Self-consistent hydrodynamical model for He II near absolute zero in the framework of stochastic mechanics

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Working in the framework of stochastic mechanics we propose a simple model, based on the hard-sphere gas approximation, for He II at $T=0$. The model seems to describe correctly the peculiar hydrodynamical behavior of He II near absolute zero and also provides good estimates for the critical velocities and the kinematic viscosity.

I. INTRODUCTION

As is well known, ^{4}He liquefies at $T=4.2$ K and remains liquid down to the lowest temperatures. Under the critical temperature $T_\lambda=2.17$ K it also exhibits a number of features highly unusual in normal liquids. Under T_λ, ^{4}He is denoted in the literature as He II.

In this work we shall concentrate our attention on the properties of He II that seem to be conserved near absolute zero: We will be mainly concerned with the He II viscosity and its behavior under rotation.

An unusual fact concerning the viscosity is that He II seems to flow without any friction in narrow channels, while a finite viscosity is observed in experiments with oscillating disks and rotating viscometers.[1] The widely accepted explanation for this paradoxal behavior is that formulated in the phenomenological two-fluid theory of Landau[2] and Tisza,[3] where He II is mathematically modeled as a mixture of two noninteracting fluids: the superfluid, ideal and irrotational, and the normal fluid, responsible for all dissipative effects. The viscosity paradox is explained in the sense that only the normal fluid is expected to interact with the walls, so that it does not flow in narrow channels while it is entrained by the oscillating disks and the walls of the viscometers. The fraction of superfluid should be zero at T_λ and increase to one as T is lowered to zero.

As a consequence, one would expect to observe that the viscosity goes to zero with T in the experiments with oscillating disks and rotating viscometers, which sometimes is not observed.[4]

An unexpected phenomenon also occurs in experiments with He II in rotating buckets[5] where, above some critical angular velocity, the liquid helium rotates at every T as a normal fluid, i.e., it exhibits the classical meniscus, typical of rigid-body rotation.[6]

The widely accepted means of overcoming this difficulty is contained in the quantized vortex-line theory by Onsager[7] and Feynman,[8] where the ideas of purely irrotational flow and rigid-body rotation are reconciled by showing how it is possible for a dense array of microscopic quantized vortex lines (which individually give a curl-free velocity field) to simulate a macroscopic rigid-body motion.

An interesting aspect of this theory is that it predicts that the circulation, as well as the angular momentum of the He II near absolute zero, should be quantized. Actually the possibility of quantized circulations was first observed in Vinen's classical experiment[9] where ^{4}He was rotated to the equilibrium at $T>T_\lambda$ and then slowly cooled under the λ point to $T=1.3$ K (where the percent of superfluid is about 96%), when the bucket was stopped. Quantized circulation was observed for the most stable states, corresponding to persistent currents. This result was then confirmed by Whitmore and Zimmermann[10] and Hess and Fairbank.[11]

Obviously we do not pretend to give here an exhaustive description of the various phenomena observed in the course of the wide experimental work on He II. We are simply trying, referring the reader to classical and confirmed experiments, to roughly summarize the main features of the strictly hydrodynamical behavior of He II near absolute zero, emphasizing that the currently accepted explanations of the observed phenomena are based only on phenomenological and semiqualitative theories.

The aim of this paper is to show that it is possible, working in the framework of stochastic mechanics, to derive from first principles a hydrodynamical model of He II at $T=0$ which qualitatively describes the unusual behavior sketched out above and which also gives good estimates for some typical hydrodynamical parameters, such as the critical velocities and kinematic viscosity.

II. KINEMATIC AND DYNAMICAL ASSUMPTIONS

As is known, stochastic mechanics, starting from conceptually simple assumptions, provides a description of quantum phenomena in terms of stochastic processes.[12]

In the case of a quantum particle of mass m, the basic kinematic assumption is that its position $\mathbf{q}(t)$ performs a diffusion process with coefficient $\hbar/m$. That is, its sample paths are constructed by means of a standard Wiener process $\mathbf{w}(t)$ by the rule

$$d\mathbf{q}(t)=\mathbf{v}_+[\mathbf{q}(t),t]dt+\left[\frac{\hbar}{m}\right]^{1/2}d\mathbf{w}(t)\ . \tag{1}$$

The Brownian term $(\hbar/m)^{1/2}d\mathbf{w}(t)$ represents a random contribution to the infinitesimal displacement of the particle that models the effect of the quantum fluctuations, while the drift $\mathbf{v}_+$ has to be determined by dynamical constraints.

As in the classical case, dynamics can be introduced in different ways: either by means of a generalization of Newton's second law, as was originally introduced by Nelson,[13] or by exploiting variational principles.[12,14–16]

In this paper we shall exploit the variational principle formulated in Ref. 16. The particularity of such an approach lies in providing a class of solutions broader than the corresponding one for the Schrödinger equation.

One of the main points in this work is that the nonconventional solutions seem to nicely overcome some peculiar difficulties in a self-consistent hydrodynamical description of He II near absolute zero.

In the method proposed in Ref. 16, the drift field $\mathbf{v}_+$ is determined by the requirement that the average of the discretized classical action over all possible sample paths be stationary for a given class of admissible variations. A variation $\delta\mathbf{q}(t)$ is called admissible if $\mathbf{q}(t)+\delta\mathbf{q}(t)$ is still a diffusion with the same coefficient $\hbar/m$.

To be more precise, in the case of a scalar potential Φ, the variational principle requires that, considering an equipartition of an arbitrary interval $[t_a,t_b]$, then

$$\lim_{N\to\infty}\delta E\left[\sum_{i=1}^{N}\left[\frac{\frac{1}{2}m(\mathbf{q}(t_i+\Delta)-\mathbf{q}(t_i))^2}{\Delta^2}-\Phi(\mathbf{q}(t_i),t_i)\right]+p_{t_b}\mathbf{q}(t_b)\right]=o(\delta\mathbf{q}),\quad \delta\mathbf{q}(ta)=0\ , \tag{2}$$

where $\Delta=(t_b-t_a)/N$ and p_{t_b} plays the role of a Lagrangian multiplier.

The boundary term $p_{t_b}q(t_b)$ is added since the mathematical properties of the class of admissible variations do not allow the further constraint $\delta\mathbf{q}(t_b)=0$.

The necessary and sufficient conditions in order that $\mathbf{q}(t)$ satisfy (2), without any additional constraint on the drift, can be written as follows (the details of the calculation are given in Ref. 17):

$$\begin{aligned}&\frac{\partial}{\partial t}\rho=-\nabla(\rho\mathbf{v})\ ,\\&\frac{\partial}{\partial t}\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{v}-\left[\frac{\hbar}{2m}\nabla^2\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right]\\&\qquad+\left[\mathbf{u}+\frac{\hbar}{2m}\nabla\right]\times\nabla\times\mathbf{v}=-\frac{1}{m}\nabla\Phi\ ,\end{aligned} \tag{3}$$

where ρ denotes the probability density of $\mathbf{q}(t)$; $\mathbf{v}$ its current velocity, equal to $\mathbf{v}_+-(\hbar/2m)\nabla\ln\rho$; and $\mathbf{u}$, often called the "osmotic velocity," is a shorthand notation for $(\hbar/2m)\nabla\ln\rho$.[18]

Therefore, the subclass of solutions with the additional constraint $\nabla\times\mathbf{v}=0$ corresponds to the solutions of Madelung's fluid equations; that is, to the solutions of the analogous Schrödinger equation with the positions $\mathbf{v}\equiv\nabla S/m$ and $\psi\equiv\rho^{1/2}e^{iS/\hbar}$.

If Φ does not depend on t, one has nondissipative behavior since, denoting by H the Hamiltonian operator,

$$\frac{d}{dt}E(\tfrac{1}{2}m\mathbf{v}^2+\tfrac{1}{2}m\mathbf{u}^2+\Phi)=\frac{d}{dt}\langle\psi\,|\,H\,|\,\psi\rangle=0\ .$$

Equations (3), moreover, give rise to the possibility of new solutions, corresponding to the case when $\nabla\times\mathbf{v}\neq0$.

III. HYDRODYNAMICS OF He II AT $T=0$

Equations (3) are connected to a possible theory of liquid helium at $T=0$ through the old idea of London that at $T=0$ He II should be considered as an ensemble of Bose particles that coherently move in the same quantum state with a macroscopic de Broglie wavelength.[19] The conjecture was based on the belief that the basic phenomenon involved in the λ transition was something analogous to the Bose-Einstein condensation for an ideal Bose gas.

We shall first give a rough argument in order to show how London's idea can be physically motivated by exploiting the explicit model of quantum fluctuations given in the kinematic assumption of stochastic mechanics. In fact, from the shape of the atom-atom interaction potential and from the estimate of the mean interatomic distance, it can be guessed that the effective potential "seen" by the single He II atom at $T=0$ is approximately that of a hard-sphere gas (see, for example, Ref. 19). This guess is also supported by the excellent agreement between the numerical Monte Carlo simulations (see, for example, Ref. 20) of the structure factor for a quantum hard-sphere system, and the same factor experimentally obtained for He II.

Let us now consider two atoms, denoted by A and B, and assume that at some time t they are in the same stationary quantum state, so that they have the same time-independent drift $\mathbf{v}_+$. We will suppose that such a drift does not appreciably vary on distances of the order of the diameter of the hard sphere.

If, due to the Brownian fluctuation, they collide for the first time at t^*, their mean velocities in the time interval $[t^*-\Delta t,t^*]$ are, respectively,

$$\begin{aligned}\mathbf{v}_A^i&=\mathbf{v}_+(x^*)+\left[\frac{\hbar}{m}\right]^{1/2}\frac{\mathbf{w}^A(t^*)-\mathbf{w}^A(t^*-\Delta t)}{\Delta t}+\frac{o(\Delta t)}{\Delta t}\ ,\\\mathbf{v}_B^i&=\mathbf{v}_+(x^*)+\left[\frac{\hbar}{m}\right]^{1/2}\frac{\mathbf{w}^B(t^*)-\mathbf{w}^B(t^*-\Delta t)}{\Delta t}+\frac{o(\Delta t)}{\Delta t}\ ,\end{aligned} \tag{4}$$

where x^* is the "point" of collision and $\mathbf{w}^A$ and $\mathbf{w}^B$ are two independent standard Wiener processes.

Now, justified by the fact that during the collision the actions are much greater than $\hbar$, we model the scattering, considering the hard spheres as perfectly smooth, in purely classical terms. Therefore, denoting by $(\hat{\mathbf{r}},\hat{\mathbf{y}},\hat{\mathbf{z}})$ an orthogonal frame, where $\hat{\mathbf{r}}$ is parallel to the line joining the centers of A and B, we have that the mean velocity of the atom A in the interval $[t^*, t^*+\Delta t]$ is approximately given by

$$\mathbf{v}_A^f=\mathbf{v}_+(x^*)+\left[\frac{\hbar}{m}\right]^{1/2}\frac{1}{\Delta t}\{[\mathbf{w}^B(t^*)-\mathbf{w}^B(t^*-\Delta t)]_r\hat{\mathbf{r}}+[\mathbf{w}^A(t^*+\Delta t)-\mathbf{w}^A(t^*)]_y\hat{\mathbf{y}}+[\mathbf{w}^A(t^*+\Delta t)-\mathbf{w}^A(t^*)]_z\hat{\mathbf{z}}\}+\frac{o(\Delta t)}{\Delta t}\ . \tag{5}$$

That is, the two hard spheres exchange their mean Brownian velocities along $\hat{\mathbf{r}}$.

But this is the same as saying that

$$\mathbf{q}_A(t^*+\Delta t)-\mathbf{q}_A(t^*)=\mathbf{v}_+(x^*)\Delta t+\left[\frac{\hbar}{m}\right]^{1/2}\{[\mathbf{w}^B(t^*)-\mathbf{w}^B(t^*-\Delta t)]_r\hat{\mathbf{r}}+[\mathbf{w}^A(t^*+\Delta t)-\mathbf{w}^A(t^*)]_y\hat{\mathbf{y}}+[\mathbf{w}^A(t^*+\Delta t)-\mathbf{w}^A(t^*)]_z\hat{\mathbf{z}}\}+o(\Delta t)\ , \tag{6}$$

so that one can see that the only difference in the law of constructing the sample paths of the atom A, with respect to the case of no collision, is in replacing $\mathbf{w}^A(t^*+\Delta t)-\mathbf{w}^A(t^*)$ with $\mathbf{w}^B(t^*)-\mathbf{w}^B(t^*-\Delta t)$.

Thus, since the probability density of the increments of a Wiener process is independent of t, and $\mathbf{w}^A$ and $\mathbf{w}^B$ have the same probability density, we can conclude that not only is the drift $\mathbf{v}_+$ unchanged after the collision, but also that the transition probability from x^* at time t^* to x at time $t^*+\Delta t$ remains the same.

The argument shows that *it is possible* for all He II atoms to be in the same quantum state, provided the corresponding drift field does not appreciably vary on the dimension of the "hard spheres."

So motivated, we shall take London's idea as a working assumption and extend the notion of quantum state to any couple $\{\rho,\mathbf{v}\}$ that is a solution of (3). For our purpose, Φ in (3) merely represents the external force acting on the single He II atom. In the course of this work we shall assume that the only external field is gravity.

Let us now face, in the framework of the tentative model put forth above, the problem of describing the flow of He II at $T=0$ in a capillary and its behavior under rotation. To this end, we must keep in mind that (3) are the equations for the density and the current velocity fields common to the "unperturbed" atoms, i.e., to the atoms which have interacted only with other atoms having the same drift.

The equations are considered "macroscopic" in the sense that, if London's assumption holds, the number of atoms having in common at $T=0$ the fields ρ and $\mathbf{v}$ is approximately, at any time, the total number of atoms in the container. In addition, possible solutions of (5) that appreciably vary on distances of the order of the diameter of the hard spheres (≈ 2 Å) are not taken into account.

Consider first the flow in a capillary: denoting by (r,θ,z) a suitable set of polar cylindrical coordinates, we shall neglect the effects of the gravity acceleration in any direction perpendicular to the longitudinal axis of the capillary, so that $\nabla\Phi\equiv[(\partial/\partial z)\Phi]\hat{\mathbf{z}}$. Since, of course, Φ is independent of time, it is clear that, provided the boundary conditions allow it, there exists the possibility of purely nondissipative motions, described by solutions in the subclass with $\nabla\times\mathbf{v}=0$. Therefore, the main problem is giving physically meaningful conditions at the walls.

In the phenomenological two-fluid model the superfluid is assumed not to interact with the walls. This can in fact be accepted since, in such a model, the superfluid component of the He II is a mathematical object and not a physical fluid!

But in our case we deal with the physical fluid, so the assumption that it does not interact with the walls is not tenable: actually He II "wets" the walls (as is quite clear, for example, in the formation of superfluid films).

Therefore, we shall assume that the walls of the pipe are actually coated by helium atoms and that atoms flowing in the pipe will occasionally interact with them. Due to Brownian fluctuations, also an unperturbed atom, drifting parallel to the pipe, will interact with the atoms that coat the walls and, in conclusion, due to collisions with other atoms which now do have a different drift, the disturbance will propagate to all regions of the pipe. As a consequence, the picture with all the drifts parallel to the pipe is no longer stable. The situation is even worse in the case when the unperturbed drift is not parallel to the walls.

Since all collisions, in our simplified picture, are assumed perfectly elastic, we can try to make some guess as to what happens, observing that we could consider the component of the velocity drift along the longitudinal axis of the pipe as "unperturbed" if the fluctuations of the components in the orthogonal plane are smaller, or at most, of the same order as the quantum ones.

In order to make this statement more precise, let us now consider a Brownian displacement of length s in the radial direction; that is, such that, for some time interval μ_s, one has

$$\left[\frac{\hbar}{m}\right]^{1/2}(\mathbf{w}_{t+\mu_s}-\mathbf{w}_t)=s\ . \tag{7}$$

Since $(\mathbf{w}_{t+\mu_s}-\mathbf{w}_t)$ is of the order of $(\mu_s)^{1/2}$, one can estimate $\mu_s \approx s^2 m/\hbar$. Therefore, the quantum fluctuations of the positions of the atoms inside the capillary in the radial direction are approximately of the same size as those produced by the walls in the time interval μ_s if $|\delta v_r| \approx \hbar/ms$.

If we now observe that the minimum value of $\hbar/ms$ corresponds to $s=d$, where d denotes the diameter of the capillary, and that the maximum possible value of $|\delta v_r|$ is the maximum absolute value of the unperturbed drift, we expect, reasoning in the same way for δv_θ, that liquid helium flows *as if the walls were perfectly smooth*, provided the maximum absolute value of the drift does not exceed $\hbar/md$.

Assuming that the density is approximately constant in a relevant region of the capillary, so that the drift coincides with the current velocity in this region, we get, for small d, a rough estimate of the critical velocity as

$$v_c \approx \frac{\hbar}{md}\ . \tag{8}$$

Remarkably enough, (8) is quite the same formula as that deduced from the result of the first experiments of He II flowing in narrow channels.[21] So, we can conclude that, under v_c, He II at $T=0$ *can* flow in a capillary as an ideal frictionless fluid.

Following our assumptions we must now study the case when $\nabla\times\mathbf{v}\neq 0$. The equations are now quite unfamiliar and more complicated, but we can get some physical information considering the fluid as approximately incompressible. Putting $\mathbf{u}=0$ and $\operatorname{div}\mathbf{v}=0$ in (3), we find, for the flow in a capillary, the Navier-Stokes—type equation

$$\frac{\partial}{\partial t}\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\frac{\hbar}{2m}\nabla^2\mathbf{v}-\frac{1}{m}\left[\frac{\partial\Phi}{\partial z}\right]\hat{\mathbf{z}}\ . \tag{9}$$

The problem of finding a steady-state solution of (9) in a cylindrical symmetry is then reduced to that of solving the ordinary differential equation $(\hbar/2m)(d^2v_z/dr)=\text{const}$, whose solutions, for any physically meaningful condition at the walls, exhibit the classical velocity profile with a maximum for $r=0$ and a minimum at the wall.

Therefore, we conclude that the steady-state solutions of (9) model some dissipative effect by the wall and then, at least in the incompressibility approximation, we discard, in order to describe the flow in a capillary, the solution of (3) with $\nabla\times\mathbf{v}\neq 0$, for current velocities less than the critical one. Of course this does not mean that (9) is correct for velocities greater than v_c since in that case the assumption that all atoms are in the same "quantum state" does not hold.

Conversely, to the authors' opinion, the situation of liquid helium over the critical velocity has some analogies with that of a normal fluid at *finite* temperatures. In fact, it has been proven (see Refs. 22 and 23) that a thermal quantum mixture can be modeled in stochastic mechanics as a mixture of Markov processes with random drifts.

Finally let us study the case of liquid helium at $T=0$ rotating in a cylindrical bucket of radius R. In the case of $\nabla\times\mathbf{v}=0$ the problem of searching the steady-state solutions is reduced to that of solving a stationary Schrödinger equation in cylindrical symmetry. The solutions are then of the type

$$\phi_{lk}(r,\theta,z)=J_l(kr)e^{il\theta}Z_k(z)\quad (l=0,\pm1,\pm2,\ldots)\ , \tag{10}$$

where J_l are suitable Bessel functions.

We immediately find that the current velocity is $\mathbf{v}\equiv v_\theta(r)\hat{\boldsymbol{\theta}}\equiv(\hbar/m)\nabla(l\theta),\ \equiv l(\hbar/mr)\hat{\boldsymbol{\theta}}$ corresponding to an (*irrotational*) flow around a vortex of quantized strength $l\hbar/m$. Since the velocity goes to infinity for $r=0$ we shall assume, as is usually done in the literature, that the vortex has a hollow core. Consider now the case of Vinen's experiment,[9] where He II still rotates after the container has been stopped.

Repeating for the annular flow the rough argument we used for the flow in a capillary, we get that the wall of the cylinder can be considered "perfectly smooth" if the current velocity of the atoms near the wall, relative to the wall itself, is less than $\hbar/[m(R-a)]$, where a denotes the radius of the core. Consequently, we have the constraint $l<R/(R-a)$. Thus, in our rough estimate, we expect persistent currents with quantized circulation at least for $l=1$.

Still considering Vinen's experiment let us study the situation when liquid helium is cooled under the λ point, keeping the bucket in rotation with angular velocity Ω. In this case the most natural boundary condition at the equilibrium is $v_\theta(R)=\Omega R$, which is not compatible with $\nabla\times\mathbf{v}=0$.

To get an approximate solution with $\nabla\times\mathbf{v}\neq 0$, we consider again the fluid as nearly incompressible and neglect the external field, reducing the problem to finding a steady-state solution of the familiar equation

$$\frac{\partial}{\partial t}\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{v}=(\hbar/2m)\nabla^2\mathbf{v}\ . \tag{11}$$

As is well known the energetically preferred solution of (11) in the approximation of neglecting the effects of centrifugal force, is $\mathbf{v}\equiv v_\theta(r)\hat{\boldsymbol{\theta}}=\Omega r\hat{\boldsymbol{\theta}}$, corresponding to the rigid-body rotation, that is in fact observed. Of course, this is also the case of the celebrated Osborne experiment, where the rigid-body rotation of He II was first observed. In this experiment He II, initially at rest, was then entrained into rotation by the walls of the container.

Since this procedure was wholly performed at the lowest temperature, one would perhaps expect that the motion of He II could be described at every time by some solution of (3). This is not true. We can only say that, under some critical angular velocity, surely no rigid-body rotation can be observed. By the same argument exploited above, such an angular velocity is roughly estimated as

$$\Omega_c=\frac{\hbar}{2mR^2}\ , \tag{12}$$

corresponding to the angular velocity necessary to the wall in order to produce a disturbance of the same order of the quantum one in all the regions of the bucket.

Assuming (8) as experimental evidence, the same formula was also derived by London through a more complicated argument.[19(b)]

For values of Ω greater than Ω_c we expect that the atoms begin to have nonzero random drifts. As observed above, the state of liquid helium is then somewhat similar to that of a normal liquid at finite temperature and we expect that it *can* be entrained by the walls. Indeed, to the authors' knowledge, in no Osborne-type experiment has a rotation for $\Omega \leq \Omega_c$ been observed.

The steady-state solution of (11) corresponds only to the equilibrium situation, on the assumption that in this case approximately all drifts are "ordered," if the temperature is kept constant at the lowest values.

To confirm this picture let us return to Eq. (11): we observe that it directly gives the value $\hbar/2m$ for the kinematic viscosity of the He II near $T=0$ under (forced) rotation, which actually, through the observation of the depth of the meniscus, is a *measurable* parameter.

It is notable that the order of magnitude of such an estimate agrees with experiment.[24] (The observation that the order of magnitude of kinematic viscosity of He II was the same as $\hbar/m$ is due to Onsager.[25])

In addition, one can also guess from (11) that there should exist, for He II near $T=0$, under forced rotation, a second critical angular velocity, corresponding to the transition to turbulent motion, as well as other types of phenomena such as creation of vorticity at the boundary and instabilities.[26]

Indeed from the dimensionless form of (11) one gets the (*quantum*) Reynolds number

$$R_e = L_c V_c \Big/ \frac{\hbar}{2m} , \tag{13}$$

where L_c and V_c are, respectively, a length and a velocity characteristic of the experiment, so that we could expect, in analogy to classical hydrodynamics, a transition to a turbulent behavior for high Reynolds numbers.

Leaving out of consideration a detailed analysis of this phenomenon, we only stress that superfluid second-critical velocities always seem to be observed and that the corresponding quantum Reynolds numbers, that we have deduced from (13), for experiments with rotating viscometers (see Refs. 27 and 28), approximately lie in the range 10^3-10^4, which is comparable with that of a classical fluid.

IV. CONCLUSIONS

Considering the whole emerging picture, we feel that our tentative simplified model seems to work both from a qualitative and a quantitative point of view. In particular, both London's conjecture and the proposed extension of the notion of quantum state seem physically meaningful.

To theoretically confirm London's conjecture we have also given a simple argument that is uniquely based on the peculiar shape of the atom-atom interaction potential for ^{4}He and the kinematic assumption of stochastic mechanics. The argument actually stems from a simplified microscopic model of the liquid helium at $T=0$ that, in spite of crude approximations, has the notable property of directly providing sensible estimates of the critical velocities for various experimental situations.

It is worth stressing that the result is not achieved by the previous theories: in Landau's two-fluid model the estimated critical velocity for the superfluid in a narrow channel is orders of magnitude greater than the observed ones and the quantized vortex theory provides a formula somewhat similar to (8), but containing an unknown parameter (for a discussion on this point see, for example, Lane[1(b)] and Putterman.[1(c)])

As far as the extension of the notion of a quantum state is concerned, the first thing to say is that it naturally comes from the variational principle and it seems confirmed by the experiment on He II. Moreover, such an extension looks necessary in order to make the whole theory self-consistent.

Were we restricted only to the standard states, we should have to insert some "*ad hoc*" mechanism to explain the rigid-body rotation. If we exploit the conjecture of the microscopic quantized vortex lines, we could not explain the experimental value of the kinematic viscosity. In addition, one should stress that the quantitative predictions of the superfluid second-critical velocities given in the framework of such a theory do not agree with experiment [see Refs. 6(b), 27, and 28].

Anyway, it is also worth noting that the formation in rotating helium of *microscopic* quantum vortex lines is in principle not forbidden by our proposed model, since the velocity fields we have considered are in fact macroscopic.

A number of mathematical problems has been left aside in the course of this work, the main one perhaps concerning the stability properties of the solutions of (3). This will be the subject of further work, as well as the extension of the model to finite temperature.

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