PHYSICAL REVIEW B VOLUME 35, NUMBER 4 1 FEBRUARY 1987

Interfacial structure and kinetics of ordering in weak ferromagnets

Debashish Chowdhury and J. D. Gunton
Department of Physics, Temple University, Philadelphia, Pennsylvania 19122
(Received 19 September 1986; revised manuscript received 27 October 1987)

The kinetics of domain growth in weak ferromagnets is studied by use of an appropriate set of Langevin equations. Because of the coupling between the order parameters, which arises from the Dzyaloshinsky-Moriya anisotropic interaction, the time evolution of the various correlation functions is coupled even in the linear theory. An analysis of the late stage of the evolution, following a critical quench, in terms of the motion of the interfaces is shown to lead to a $t^{1/2}$ domain-growth law. We have also studied the growth following an off-critical quench from a very high temperature.

I. INTRODUCTION

The dynamical growth of an ordered structure from a disordered one in thermodynamic systems is a subject of considerable current interest. Although the physics of pattern formation in systems in quasiequilibrium is quite well understood in terms of universal features revealed by renormalization group techniques, the evolution of equilibrium structures from metastable or unstable states to stable equilibrium states remains much less understood.[1–3] Approximate theories of several simple models have been complemented by Monte Carlo computer simulation to gain insight into the universal as well as the nonuniversal aspects of the "growth laws" in different time regimes.

In the case of simple systems with one-component nonconserved order parameters (e.g., the Ising model) the system is unstable against long-wavelength fluctuations at very early times (near $t=0$) following a critical quench from a high temperature to a temperature below the two-phase coexistence curve. At later times order develops, with the dynamical evolution driven by the curvature of the interfaces separating the phases. The velocity v of these interfaces is proportional to the corresponding local mean curvature. In the latter regime the average linear size $R(t)$ of the domains grows as $R(t) \sim t^{1/2}$.

The phenomenon of spinodal decomposition in binary alloys is concerned with the decay of unstable states, such as those created by quenching the system at a critical concentration. Similar phenomena have been studied also for several other systems with conserved and with non-conserved order parameters. However, most of the theoretical attention has been focused so far on systems with a one-component scalar order parameter, although there are some exceptions, such as ^{3}He-^{4}He mixtures and metamagnets,[4–7] the n-vector model in the large-n limit[8–10] and the clock model.[11] The most crucial feature of the n-vector model that makes it more interesting, albeit more difficult, is the continuous symmetry of the Hamiltonian that leads to the Nambu-Goldstone modes. In this paper we shall study the growth of ordered domains in the so-called weak ferromagnets[12,13] (referred to as WFM hereafter) which are described by multicomponent order parameters. Examples of such WFM are α-Fe_2O_3, rare-earth orthoferrites, etc., which have important industrial applications

II. FREE-ENERGY FUNCTIONAL AND THE EQUILIBRIUM STATES OF THE WFM

The magnitude of the spontaneous magnetization in a WFM is very small (about 10^{-2}–10^{-5} of the normal value in comparable ferromagnets). This magnetization arises from the special feature of the anisotropy in these systems. The magnetizations in the two sublattices are nearly, but not exactly, antiparallel to each other as we shall see later. The general approach in the continuum theories of spinodal decomposition consists of two essential ingredients, namely, a coarse-grained free-energy functional and the corresponding Langevin equation for the dynamical evolution of the correlation functions. Substitution of the former into the latter leads to the equation(s) of motion, the solution(s) of which give the growth laws. We shall also follow this basic scheme here. Let $\mathbf{M}_1$ and $\mathbf{M}_2$ denote the magnetizations of the two sublattices and define

$$\mathbf{M}=(\mathbf{M}_1+\mathbf{M}_2)/(2M_0)$$

and

$$\mathbf{L}=(\mathbf{M}_1-\mathbf{M}_2)/(2\mathbf{M}_0) .$$

The condition that the magnitude of the individual sublattice magnetizations are constant, viz., $\mathbf{M}_1^2=\mathbf{M}_2^2=\mathbf{M}_0^2$, implies the constraint $\mathbf{M}^2+\mathbf{L}^2=1$. The free-energy functional for a WFM is, in general, given by

$$F=F_L+F_M+F_{LM}$$

where F_L and F_M are the contributions from $\mathbf{L}$ and $\mathbf{M}$, respectively, and F_{LM} represents the coupling between $\mathbf{L}$ and $\mathbf{M}$. The coupling term in this model arises from the anisotropic exchange interaction and its explicit form depends on the symmetry of the crystal. The Dzyaloshinsky-Moriya (DM) anisotropic exchange interaction plays a crucial role in the WFM and is responsible for the appearance of a finite (albeit small) spontaneous magnetization in these systems in which the true exchange-driven transition is antiferromagnetic. The contribution to the microscopic Hamiltonian from the DM anisotropic interaction is given by

$$\mathscr{H}_{\rm DM} = -\mathbf{d}_{ij}\cdot(\mathbf{S}_i \times \mathbf{S}_j)$$

where d_{ij} determines the strength of the anisotropic interaction and $\mathbf{S}_i$ and $\mathbf{S}_j$ are the spin vectors at the ith and jth lattice sites, respectively. Note that the latter interaction favors an orthogonal orientation of the neighboring spins. For α-Fe_2O_3, the coarse-grained Landau-Ginzburg free-energy functional is given by[14]

$$F = \int d^d r[(A/2)\mathbf{L}^2 + (\alpha/2)L_z^2 + (B/2)\mathbf{M}^2 + (\beta/2)M_z^2 + (D/4)(\mathbf{L}^2)^2 + K(L_x M_y - L_y M_x) + (g_1/2)|\nabla \mathbf{L}|^2 + (g_2/2)|\nabla \mathbf{M}|^2], \quad (1)$$

where A, B, α, β, D, g_1, and g_2 are phenomenological parameters and K is the measure of the strength of the anisotropic contribution to the free energy, where the corresponding term arises from the DM interaction. Symmetry arguments originally presented by Dzyaloshinsky[14] show that there is no $\mathbf{M}^4$ term in (1) which rules out the possibility of any exchange-driven ferromagnetic transition. In the absence of an external magnetic field the term proportional to g_2 is small compared to that proportional to g_1.

Let us first briefly summarize the main features of the static equilibrium state(s) possible for the free-energy functional (1). Minimizing (1) for constant $|\mathbf{L}|$, we get[14] two sets of solutions. The first set of solutions is given by

$$M_x = M_y = M_z = 0 = L_x = L_y$$

and (2)

$$L_z \neq 0 ,$$

which corresponds to a pure antiferromagnet. The second solution is

$$M_z = 0 = L_z, \quad M_x = (K/B)L_y, \quad M_y = -(K/B)L_x \quad (3)$$

and corresponds to the weak ferromagnet. The latter solution is the stable state of α-Fe_2O_3 in the temperature range $250° \le T \le 950°$. Thus, the nonzero magnetization in the state (3) arises from the coupling K. In other words, the WFM state is a consequence of the absence of the $\mathbf{M}^4$ term and of the special form of the anisotropy. Since $|K|/B \sim 10^{-2}$–10^{-5}, $|M|/|L| \sim 10^{-2}$–10^{-5}, as stated earlier. Note that in the WFM phase

$$|\mathbf{L}| = \{-[A - (K^2/B)]/D\}^{1/2} . \quad (4)$$

Therefore, the transition temperature is given by $[A-(K^2/B)]=0$, i.e., the coupling of $\mathbf{L}$ with $\mathbf{M}$ shifts the transition temperature by a term proportional to K^2.

III. EQUATIONS OF MOTION

For simplicity, let us introduce the notation

$$\underline{Q} = \begin{bmatrix} L_x \\ L_y \\ L_z \\ M_x \\ M_y \\ M_z \end{bmatrix},$$

where the six components of $\mathbf{L}$ and $\mathbf{M}$ constitute the six components of a generalized order parameter Q. The stochastic equation of motion (Langevin equation) is

$$\frac{\partial Q_\mu(\mathbf{r},t)}{\partial t} = -\Gamma(\delta F/\delta Q_\mu(\mathbf{r},t)) + \zeta(\mathbf{r},t) \quad \text{for } \mu = 1,2,\ldots,6 , \quad (5)$$

where for simplicity we have assumed the same kinetic coefficient Γ for all the six components and $\delta/\delta Q_\mu$ denotes the functional derivative with respect to Q_μ. All the subsequent equations in this paper can be easily generalized to different Γ for different Q_μ. The white noise in (5) is assumed to satisfy

$$\langle \zeta(\mathbf{r},t) \rangle = 0 \quad (6a)$$

and

$$\langle \zeta(\mathbf{r},t)\zeta(\mathbf{r}',t') \rangle = 2\Gamma\delta(\mathbf{r}-\mathbf{r}')\delta(t-t') . \quad (6b)$$

Let us define the equal-time correlation functions

$$C_{\mu\nu}(\mathbf{r}-\mathbf{r}') = \langle Q_\mu(\mathbf{r},t) Q_\nu(\mathbf{r}',t) \rangle = \int D[Q]\, Q_\mu(\mathbf{r}) Q_\nu(\mathbf{r}') P(Q,t) , \quad (7)$$

where $\int D[Q]$ denotes a functional integral over the fields $\{Q\}$, and P satisfies a Fokker-Planck equation. Generalizing the standard techniques for models with single-component order parameters to the model (1) with a six-component order parameter, we get

$$\frac{\partial C_{\mu\nu}(\mathbf{r},t)}{\partial t} = -\Gamma \int D[Q] P(Q,t)\{Q_\mu(0)(\delta F/\delta Q_\nu) + Q_\nu(0)[\delta F/\delta Q_\mu(\mathbf{r})]\} + 2\Gamma\delta(\mathbf{r})\delta_{\mu\nu} . \quad (8)$$

IV. LINEAR THEORY

For the computation of the specific elements of the matrix C we need to use the particular expression (1) for F. Note that the nonlinear terms in the equations of motion, which arise from the quartic term in the free-energy functional, play a very important role in the time-evolution of the system. It is these nonlinear terms that suppress the instabilities in the n-vector model.[8,9] However, in order to get an insight into the nature of the instability during the very early stage of the evolution, we shall carry out a standard linear stability analysis. The formal structure of the time evolution of the correlation functions for the WFM turn out to be more interesting, even in the linear theory, than those for the n-vector models (including $n=1$) because of the dynamic coupling of the various elements of C in the former. We first expand the free energy around $\mathbf{L}=0$ and $\mathbf{M}=0$, and keep terms up to the second derivative, i.e.,

$$F=F_0+(1/2)\sum_{\beta}\sum_{\alpha}\left[\frac{\partial^2}{\partial Q_\alpha \partial Q_\beta}\right]Q_\alpha Q_\beta + (\text{higher-order terms}) \quad (9)$$

where F_0 is the free energy at $\mathbf{L}=\mathbf{0}$, $\mathbf{M}=\mathbf{0}$, and the first derivative of F vanishes because of the extremization condition. Using (9) we get

$$\frac{\partial C_{11}(\mathbf{r},t)}{\partial t}=\frac{\partial}{\partial t}\langle L_x(\mathbf{r})L_x(0)\rangle=-2\Gamma[-g_1\bar{C}_{11}(\mathbf{r},t)+AC_{11}(\mathbf{r},t)+KC_{15}(\mathbf{r},t)]\ , \quad (10)$$

$$\frac{\partial C_{55}(\mathbf{r},t)}{\partial t}=\frac{\partial}{\partial t}\langle M_y(\mathbf{r})M_y(0)\rangle=-2\Gamma[-g_2\bar{C}_{55}(\mathbf{r},t)+BC_{55}(\mathbf{r},t)+KC_{15}(\mathbf{r},t)]\ , \quad (11)$$

$$\frac{\partial C_{15}(\mathbf{r},t)}{\partial t}=\frac{\partial}{\partial t}\langle L_x(\mathbf{r})M_y(0)\rangle=-\Gamma[-g_1\bar{C}_{15}(\mathbf{r},t)+AC_{15}(\mathbf{r},t)+KC_{55}(\mathbf{r},t)-g_2\bar{C}_{15}(\mathbf{r},t)+BC_{15}(\mathbf{r},t)+KC_{11}(\mathbf{r},t)]\ , \quad (12)$$

where

$$\bar{C}_{11}(\mathbf{r},t)=\langle L_x(0)\nabla_r^2 L_x(\mathbf{r})\rangle\ ,$$

$$\bar{C}_{55}(\mathbf{r},t)=\langle M_y(0)\nabla_r^2 M_y(\mathbf{r})\rangle\ ,$$

$$\bar{C}_{15}(\mathbf{r},t)=\langle M_y(0)\nabla_r^2 L_x(\mathbf{r})\rangle\ .$$

Note that the equations of motion of the three elements C_{11}, C_{55}, and C_{15} are coupled to each other but not to those of the other elements of the matrix C. The equations of motion for the three elements C_{22}, C_{44}, and C_{24} form another closed set. However, we shall focus our attention only on the set (10)–(12). Upon taking the Fourier-Laplace transform

$$\tilde{C}_{\mu\nu}(\mathbf{k},\omega)=(1/2\pi)\int d\mathbf{r}\exp(i\mathbf{k}\cdot\mathbf{r}-\omega t)C_{\mu\nu}(\mathbf{r},t)\ ,$$

we get

$$\underline{M}\begin{bmatrix}\tilde{C}_{11}\\ \tilde{C}_{55}\\ \tilde{C}_{15}\end{bmatrix}=\begin{bmatrix}C_{11}(\mathbf{k},t=0)\\ C_{55}(\mathbf{k},t=0)\\ C_{15}(\mathbf{k},t=0)\end{bmatrix}\ , \quad (13)$$

where

$$\underline{M}=\begin{bmatrix}(\omega+G_1) & 0 & 2\Gamma K\\ 0 & (\omega+G_2) & 2\Gamma K\\ \Gamma K & \Gamma K & (\omega+G_3)\end{bmatrix}\ , \quad (14)$$

with

$$G_1=2\Gamma(g_1k^2+A)\ , \quad (15)$$

$$G_2=2\Gamma(g_2k^2+B)\ , \quad (16)$$

and

$$G_3=\Gamma((g_1+g_2)k^2+(A+B))\ . \quad (17)$$

We cannot, in general, solve Eqs. (13) analytically, although it is, of course, possible to write formal solutions as shown in Appendix A. However, in order to obtain the explicit form of the early-time growth (or decay) of the correlation functions C_{11}, C_{55}, and C_{15} we need to find out the eigenvalues of the matrix

$$\underline{N}=\begin{bmatrix}G_1 & 0 & 2\Gamma K\\ 0 & G_2 & 2\Gamma K\\ \Gamma K & \Gamma K & G_3\end{bmatrix}$$

[or, equivalently, the values of ω satisfying the equation $\det\underline{M}=(\omega-\lambda_1)(\omega-\lambda_2)(\omega-\lambda_3)=0$], for various values of k. Note that negative value of an eigenvalue indicates an instability of the corresponding correlation function. To get an insight into the orders of the magnitudes of the quantities involved let us carry out an elementary analysis of the eigenvalues of N. Let us compute the eigenvalues ω in the units of Γ (i.e., $\Gamma=1$) and the energy in the units in which $g_1=1$. We shall also assume $g_2\sim O(1)$. A is negative and of $O(1)$. B is assumed to be positive and of $O(1)$. Since $K/B\sim O(10^{-2}-10^{-5})$ we assume $K\sim 10^{-2}-10^{-5}$. Then, $G_1\sim O(1)$, $G_2\sim O(1)$, and $G_3\sim O(1)$ for a wave number $k\sim O(1)$; but $\Gamma^2K^2\sim 10^{-4}-10^{-10}$. Therefore, in the weak-coupling approximation, the terms involving Γ^2K^2 can be dropped from the determinant M. More precisely, we assume that $\Gamma^2K^2\sim O(\epsilon^2)$, where ϵ is a small number, whereas G_2G_3 and $G_1G_3\sim O(1)$. We shall retain terms up to the order ϵ but neglect higher-order terms. Note that the condition $G_1G_3\gg 2\Gamma^2K^2$ would not hold very close to the transition temperature because $G_1\to 0$ for $k\to 0$ at T_c. Therefore, in this section we shall focus our attention on temperature regimes not too close to T_c. Under this approximation (see Appendix A),

$$\det\underline{M}=(\omega+G_1)(\omega+G_2)(\omega+G_3)\ ,$$

and therefore,

$$\lambda_1=-G_1,\ \ \lambda_2=-G_2,\ \ \lambda_3=-G_3\ . \quad (18)$$

Also note that under this weak-coupling approximation the transition temperature is given by $A=0$. Therefore, for sufficiently small values of k, G_1 is negative. Moreover, depending on the relative values of A and B, G_3 can also be negative. But G_2 is always positive provided B remains positive. Substituting the values (18) of λ_1, λ_2, and λ_3 into (A1), (A2), and (A3) we get

$$C_{11}(\mathbf{k},t)=\{[(G_1+G_2)(G_1+G_3)]/[(G_1-G_2)(G_1-G_3)]\}e^{-G_1t}C_{11}(\mathbf{k},t=0)$$
$$+\{[2G_2(G_2+G_3)]/[(G_2-G_3)(G_2-G_1)]\}e^{-G_2t}C_{11}(\mathbf{k},t=0)$$
$$+\{[2G_3(G_2+G_3)]/[(G_3-G_1)(G_3-G_2)]\}e^{-G_3t}C_{11}(\mathbf{k},t=0)$$
$$+[2\Gamma K/(G_3-G_1)](e^{-G_3t}-e^{-G_1t})C_{15}(\mathbf{k},t=0)\ , \qquad (19)$$

$$C_{55}(\mathbf{k},t)=\{[2G_1(G_1+G_3)]/[(G_1-G_2)(G_1-G_3)]\}e^{-G_1t}C_{55}(\mathbf{k},t=0)$$
$$+\{[(G_1+G_2)(G_2+G_3)]/[(G_2-G_3)(G_2-G_1)]\}e^{-G_2t}C_{55}(\mathbf{k},t=0)$$
$$+\{[2G_3(G_1+G_3)]/[(G_3-G_1)(G_3-G_2)]\}e^{-G_3t}C_{55}(\mathbf{k},t=0)$$
$$+[2\Gamma K/(G_3-G_2)](e^{-G_3t}-e^{-G_2t})C_{15}(\mathbf{k},t=0)\ , \qquad (20)$$

$$C_{15}(\mathbf{k},t)=\{[2G_1(G_1+G_2)]/[(G_1-G_2)(G_1-G_3)]\}e^{-G_1t}C_{15}(\mathbf{k},t=0)$$
$$+\{[2G_2(G_1+G_2)]/[(G_2-G_3)(G_2-G_1)]\}e^{-G_2t}C_{15}(\mathbf{k},t=0)$$
$$+\{[(G_1+G_3)(G_2+G_3)]/[(G_3-G_1)(G_3-G_2)]\}e^{-G_3t}C_{15}(\mathbf{k},t=0)$$
$$+[\Gamma K/(G_3-G_1)](e^{-G_3t}-e^{-G_1t})C_{11}(\mathbf{k},t=0)+[\Gamma K/(G_3-G_2)](e^{-G_3t}-e^{-G_2t})C_{55}(\mathbf{k},t=0)\ . \qquad (21)$$

Note that the solutions (19)—(21) reduce to the correct $t=0$ limits. Now let us examine the signs of G_1, G_2, G_3, and that of the time-independent amplitudes of the various terms in (19), (20), and (21). We are interested in the regime where A is negative. Therefore, for sufficiently small k, we have $G_1<0$, thereby signaling long-wavelength instabilities of the correlation functions. In order to determine the signs of the amplitudes we have to specify the signs and the relative magnitudes of G_1, G_2, and G_3. For simplicity let us focus our attention only on the $k=0$ modes. First of all, we shall not consider temperatures too close to the transition temperature where A is very small, because in this regime the weak-coupling approximation will be violated and the solutions (19)—(21) cannot hold there. Thus, we are interested in the temperature regime where $|A| \gg B$ (indeed, the condition $|A|>2B$ is sufficient). In the latter regime, $G_3<0$, and

$$|G_1|>G_2,\quad |G_1|>|G_3|,\quad |G_3|>G_2\ .$$

Therefore, taking the inequalities into account (together with the fact that ΓK is small in the weak-coupling approximation), one can easily check that the amplitudes of the dominant instability are positive for both $C_{11}(\mathbf{k},t)$ and $C_{55}(\mathbf{k},t)$, as they should be.

Thus, the linear stability analysis shows an expected instability of the system for sufficiently small k and the system does not exhibit any mechanism for equilibration. However, the linear stability analysis neglects a crucial aspect of the evolution, namely, the well-known feedback effect of the nonlinear terms which tames the instability and drives the system toward equilibrium. This feature of the theory is also shared by one-component systems[2] as well as the n-component systems in the absence of anisotropy.[8,9] However, what makes the WFM more interesting is the coupled nature of the instability of the correlation functions. In the next section we shall carry out an approximate analysis of the late stage of the growth which takes these nonlinearities into account.

We note in passing that the linear theory of the spinodal decomposition in ^{3}He-^{4}He mixture[4] predicted a "flickering of the instability" introduced by the two second-sound modes, which arise from the conservation of entropy. Our linear theory predicts no such flickering during the ordering process in the WFM, since there is no conservation law in our model.

V. GROWTH IN THE LATE STAGES: NONLINEAR THEORY

The linear stability analysis that led to (19)—(21) holds only for very early time. As stated in the Introduction, in the late stages one gets a $t^{1/2}$ power-law growth for models with a scalar nonconserved order parameter. We shall show now that the same power law holds also in the WFM, although the analysis is more complicated in the present case. This result is due to the fact that the dynamics of the interfaces determine the late-stage growth. In order to investigate the latter problem one usually first finds the inhomogeneous steady-state solution which describes a situation in which different phases are separated from one another by interfaces. For simplicity, we shall assume that the gradient term in $\mathbf{M}$ is negligibly small. This assumption is quite reasonable in the absence of external field, as explained earlier.[30] So far as the term proportional to g_1 is concerned, we shall keep only $(g_1/2)L_z^2$ and drop the inhomogeneities along x and y directions. Moreover, since we are now interested in the evolution of the nonuniform steady-state solution to the equilibrium solution (3), we drop the terms involving L_z^2 in (1) for the subsequent discussion. Following the approach of San Miguel and Gunton[5] we minimize the total free-energy functional (1) with respect to M_x, M_y, M_z keeping L_x, L_y, L_z fixed, to obtain

$$M_x=(K/B)L_y,\quad M_y=-(K/B)L_x,\quad M_z=0\ . \qquad (22)$$

Substituting (22) into (1), we get

$$\tilde{F} = \int d^d r [(\tilde{A}/2)(L_x^2+L_y^2)+(D/4)(L_x^2+L_y^2)^2 + (g_1/2)|\nabla L|^2] \quad (23)$$

where

$$\tilde{A}/2 = [(A+K)/2]-(K^2/B) ,$$

which reduces to $(A+K)/2$ in the weak-coupling approximation. Minimizing (26) with respect to L_x and L_y, we obtain

$$\tilde{A}L_x + D(L_x^2+L_y^2)L_x - (g_1/2)\frac{\partial^2 L_x}{\partial z^2} = 0 , \quad (24)$$

$$\tilde{A}L_y + D(L_x^2+L_y^2)L_y - (g_1/2)\frac{\partial^2 L_y}{\partial z^2} = 0 . \quad (25)$$

However, since $\mathbf{M}^2 \ll \mathbf{L}^2 \simeq 1$, we should have $L_x^2+L_y^2=1$. Moreover, the static solutions summarized above lead to $\mathbf{M}^2=(K/B)^2\mathbf{L}^2$. Therefore, we impose the boundary condition $L_x=\pm 1$ at $z=\infty$ and $L_y=0$ at $z=\infty$. Under these conditions, we have[15] (see Fig. 1 for a schematic picture)

$$L_x = \tanh(z/z_0) \quad (26a)$$

and

$$L_y = 1/[\cosh(z/z_0)] . \quad (26b)$$

Therefore,

$$M_x = (K/B)\{1/[\cosh(z/z_0)]\} \quad (27a)$$

and

$$M_y = -(K/B)[\tanh(z/z_0)] , \quad (27b)$$

where $z_0=(2/a)^{1/2}$ where $a=-\tilde{A}$ (remember that $\tilde{A}$ is negative below the transition temperature). The physical implication of this set of solutions is very simple. Since $\mathbf{L}$ and $\mathbf{M}$ are, respectively, the sum and the differences of the two sublattice magnetizations, $|L_\mu|$ is maximum where $|M_\mu|$ is minimum and vice versa.

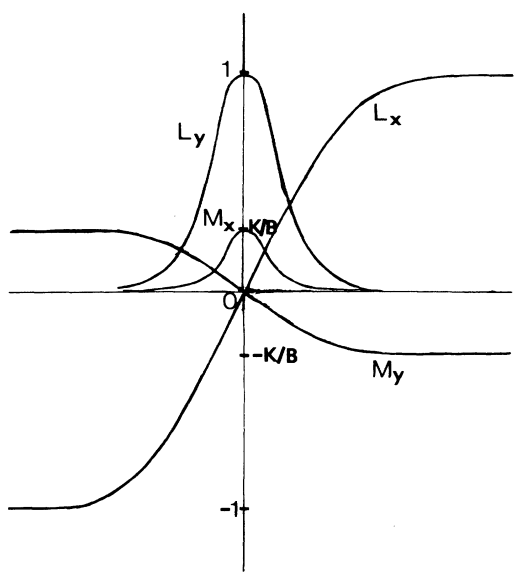

FIG. 1. Schematic representation of the geometrical structure of the interfaces described by Eqs. (26a)—(27b).

Let us now introduce a new set of coordinates $(\hat{q}_1,\hat{q}_2,\hat{q}_3)$. We divide space into surfaces of constant L_x and choose $\hat{q}_1$ to be orthogonal to this surface. Carrying out a transformation from the old coordinate system to the new coordinate system, we have

$$\nabla^2 L_x = \frac{\partial^2 L_x}{\partial g^2} - (K_1+K_2)\frac{\partial L_x}{\partial g} , \quad (28)$$

where $dg = h_1 dq_1$ denotes the displacement normal to the interface such that

$$\nabla L_x = \frac{\partial L_x}{\partial g}\hat{\mathbf{g}} ,$$

and K_1 and K_2 are the two principle curvatures of the interface. Hence, the equation of motion for L_x reduces to

$$\frac{\partial L_x}{\partial t} = -\Gamma\left[\frac{\partial f}{\partial L_x} - g_1\left[\frac{\partial^2 L_x}{\partial g^2} - (K_1+K_2)\frac{\partial L_x}{\partial g}\right]\right] ,$$

where f is the free energy excluding the gradient term. Since

$$\frac{\partial f}{\partial L_x} - g_1\frac{\partial^2 L_x}{\partial g^2} = 0$$

yields the kink solution (26a), we finally have

$$\left[\frac{\partial L_x}{\partial t}\right]_g = -\Gamma g_1(K_1+K_2)\left[\frac{\partial L_x}{\partial g}\right]_t$$

for locally planar interfaces. (This equation was first derived for a scalar order parameter with nonconserved dynamics by Allen and Cahn.[16]) Therefore, the velocity of the interface is given by

$$v = \left[\frac{\partial g}{\partial t}\right]_{L_x} \propto (K_1+K_2) .$$

Since $v \sim (dR/dt)$, where $R(t)$ is the characteristic linear size of a domain, $(dR/dt) \propto (1/R)$. Hence, finally, $R \sim t^{1/2}$ is the growth law for the weak ferromagnets.

VI. GROWTH OF DROPLETS FOLLOWING AN OFF-CRITICAL QUENCH

So far we have investigated the kinetics of domain growth in the WFM following a critical quench from a very high temperature. In this section we shall briefly discuss the kinetics of domain growth in the WFM following an off-critical quench from a very high temperature. For the theoretical description of an off-critical quench one must rewrite the free-energy functional in terms of the quantities L_x-U_x, L_y-U_y, L_z, M_x-V_x, M_y-V_y, M_z, where U_x, U_y, V_x, V_y are the equilibrium values of L_x, L_y, M_x, M_y, respectively. This introduces linear as well as cubic terms, in addition to the quadratic and quartic terms, in the free energy functional. Therefore, the "double-well" structure of the free energy becomes asymmetric. This feature changes the physics of the problem. Now droplets of one of ordered phases, surrounded by the disordered high-temperature phase, are formed. Under such circumstances one can easily show[17] that the interface profile maintains an approximately constant shape during its propagation, i.e., the kinetic equation of evolution remains invariant under the translation in time $t \to t' = t + \Delta t$, and in space $\mathbf{r} \to \mathbf{r}' = \mathbf{r} + \Delta\mathbf{r}$. Thus, choosing a reference point $\mathbf{R}(t)$ on the moving interface, one can say that the if a spatial translation $\mathbf{r} \to \mathbf{X} = \mathbf{r} - \mathbf{R}(t)$ is applied, the order parameter appears the same at all times. Consequently, the components L_x, L_y, M_x, and M_y, are functions of $\mathbf{r} - \mathbf{R}(t)$. From this general feature of the interface structure one can show[17] that the growth is dominantly linear in time. This is in contrast to the $t^{1/2}$ growth law in the case of a critical quench described above.

VII. CONCLUSION

In this paper we have investigated the time evolution of the spin correlation functions from an unstable low-temperature state of weak ferromagnets following a quench from a high-temperature equilibrium state. Because of the coupling of the order parameters, introduced by the Dzyaloshinsky-Moriya anisotropy, the correlation functions are shown to form a few separate sets where elements belonging to each set evolve in a mutually coupled manner even in a linear theory. The geometrical structure of the interfaces separating the different phases is richer than that in the isotropic n-vector and Ising models. The analysis of the late-stage growth, following a critical quench, in terms of the motion of the interfaces leads to a $t^{1/2}$ domain-growth law. A linear-growth law is obtained in the case of an off-critical quench.

ACKNOWLEDGMENTS

This work is supported by National Science Foundation (NSF) Grant No. DMR-83-12958. We would like to thank D. Stauffer for a critical reading of an earlier version of the manuscript.

APPENDIX A

The formal solutions for $C_{11}(\mathbf{k},t)$, $C_{55}(\mathbf{k},t)$, and $C_{15}(\mathbf{k},t)$ can be written as

$$C_{11}(\mathbf{k},t) = \mathscr{L}^{-1}((M_{11}/\det\underline{M})C_{11}(\mathbf{k},t=0) - (M_{21}/\det\underline{M})C_{55}(\mathbf{k},t=0) + (M_{31}/\det\underline{M})C_{15}(\mathbf{k},t=0)) ,$$

where $\mathscr{L}^{-1}$ denotes the inverse Laplace transform and

$$M_{11} = \omega^2 + \omega(G_2+G_3) + (G_2G_3 - 2\Gamma^2K^2) = \omega^2 + \mu\omega + \nu ,$$

$$M_{21} = -2\Gamma^2K^2 ,$$

$$M_{31} = -2\Gamma K(\omega + G_2) ,$$

and

$$\det\underline{M} = \omega^3 + a_2\omega^2 + a_1\omega + a_0 = (\omega-\lambda_1)(\omega-\lambda_2)(\omega-\lambda_3) ,$$

where

$$a_2 = G_1 + G_2 + G_3 ,$$

$$a_1 = G_1G_2 + G_2G_3 + G_3G_1 - 4\Gamma^2K^2 ,$$

$$a_0 = G_1G_2G_3 - 2\Gamma^2K^2(G_1+G_2) ,$$

$$C_{55}(\mathbf{k},t) = \mathscr{L}^{-1}(-(M_{12}/\det\underline{M})C_{11}(\mathbf{k},t=0) + (M_{22}/\det\underline{M})C_{55}(\mathbf{k},t=0) - (M_{32}/\det\underline{M})C_{15}(\mathbf{k},t=0)) ,$$

where

$$M_{12} = -2\Gamma^2K^2 ,$$

$$M_{22} = (\omega+G_1)(\omega+G_3) - 2\Gamma^2K^2 = \omega^2 + \mu'\omega + \nu' ,$$

$$M_{32} = 2\Gamma K(\omega + G_1) ,$$

$$C_{15}(\mathbf{k},t) = \mathscr{L}^{-1}((M_{13}/\det\underline{M})C_{11}(\mathbf{k},t=0) - (M_{23}/\det\underline{M})C_{55}(\mathbf{k},t=0) + (M_{33}/\det\underline{M})C_{15}(\mathbf{k},t=0)) ,$$

where

$$M_{13} = -\Gamma K(\omega + G_2) ,$$

$$M_{23} = \Gamma K(\omega + G_1) ,$$

$$M_{33} = (\omega+G_1)(\omega+G_2) = \omega^2 + \mu''\omega + \nu'' .$$

Taking the inverse Laplace transform, the formal solutions for $C_{11}(\mathbf{k},t)$, $C_{55}(\mathbf{k},t)$, and $C_{15}(\mathbf{k},t)$ are given by

$$\begin{aligned} C_{11}(\mathbf{k},t) = {} & [(\lambda_1^2 - \mu\lambda_1 + \nu)/\Lambda_1]e^{\lambda_1 t}C_{11}(\mathbf{k},t=0) \\ & + [(\lambda_2^2 - \mu\lambda_2 + \nu)/\Lambda_2]e^{\lambda_2 t}C_{11}(\mathbf{k},t=0) + [(\lambda_3^2 - \mu\lambda_3 + \nu)/\Lambda_3]e^{\lambda_3 t}C_{11}(\mathbf{k},t=0) \\ & + (2\Gamma^2K^2/\Lambda_1)e^{\lambda_1 t}C_{55}(\mathbf{k},t=0) + (2\Gamma^2K^2/\Lambda_2)e^{\lambda_2 t}C_{55}(\mathbf{k},t=0) \\ & + (2\Gamma^2K^2/\Lambda_3)e^{\lambda_3 t}C_{55}(\mathbf{k},t=0) - \{[2\Gamma K(\lambda_1+G_2)]/\Lambda_1\}e^{\lambda_1 t}C_{15}(\mathbf{k},t=0) \\ & - \{[2\Gamma K(\lambda_2+G_2)]/\Lambda_2\}e^{\lambda_2 t}C_{15}(\mathbf{k},t=0) - \{[2\Gamma K(\lambda_3+G_2)]/\Lambda_3\}e^{\lambda_3 t}C_{15}(\mathbf{k},t=0) , \end{aligned} \quad \text{(A1)}$$

where

$$\Lambda_1=[(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)]\ ,$$

$$\Lambda_2=[(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)]$$

and

$$\Lambda_3=[(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)]\ .$$

Similarly,

$$\begin{aligned}C_{55}(\mathbf{k},t)=&[(2\Gamma^2K^2)/\Lambda_1]e^{\lambda_1 t}C_{11}(\mathbf{k},t=0)+[(2\Gamma^2K^2)/\Lambda_2]e^{\lambda_2 t}C_{11}(\mathbf{k},t=0)+[(2\Gamma^2K^2)/\Lambda_3]e^{\lambda_3 t}C_{11}(\mathbf{k},t=0)\\&+[(\lambda_1^2-\mu'\lambda_1+\nu')/\Lambda_1]e^{\lambda_1 t}C_{55}(\mathbf{k},t=0)+[(\lambda_2^2-\mu'\lambda_2+\nu')/\Lambda_2]e^{\lambda_2 t}C_{55}(\mathbf{k},t=0)\\&+[(\lambda_3^2-\mu'\lambda_3+\nu')/\Lambda_3]e^{\lambda_3 t}C_{55}(\mathbf{k},t=0)-\{[2\Gamma K(\lambda_1+G_1)]/\Lambda_1\}e^{\lambda_1 t}C_{15}(\mathbf{k},t=0)\\&-\{[2\Gamma K(\lambda_2+G_1)]/\Lambda_2\}e^{\lambda_2 t}C_{15}(\mathbf{k},t=0)-\{[2\Gamma K(\lambda_3+G_1)]/\Lambda_3\}e^{\lambda_3 t}C_{15}(\mathbf{k},t=0)\ ,\end{aligned}\tag{A2}$$

$$\begin{aligned}C_{15}(\mathbf{k},t)=&-\{[\Gamma K(\lambda_1+G_2)]/\Lambda_1\}e^{\lambda_1 t}C_{11}(\mathbf{k},t=0)\\&-\{[\Gamma K(\lambda_2+G_2)]/\Lambda_2\}e^{\lambda_2 t}C_{11}(\mathbf{k},t=0)-\{[\Gamma K(\lambda_3+G_2)]/\Lambda_3\}e^{\lambda_3 t}C_{11}(\mathbf{k},t=0)\\&-\{[\Gamma K(\lambda_1+G_1)]/\Lambda_1\}e^{\lambda_1 t}C_{55}(\mathbf{k},t=0)-\{[\Gamma K(\lambda_2+G_1)]/\Lambda_2\}e^{\lambda_2 t}C_{55}(\mathbf{k},t=0)\\&-\{[\Gamma K(\lambda_3+G_1)]/\Lambda_3\}e^{\lambda_3 t}C_{55}(\mathbf{k},t=0)+\{[\lambda_1^2-\mu''\lambda_1+\nu'']/\Lambda_1\}e^{\lambda_1 t}C_{15}(\mathbf{k},t=0)\\&+[(\lambda_2^2-\mu''\lambda_2+\nu'')/\Lambda_2]e^{\lambda_2 t}C_{15}(\mathbf{k},t=0)+[(\lambda_3^2-\mu''\lambda_3+\nu'')/\Lambda_3]e^{\lambda_3 t}C_{15}(\mathbf{k},t=0)\ .\end{aligned}\tag{A3}$$

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