes the energy cost $\epsilon = \frac{1}{2}\gamma\theta^2$ of bending a bond or twisting a bond through a specified small angle θ. For a bond that has a width δ, γ is given, up to a constant of order unity, by

$$\begin{aligned} &\gamma \sim Y_0 \delta^{5/2}/a^{1/2} \quad \text{in } d=2 , \\ &\gamma \sim Y_0 \delta^{7/2}/a^{1/2} \quad \text{in } d=3 , \end{aligned} \tag{33}$$

where Y_0 is the Young's modulus of the transport elastic medium. These results can be understood if we again approximate a neck by a thin rectangle or cylinder of width δ and length $l \approx (\delta a)^{1/2}$, and use the classical result $\gamma = Y_0(I/l)$ for a bent-beam problem, where I is the moment of inertia of the cross section of the beam. In the three-dimensional case, for example, we have

$$I = \int_{\text{c.s.}} x^2 dx\, dy , \tag{34}$$

where the integral is taken over the cross sectional area of the elastic beam, and x is measured from the middle plane which is perpendicular to the direction of bending. Thus in $d=2$, we have $I \sim \delta^3$, and in $d=3$, we have $I \sim \delta^4$. We remark that the bond-stretching force constant varies like the conductivity, for $\delta \to 0$, and is therefore much larger than γ in this limit.

From the above, we see that the different percolation transport properties of the Swiss-cheese model can be analyzed in terms of an equivalent discrete random network with a wide distribution of bond strengths, near percolation, which was studied in some length in the preceding section. In particular, the index y of the bond conductance distribution for the electrical conductivity problem is $y = -\frac{1}{2}$ in $d=2$, and $y = \frac{1}{2}$ in $d=3$; while for the elastic moduli problem and the viscous fluid permeability problems one finds $y = \frac{3}{2}$ in $d=2$ and $y = \frac{5}{2}$ in $d=3$. These values of y in turn determine the values of the percolation transport exponents, as discussed in the preceding section.

These results are summarized in Table I(a). The numbers listed in the Table I(a) are actually the quantities $y' \equiv \max(y,0)$ in each case. The inequalities for the transport exponents derived in Sec. II may then be written

TABLE I. Estimates of the differences between the transport percolation exponents in the continuum models ($\bar{t}$, $\bar{f}$, and $\bar{e}$) and the corresponding exponents on a discrete lattice. Nonzero entries in the table correspond to the upper bounds in Eq. (5), and therefore are slight overestimates of the actual values. (a) The Swiss-cheese model; (b) the inverted Swiss-cheese or potential model.

	Conductivity $(\bar{t}-t)$	Elasticity $(\bar{f}-f)$	Permeability $(\bar{e}-t)$
		(a)	
$d=2$	0	$\frac{3}{2}$	$\frac{3}{2}$
$d=3$	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{5}{2}$
		(b)	
$d=2$	0	0	0
$d=3$	0	$\frac{1}{2}$	$\frac{1}{2}$

$$\max(t, t_1+y') \le \bar{t} \le t+y' , \tag{35a}$$

$$\max(t, t_1+y') \le \bar{e} \le t+y' , \tag{35b}$$

$$\max(f, f_1+y') \le \bar{f} \le f+y' , \tag{35c}$$

where $\bar{t}$, $\bar{e}$, and $\bar{f}$ are the exponents for the Swiss-cheese model, t and f are the exponents for conductivity and elasticity in the standard discrete lattice problem, $t_1 \equiv 1+(d-2)\nu$ is a lower bound to t, and $f_1 \equiv 1+d\nu$ is the lower bound for f first derived by Kantor and Webman.[9]

B. Inverted "Swiss-cheese" model

In the "inverted Swiss-cheese" model, or the inverted spherical-void model, the roles of the two different media are switched, i.e., the transport medium is now the space occupied by the spheres, while the region between the spheres does not support transport. A mapping onto an equivalent discrete network with bond strength distributions can be made analogously to the Swiss-cheese model. Now the nodes of the network are the sphere centers, and the bonds are lines joining the neighboring sphere centers. The shape of an occupied bond is shown for the $d=2$ case in Fig. 6. In $d=3$ the shape of a bond is the rotation of the structure in Fig. 6 along the axis of the dot-dashed line. The critical transport percolation exponents can be defined, for example, as $\Sigma \sim (q-q_c)^{\bar{t}}$, for $q \gtrsim q_c$, where q is the volume fraction of the spheres.

The strength of a bond of overlap δ_α can be analyzed as follows. We define $\delta_\alpha \equiv 2a - s_\alpha$, where s_α is the separation between the centers of the two spheres in question, and a is the sphere radius. The diameter Δ_α of the constricted region in the bond is related to δ_α by $\Delta_\alpha = (2\delta_\alpha a)^{1/2}$, where a is the radius of the spheres. For electrical conduction, or for viscous fluid flow through an aperture, we expect that the region of maximum dissipa-

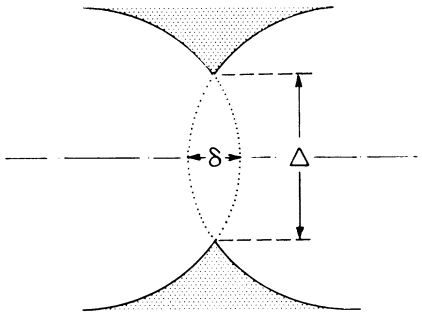

FIG. 6. Neck geometry in the inverted Swiss-cheese model.

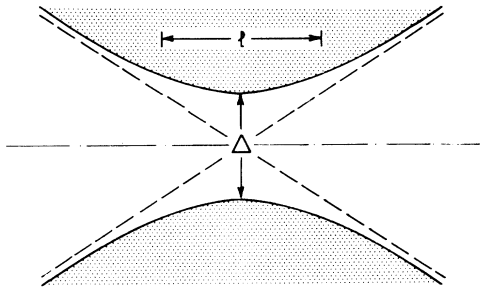

FIG. 7. Geometry of a narrow neck in the potential model.

tion will extend for a distance of order $\Delta_\alpha/2$ on either side of the aperture. This expectation can be easily justified by studying the electrical conduction through a thin insulating wall with a circular hole of radius $\Delta_\alpha/2$. It is easy to show that the electrical field differs significantly from the uniform external field only in a region of dimension $\Delta_\alpha/2$ near the hole, in both the direction perpendicular to the hole and in directions which lie within the plane of the wall. Thus we may approximate the bond by a cylinder with length l_α equal to its diameter $\Delta_\alpha \sim \delta_\alpha^{1/2}$. We therefore find that the conductance g_α for bond α, up to constant of order unity, is given in $d=2$ by $g_\alpha \approx \Delta_\alpha/l_\alpha \sim \delta_\alpha^0$, and in $d=3$ by $g_\alpha \approx \Delta_\alpha^2/l_\alpha \sim \delta_\alpha^{1/2}$. [More precisely, $g_\alpha \propto \ln(a/\delta_\alpha)$, for $d=2$.] This gives the value of the distribution index y for the electrical conduction problem $y=-1$ in $d=2$ and $y=-\frac{1}{2}$ in $d=3$, both giving rise to no correction to the conductivity exponent. For the fluid permeability problem, however, we see that the fluid flow conductance of a bond scales as $g_\alpha \approx \Delta_\alpha^3/l_\alpha \sim \delta_\alpha^{1/2}$ in $d=2$, and $g_\alpha \approx \Delta_\alpha^4/l_\alpha \sim \delta_\alpha^{3/2}$ in $d=3$, giving rise to $y=-\frac{1}{2}$ in $d=2$ and $y=\frac{1}{2}$ in $d=3$. In the elasticity problem, we notice that the bond-bending or twisting constant of a bond can again be obtained from classical beam-bending theory, with $\gamma_\alpha \propto I_\alpha/l_\alpha$. The moment of inertia for the cross section is given by $I_\alpha \sim \Delta_\alpha^3$ in $d=2$ and $I_\alpha \sim \Delta_\alpha^4$ in $d=3$. Thus $y=-\frac{1}{2}$ in $d=2$ and $y=\frac{1}{2}$ in $d=3$ for this problem. It is clear, also, that the distribution $P(\delta)$ for this model is finite for $\delta \to 0^+$. The corrections to the transport exponents can therefore be read off from the analysis of Sec. II, and the results are listed in Table I(b).

C. Potential model

In the subsection we consider yet another class of continuum models, namely the potential model, where the transport regions are the portions of space where a specified smooth stochastic "potential-function" $V(\mathbf{r})$ is less than some cutoff value V^*. A two-dimensional experimental realization of the potential model was studied by Smith and Lobb,[40] who used the intensity of a laser-speckle pattern to generate the function $V(\mathbf{r})$, and used high contrast film and photolithographic techniques to produce the two-dimensional conducting sample. Weinrib[41] later discussed the mapping of this $d=2$ potential model onto a discrete network. A similar construction was suggested earlier by Ziman,[42] for a three-dimensional model. In these constructions, one associates nodes of the discrete network with local minima of the potential function, and one associates bonds with the saddle points connecting two valleys. We assume that in the vicinity of a saddle point, the potential $V(\mathbf{r})$ may be expanded as

$$V(\mathbf{r}) = V(\mathbf{r}_\alpha) + \tfrac{1}{2} \sum_{\alpha\beta} (\mathbf{r}-\mathbf{r}_\alpha)_\alpha (\mathbf{r}-\mathbf{r}_\alpha)_\beta U_{\alpha\beta} , \tag{35}$$

where $\mathbf{r}_\alpha$ is the position of the saddle point, and $U_{\alpha\beta}$, the matrix of second derivatives of V, has one negative eigenvalue λ_1, and $(d-1)$ positive eigenvalues, $\lambda_2, \ldots, \lambda_d$. If $h_\alpha \equiv V^* - V(\mathbf{r}_\alpha)$ is negative, the bond at $\mathbf{r}_\alpha$ is unoccupied; but if h_α is positive, the bond is occupied. At its narrowest point, the bond has an elliptical cross section with the principal diameters $\Delta_2 = (8h_\alpha/\lambda_2)^{1/2}$, $\Delta_3 = (8h_\alpha/\lambda_3)^{1/2}$, etc. (see Fig. 7). If h_α is small, the length l_α of the constricted region of the bond is given by $l_\alpha \approx [8h_\alpha/(-\lambda_1)]^{1/2}$, which is of the same order as the diameter Δ_2, if we assume that all eigenvalues of $U_{\alpha\beta}$ have similar magnitudes.

Now we can proceed to compute the transport bond strengths distribution exponents y of the equivalent discrete network corresponding to this continuum model. As in the inverted Swiss-cheese model, we may replace a bond by a cylinder of length l and diameter Δ_2. Then the electrical conductance g_α of a bond with height parameter h_α is proportional to $h_\alpha^{1/2}$ in $d=3$ and h_α^0 in $d=2$. For the fluid flow through the bond we find $g_\alpha \propto h_\alpha^{3/2}$ in $d=3$ and $g_\alpha \propto h_\alpha$ in $d=2$. Similarly, the bond-bending and torsion constants obey $\gamma_\alpha \propto h_\alpha^{3/2}$ in $d=3$ and $\gamma_\alpha \propto h_\alpha$ in $d=2$.

Finally, we note that except for pathological cases,[43] we may assume that the distribution of saddle-point values $V(\mathbf{r}_\alpha)$ is regular in the vicinity of the critical value, so that the probability distribution $P(h_\alpha)$ has a finite value $P(0)$ in the limit $h_\alpha \to 0^+$. Thus we see that formally the potential model is equivalent to the inverted Swiss-cheese model discussed in the last subsection, in so far as its percolation transport properties are concerned, and the exponents are those indicated in Table I(b).

IV. SUMMARY AND DISCUSSIONS

We are now ready to summarize our results and discuss some of their consequences.

A. Summary

We may summarize our theoretical results as follows. If we consider a network in which a fraction p of the bonds are randomly present, with bond strengths g_α or γ_α that vary as δ_α^{y+1}, where the δ_α are independent random variables with a probability distribution that remains finite in the limit $\delta \to 0$, and if t, e, and $\bar{f}$ are the exponents for the vanishing of the network conductivity, viscous fluid permeability and elastic moduli as $p \to p_c$, then for $y > 0$ we have the inequalities

$$\begin{aligned} &\max(t_1 + y, t) \le \bar{t} \le t + y \ , \\ &\max(t_1 + y, t) \le \bar{e} \le t + y \ , \\ &\max(f_1 + y, t) \le \bar{f} \le f + y \ , \end{aligned} \tag{36}$$

where $t_1 \equiv 1 + (d-2)\nu \le t$ is the nodes-links-blobs lower bound to the true lattice percolation conductivity exponent t, and $f_1 \equiv 1 + d\nu \le f$ is the similar lower bound to the true lattice elasticity exponent f. (As discussed in Sec. II, some of these inequalities are rigorous but others not.) Insofar as the differences $(t - t_1)$ and $(f - f_1)$ are small, these inequalities determine $\bar{t}$, $\bar{e}$, and $\bar{f}$ to a reasonable accuracy. According to the arguments of Ref. 21 and the results of Refs. 22 and 23, it seems likely that exponents $\bar{t}$, $\bar{e}$, and $\bar{f}$ are precisely given by the lower bounds in Eq. (36), in most cases. For $y \le 0$, we expect $\bar{t} = t$, $\bar{e} = t$, and $\bar{f} = f$. In order to obtain the results of Table I for the continuum models, we have simply used the values of y appropriate to the narrow bonds (necks) in each model.

B. When the channel size does not extend to zero

According to the analyses given above, interesting continuum corrections to the discrete percolation transport exponents can occur only if the distribution of the bond width parameter δ (or h) extends all the way down to $\delta \to 0$. If, in a particular continuum percolation system under study the bonds are restricted to be larger than a certain minimum width, then the transport exponents very close to the percolation threshold should be the same as their counterparts in the standard discrete lattice models. One may, however, observe the effect of the bond-strength distributions in the behavior of the transport coefficients when systems are near but not too close to percolation threshold.

We can illustrate our point for the following sterotypical example. Let us consider a restricted Swiss-cheese model, in which all the narrow channels are constrained to have a width δ larger than δ_r. First we can again map the problem onto a discrete network of fraction p occupied bonds, with bond strength $g \sim \delta^{y+1}$ with the constraint $\delta > \delta_r$. From linear chain analysis in Sec. II, we know that when the continuum corrections are relevant (i.e., $y > 0$), the system's transport is mostly determined by channels of the width $\delta_{\min} \sim a / L_1$, where a is the radius of punched spheres, and $L_1 \sim 1/(p - p_c)$. Thus we see that when δ_r is sufficiently small, we can find a range of p values for which conditions that system is in the critical regime (i.e., scaling laws apply) and $(p - p_c) > \delta_r / a$ are both satisfied. For this range of p, it is apparent that one can still use the new continuum exponents $\bar{t}$ etc. to describe the transport properties of the system. But when the system is really close to threshold, i.e., when $(p - p_c) < \delta_r / a$, the resistance of a chain will be determined by the singly connected bonds whose width is of order δ_r, and the number of such bonds on each chain will be of order $L_1 \delta_r / a$. Thus extremely close to p_c, we need to use the standard lattice exponents t, etc. to describe the system.

The above crossover phenomenon can be described by the following scaling function, as $\Delta p \to 0$:

$$\Sigma \sim (\Delta p)^t F\left[\frac{a(\Delta p)}{\delta_r}\right] , \tag{37}$$

where the universal function $F(x)$ satisfies the asymptotic relations $F(x) \sim \text{const}$ when $x \ll 1$, and $F(x) \sim x^y$ when $x \gg 1$.

C. Universality for transport percolation

We have seen that the percolation transport critical exponents for the discrete lattice model (or continuum systems with finite minimum neck width), the Swiss-cheese model, and the inverted Swiss-cheese model (or the potential model) can be different in a given dimension. This implies that the concept of universality of critical phenomenon, when one considers the problem of percolation transport, has to be used with care. Singular distributions of microscopic transport strength can play a very important role in determining the macroscopic transport critical exponents.

D. Experiments and numerical simulations

As we mentioned above, Smith and Lobb[40] have studied experimentally the electrical conductivity of a two-dimensional continuum system falling in the class of the potential model described above. Their result for the conductivity exponent $\bar{t} \approx 1.30$ is in excellent agreement with the lattice value for t, as we would expect from our analysis, summarized in Table I.

One way to observe experimentally the difference between continuum and standard discrete percolation exponents is to measure the elastic modulus of a sheet with randomly located circular holes (two-dimensional Swiss-cheese model). This experiment was performed by Benguigui,[44] who finds $\bar{f} \approx 5$. Lobb and Forrester[45] repeated this experiment more recently and carefully. Both groups' results show reasonable agreement to our theories presented in the present paper. The system studied earlier by Benguigui[15] is not of this type, however, as the holes were centered at random sites on a discrete lattice, and there are consequently no narrow necks in the system. We expect the elasticity exponent of this experiment to be f, the lattice exponent, in agreement with his findings,

$f \approx 3.5$. The conductivity experiments of Last and Thouless,[46] which first demonstrated that the two-dimensional conductivity exponent is greater than unity, were also carried out in a geometry without narrow necks. It is clear that one would expect to find the conventional lattice exponent t in this case; but the accuracy of the Last-Thouless experiment did not allow a quantitative estimate.

Sen *et al.*[47] and Bunde *et al.*[48] have recently performed numerical simulations of the electrical conductivity problem on a discrete lattice percolation model with power-law bond-strength distribution, of the form discussed in Sec. II above. Within the numerical uncertainties, they have verified the corrections to the standard discrete lattice exponent of our analysis and of Refs. 18–22, i.e., $\bar{t} \approx t + y$, for $y > 0$.

Roberts and Schwartz[49] have studied numerically the electrical conductivity and fluid permeability of a porous sedimentary rock, using a model similar to our Swiss-cheese model in three dimensions, except that the centers of their interpenetrating insulating spheres were chosen originally from a Bernal distribution, rather than completely at random. We would not expect this additional short-range correlation to affect the critical exponents. As noted by these authors, however, the volume fraction of conductor in these models is very small ($\approx 3\%$) at the percolation threshold, regardless of whether the Bernal distribution or the random distribution was used for the hole centers. We expect that the critical exponent may be observable only for q very close to q_c. Roberts and Schwartz do not investigate the critical properties, but study instead a wide range of conducting volume fractions above percolation. We note that the analysis of Roberts and Schwartz involved mapping onto a discrete network, similar to ours, with bond strengths determined by the cross-sectional area of the necks.

If one adopts this model to describe sedimentary rocks of low porosity one can write, in terms of the porosity $\phi \equiv 1 - q$, that $\Sigma \sim (\phi - \phi_c)^{\bar{t}}$ and $\kappa \sim (\phi - \phi_c)^{\bar{e}}$. According to Table I, we have $\bar{t} \approx 2.5$ and $\bar{e} \approx 4.5$. This implies a relation $\kappa \sim \Sigma^{1.8}$, which is not far from experimental findings on many types of rocks.[6] Experiments on rocks, however, are typically made in the regime $(\phi - \phi_c)/\phi_c > 1$, so we have no right, *a priori*, to expect that asymptotic critical exponents will apply. [The measured conductivities of porous rocks are generally fit to an empirical relation (Archie's law) of the form $\Sigma \propto \phi^m$ which assumes $\phi_c \approx 0$ and a typical value of m is ≈ 1.5.] Nevertheless, the analysis of Roberts and Schwartz suggests that the conductivity is dominated by the decreasing cross-sectional area of narrow passages in the entire range $3\% \lesssim \phi \lesssim 30\%$, and the difference between the conductivity and permeability behavior is qualitatively explained by the different dependences on the areas of the passages.

Wong, Koplik and Tomanic[50] have studied a network model of conducting pipes, in which all bonds on a regular network are present, but there is a wide distribution of pipe radii. The exponents of their model depend on parameters in the distribution, but they find also typically a permeability exponent about twice the conductivity exponent, which is not far from our finding.

Since the publication of Ref. 1, there have been many interesting developments in regard to continuum percolation transport properties. Among them are a renormalization group analysis of the conductivity exponent $\bar{t}$ of a hierarchical network with a broad distribution of bond strengths,[51] an experimental study of electrical conduction in 2D continuum percolation systems,[52] a theoretical study of the resistor-superconductor networks with distribution of strengths in one dimension,[53] a theoretical study of the frequency-dependent transport properties in continuum percolation systems,[54] a numerical simulation work on conductivity and permeability of continuum percolation models,[55] and much experimental[56] and theoretical[57] work on the amplitude of $1/f$ noise in continuum percolation, systems. The ideas contained in the present paper are useful for understanding many of these problems.

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