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Theory for dirty superconductors. I. Strong-coupling equations

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A strong-coupling theory for dirty bulk superconductors is developed, which in particular constitutes a unifying description of most disorder effects considered so far separately within strong coupling or Bardeen-Cooper-Schrieffer—type formalisms. This is achieved by calculation of all six Hartree-Fock self-energy contributions (four exchange and two direct ones) within an exact eigenstate formalism, which yields disorder renormalization of vertices formally exactly in terms of electron-correlation functions. Compared with standard dirty-limit theory, this extends the region of validity substantially toward strong disorder. The most important points are that Hartree diagrams are crucial, and that one more renormalization function arising from the normal self-energy has to be introduced. A systematic procedure for calculation of the disorder-induced vertex corrections is given as well as a detailed discussion of how previous approaches are derived as special cases of the present theory. Among these are the increase of λ due to transverse phonons found by Keck and Schmid, and various effects discussed recently in connection with the T_c degradation observed in high-T_c superconductors. A preliminary discussion of new effects includes a novel diffusion enhancement of λ and T_c-decreasing effects due to the normal self-energy. All these effects are found to be of comparable magnitude. The structure of the theory favors disorder-induced T_c degradation for strong coupling, and enhancement of T_c for weak coupling.

I. INTRODUCTION

The strong-coupling theory for clean superconductors is one of the most successful chapters of condensed-matter physics. Soon after the breakthrough achieved by Bardeen, Cooper, and Schrieffer[1] (BCS) it was realized by Eliashberg[2] that the phenomenological attractive interaction of BCS could be replaced by a more realistic interaction on a microscopic level, which takes care of the the retarded nature of the electron-phonon interaction. Additionally, Eliashberg's theory includes the quasiparticle damping which had been neglected in the Fermi-liquid arguments of BCS. This was achieved by writing down a coupled-field theory for the fermion-boson problem with the help of the Green's-function technique introduced by Gorkov,[3] and evaluating the self-energies in the Hartree-Fock approximation. Later this scheme was generalized to take into account the repulsive Coulomb interaction between the electrons as well.[4–6] In the formulation of Scalapino, Schrieffer, and Wilkins[6] (SSW), this theory still constitutes the state of the art in superconductivity theory. It turned out to be quantitatively correct, which is particularly impressive in the comparison with tunneling experiments.[6] The approximate solution for T_c given by McMillan[7] has become a most popular tool which is reliable for a large class of materials. The key for understanding this somewhat miraculous success is a theorem found by Migdal[8] for the normal state, which shows that the expansion parameter for the perturbation theory is not the electron-phonon coupling, but rather the square root of the ratio of electronic and ionic mass, which is of the order of 10^{-2}. This has been realized to be true for the superconducting state as well by Eliashberg[2] and Nambu.[9]

For impure superconductors, progress has been less impressive. It soon became clear that because of the s-wave pairing in the superconducting state, very low concentrations of paramagnetic impurities drastically reduce T_c, and finally completely suppress superconductivity.[10] On the other hand, there is no obvious reason why nonmagnetic impurities should affect the Cooper pairs, and indeed Anderson[11] showed that a BCS-type ground state can be formed with the exact electronic states including modifications by impurities. As a result, one expects no dependence at all of T_c on nonmagnetic disorder,[12,13] a fact often referred to as Anderson's theorem. There are effects on the behavior in a magnetic field, though, since even in BCS theory the critical field, in contrast to T_c, is determined by a density correlation function at finite wave numbers, which is sensitive to disorder. This has been worked out in detail,[14] and constitutes the standard dirty-limit theory of superconductivity.

Though Anderson's theorem is undoubtedly true within BCS theory, it gradually became clear that the microscopic interactions do depend on disorder in a rather complicated way. More precisely, there has been increasing experimental evidence for a disorder dependence of T_c which cannot be explained by changes in the electronic density of states or in the phonon spectrum. First it was noticed that many weak-coupling superconductors show an enhancement of T_c if disorder is introduced.[15] These are relatively small effects, typically of the order of 10–20%. There are also some examples of a more drastic rise of T_c in low-T_c materials, the most prominent being Mo_3Ge,[16] granular Al,[17,15] and amorphous Ga.[15] However, these are considered exceptions rather than typical behavior. It was also found[15] that low-T_c strong coupling materials, such as Hg and Pb, show a weak decrease of T_c with increasing disorder. More recently, interest in the disorder dependence of T_c has arisen as a byproduct of the successful fabrication of high-T_c ($T_c \gtrsim 10$ K) materials, most of which are fairly dirty by metallic standards, with resistivities ρ of 10–40 $\mu\Omega$ cm. It turns out that invariably the highest T_c is obtained for the cleanest specimen. If disorder is increased, by means of irradiation of otherwise, T_c drops dramatically.[18] Thereby a number of correlations are observed, the most obvious of which is that the decrease is more rapid the higher T_c in the clean limit, while for $T_c > 10$ K no example is known where T_c would rise with disorder. The nature of the disorder seems to be unimportant: For T_c plotted against the resistivity just above T_c (or against the extrapolated residual resistivity) one obtains a single curve for each material, regardless of how the disorder has been introduced. This is sometimes called the "universal" T_c degradation. The degradation phenomenon is not specific to any particular class of materials, but rather is observed for a wide range of substances.[18]

There have been a number of theoretical attempts to deal with the problem of disorder-induced changes of T_c. Since there are strong reasons to believe that disorder-induced renormalizations of the basic interactions are responsible for the effect, it is clear that one should work within the frame of strong-coupling theory. A pioneering step has been the work of Keck and Schmid,[19] who tried to explain the small increase of T_c in the low-T_c materials mentioned above. They generalized the original Eliashberg equations (without the Coulomb interaction) by calculating the self-energy in an exact eigenstate formalism. After reducing the disorder dependence to an electronic stress correlation function entering the Eliashberg function $\alpha^2F(\omega)$, they calculated the correlation function in what amounts to a kinetic theory description. The result accounts well for the initial rise of T_c up to $\rho \lesssim 10$ $\mu\Omega$ cm, but fails completely at higher disorder. Technically, their calculation considers impurity corrections of the phonon part of the anomalous self-energy, W, and of the renormalization function Z.

Contrary to Keck and Schmid, Anderson *et al.*[20] concentrated on the Coulomb pseudopotential, i.e., the Coulomb contribution to W, in order to explain the T_c-degradation effect. They assumed that the phonon parts of the self-energies do not change, and neglected changes of the renormalization function Z. They found that electronic diffusion (as opposed to free motion) does not change the Coulomb pseudopotential significantly, essentially since impurity corrections enter μ, so that

$$\mu^* = \mu/[1+\mu \ln(E_F/\hbar\omega_0)]$$

is changed little. They then considered systems close to an Anderson transition,[21] and in this regime they found a strong enhancement of μ^*. However, Gurvitch[22] gives experimental evidence that the materials considered by Anderson *et al.* are still far from an Anderson transition. Entin-Wohlman *et al.*[22] have also questioned the relevance of the Anderson localization corrections for the materials under consideration. They considered a linear chain model for $A-15$ superconductors, and propose an enhancement of μ^* due to diffusion in the quasi-one-dimensional structure. They also assumed phonon contributions to be unaffected.

Wysokinski and Kuzemsky[23] have studied a model for transition metal alloys, and treated the disorder by means of a coherent potential approximation. Their result for the renormalized Coulomb potential includes the diffusion corrections found in Ref. 20, but the implications of this result were not discussed. They also obtain a renormalization of the electron-phonon coupling.

There has also been a considerable amount of work based on BCS-Bogoliubov models. Fukuyama and coworkers[24] have studied corrections to both the pair propagator and the schematic repulsive interaction in the weak disorder regime, they found a suppression of T_c. Kapitulnik and Kotlier[25] have considered a Ginzburg-Landau theory in the scaling regime near an Anderson transition. Neglecting Coulomb effects, they found a sharp decrease of the coherence length, which implies that fluctuations effects should be important in this regime. The present author has studied[26] a BCS-Bogoliubov model with interaction corrections to the pair propagator and to the repulsive potential taken into account, in a spirit similar to that of Ref. 24. Technically speaking, this work considered diffusion corrections of the type found by Altshuler and Aronov[27] to the Coulomb part of the normal self-energy. The result for T_c explains the degradation with reasonable parameters, but since the phenomenological model neglects all effects on the interaction parameters found in the works described above, the theory is clearly not entirely satisfactory.

In this situation it is obvious that one should try a unifying description of what has been achieved so far. This could be done by starting from SSW theory formulated in an exact eigenstate representation to incorporate disorder in terms of correlation functions, as was done in Refs. 19 and 20. This way one can easily combine these two results. In addition, however, it would be necessary to keep the full normal self-energy (not just the part usually denoted by Z) in order to incorporate the effects of Refs. 24 and 26. This means that one would have to introduce at least one more renormalization function. It would also be necessary to include the Hartree contributions to both the phonon and the Coulomb part of the normal self-energy, since contrary to standard theory they do not vanish if strong disorder is present. This way one would obtain coupled integral equations for three quantities rather than two, and they would be integral equations in energy as well as in frequency. Since disorder would have been included formally exactly in terms of (*a priori* unknown) correlation functions, these equations should be of the same accuracy as the standard theory for clean and weakly disordered systems.

It is the aim of the present paper to perform part of the program sketched above. We will succeed in obtaining a unified description of most of the effects discussed previously, but we have so far been unable to obtain an accurate theory for arbitrary disorder. The main obstacle has been that we have been unable to include the dynamically screened Coulomb potential in the presence of disorder, and have used a static screening approximation instead. The resulting theory should be reliable for small and intermediate disorder (for resistivities $\rho \lesssim 100$ $\mu\Omega$ cm), but near a metal insulator transition presumably still other effects would come into play. The paper is organized as follows. In Sec. II we derive and discuss the basic electron-impurity, electron-electron, and electron-phonon interactions. Impurity renormalizations of the respective vertices are briefly examined diagrammatically for later comparison with results obtained with the correlation function technique. In Sec. III we define the relevant Green's functions and self-energies, and calculate the latter with the exact eigenstate formalism, which includes the impurity renormalizations. In Sec. IV we perform various frequency summations to derive the strong coupling equations. Section V demonstrates how the previous theories can be obtained as special cases of the present one.

The conclusion is that all previous approaches have neglected effects equal to or larger than the ones considered, and are therefore not reliable. The present theory also includes a number of new affects, which also are of comparable magnitude. Among them is a novel diffusion enhancement of the electron-phonon coupling parameter λ, and effects due to the normal self-energy which tend to decrease T_c. We briefly discuss these new affects, and show that they tend to yield an increase of T_c with disorder for weak coupling materials, and a decrease for strong coupling ones. This is in accord with the experimental facts. A more detailed discussion and comparison with experiment will be given in a separate paper.[28]

Finally we emphasize that we will deal with a theory for bulk materials only. Phenomena in superconducting thin films are perhaps even more interesting, but apparently also more difficult for a theoretical description.

II. EFFECTIVE INTERACTIONS

We consider a system of electrons which interact with static impurities, among each other via a screed Coulomb interaction, and with lattice vibrations via an effective electron-phonon interaction. The electron-impurity interaction is the least sophisticated one, and it influences the other two, so we start with its description.

A. Electron-impurity interaction

The free motion of electrons disturbed by scattering off impurities is described by the Hamiltonian

$$H_{e-i}=\sum_{\sigma}\int d\mathbf{x}\,\Psi_{\sigma}^{\dagger}(\mathbf{x})\left[\frac{-1}{2m}\nabla_{\mathbf{x}}^{2}+\sum_{i}V(\mathbf{x}-\mathbf{x}_i)\right]\Psi_{\sigma}(\mathbf{x})\ . \tag{2.1}$$

Here the $\Psi,\Psi^{\dagger}$ are the usual electron field operators, $\mathbf{x}_i$ denotes the random sites of the impurities, and V is the electron-impurity scattering potential. We will find it convenient to work within the so-called exact eigenstate representation.[11,19] Thereby one assumes that H_{e-i} has been diagonalized with a complete set of orthonormal eigenfunctions $\{\psi_n(\mathbf{x})\}$, and corresponding eigenenergies E_n. Occasionally we will use standard diagrammatic perturbation theory[29] to deal with impurity corrections to various quantities. Accordingly, we define the one-particle Green's function

$$G(\mathbf{x}-\mathbf{x}',\tau-\tau')=-\langle\langle T_{\tau}\Psi_{\sigma}(\mathbf{x},\tau)\Psi_{\sigma}^{\dagger}(\mathbf{x}',\tau')\rangle_{H_{e-i}}\rangle_{\text{av}}\ . \tag{2.2}$$

Here T_τ is the ordering operator for the imaginary time τ, $\langle\cdots\rangle_{H_{e-i}}$ means a thermodynamic average formed with the electron-impurity Hamiltonian, and $\langle\cdots\rangle_{\text{av}}$ denotes the impurity average. The Fourier transform of G can be written as

$$G(\mathbf{k},i\omega_n)=[i\omega_n-\mathbf{k}^2/2m+\Sigma(\mathbf{k},i\omega_n)]^{-1}\ . \tag{2.3a}$$

Here $\omega_n=2\pi T(n+\frac{1}{2})$ is the Matsubara frequency, and we measure energies from the Fermi level. In diagrammatic arguments concerning impurity averages, we restrict ourselves to pointlike scatterers. The self-energy reads in lowest order

$$\Sigma(\mathbf{k},i\omega_n)=(i/2\tau_i)\operatorname{sgn}(\omega_n)\ , \tag{2.3b}$$

with the single-particle decay rate

$$1/\tau_i=2\pi N_F U_0\ . \tag{2.3c}$$

Here N_F is the density of states per spin at the Fermi level, $U_0\equiv n_i\,|\,V(\mathbf{k})\,|^2$ with n_i the impurity density, and $V(\mathbf{k})$ the Fourier transform of the scattering potential.

B. Screened Coulomb interaction

Originally, the electrons interact via a bare Coulomb potential

$$v(\mathbf{q})=(1-\delta_{\mathbf{q},0})4\pi e^2/q^2\ .$$

It is convenient to include screening from the very beginning. We write down an effective Coulomb potential[29,30]

$$V_c(\mathbf{q},i\Omega_m)=v(\mathbf{q})/\epsilon(\mathbf{q},i\Omega_m)\ , \tag{2.4}$$

where the dielectric function ϵ is defined in terms of a screened density-density response function,

$$\epsilon(\mathbf{q},i\Omega_m)=1+v(\mathbf{q})\chi_{d-d}^{sc}(\mathbf{q},i\Omega_m)\ , \tag{2.5}$$

with $\Omega_m=2\pi Tm$. χ_{d-d}^{sc} is irreducible with respect to the Coulomb interaction, but contains the full disorder corrections due to H_{e-i}. As a consequence, χ_{d-d}^{sc} behaves rather delicately in the limit of small frequencies and wavenumbers. In the present paper, we ignore the subtleties which presumably arise from this point, and simply use the Thomas-Fermi approximation. That is, we replace χ_{d-d}^{sc} by the response of the noninteracting system in the static long-wave-length limit: $\chi_{d-d}^{sc}(\mathbf{q},i\Omega_m)\simeq\chi_{d-d}^{(0)}(\mathbf{q},i0)=2N_F$.[30,31] Then we have for the effective Coulomb potential

$$V_c(\mathbf{q})\equiv V_c(\mathbf{q},i0)\simeq(1-\delta_{\mathbf{q},0})\frac{1}{2N_F}\frac{\kappa^2}{q^2+\kappa^2}\ , \tag{2.6}$$

where $\kappa=(4\pi e^2 2N_F)^{1/2}$ is the Thomas-Fermi screening wave number. This effective Coulomb interaction couples to the electron density,

$$H_{e-e}^{\text{eff}}=\sum_{\sigma,\sigma'}\int d\mathbf{x}\,d\mathbf{y}\,\Psi_{\sigma}^{\dagger}(\mathbf{x})\Psi_{\sigma}(\mathbf{x})V_c(\mathbf{x}-\mathbf{y})\Psi_{\sigma'}^{\dagger}(\mathbf{y})\Psi_{\sigma'}(\mathbf{y})\ . \tag{2.7}$$

Consequently, the bare electron-Coulomb vertex is very simple. If we combine space and time arguments by the symbol $1 \equiv (\mathbf{x}_1, \tau_1)$ etc., the bare vertex for the scattering of an electron from point 1 to point 3 by a potential at point 2, reads

$$\Gamma^c(1,2,3) = \delta(1-2)\delta(2-3) \, , \tag{2.8}$$

where

$$\delta(1-2) \equiv \delta(\mathbf{x}_1 - \mathbf{x}_2)\delta(\tau_1 - \tau_2) \, ,$$

etc. In Fourier space, scattering of an electron from $\mathbf{k}+\mathbf{q}/2$ to $\mathbf{k}-\mathbf{q}/2$ is associated with a factor

$$\Gamma^c(\mathbf{k},\mathbf{q}) = 1 \, . \tag{2.8$'$}$$

For later reference, we briefly recall that the Coulomb vertex is very sensitive to disorder.[27] Figure 1(a) shows renormalization of Γ^c by means of impurity ladder diagrams [diffusion-propagator renormalization (DPR)]. Γ^c is denoted by a dot, the renormalized vertex $\tilde{\Gamma}^c$ by a circled dot, and straight lines denote electron propagators as given by Eq. (2.2). V_c is symbolized by a dashed-dotted line, and the electron-impurity interaction is shown as dashed lines, where each cross carriers a factor U_0. Evaluation of Fig. 1(a) is straightforward and shows that the Coulomb vertex at small wave numbers is strongly enhanced by disorder due to the diffusion pole structure inherent in the vertex correction. This becomes particularly clear if we sum $\Delta\tilde{\Gamma}^c \equiv \tilde{\Gamma}^c - \Gamma^c$ over $\mathbf{k}$ and $\mathbf{q}$:

$$\begin{aligned}\Delta\tilde{\gamma}_c(i\Omega_m) &\equiv \sum_{\mathbf{k},\mathbf{q}} \Delta\tilde{\Gamma}^c(\mathbf{k},\mathbf{q};i\Omega_m) \\ &= \sum_{\mathbf{q}} I_0(\mathbf{q},i\Omega_m)/[1 - I_0(\mathbf{q},i\Omega_m)] \, , \end{aligned} \tag{2.9}$$

where I_0 is the first of a series of integrals defined as[32]

$$\begin{aligned} I_l(\mathbf{q},i\Omega_m) = U_0 \sum_{\mathbf{p}} (\mathbf{pq}/pq)^l G(\mathbf{p}+\mathbf{q}/2, i\omega_n) \\ \times G(\mathbf{p}-\mathbf{q}/2, i\omega_n - i\Omega_n) \, . \end{aligned} \tag{2.10}$$

For $\omega_n(\omega_n - \Omega_m) < 0$, and l even we have at small q and Ω,

$$I_l(\mathbf{q},i\Omega_m) = \frac{1}{l+1}\left[1 - \tau_i\left[|\Omega_m| + Dq^2\frac{3l+3}{l+3}\right] + \cdots\right] \, , \tag{2.11}$$

where $D = v_F^2\tau_i/3$ is the diffusion constant with Fermi velocity v_F. At $\Omega = 0$, $\Delta\tilde{\gamma}_c$ is proportional to $1/D$, and for small Ω it rises like $|\Omega|^{1/2}$.

Another impurity contribution to the renormalization of Γ^c is obtained by replacing the ladder in the Fig. 1(a) by the maximally crossed ladder, known from the Anderson-localization problem,[21,33] shown in Fig. 1(b) [Cooper-propagator renormalization (CPR)]. Denoting the resulting vertex correction by $\Delta\tilde{\tilde{\Gamma}}^c$, we obtain

$$\begin{aligned}\Delta\tilde{\tilde{\gamma}}_c(i\Omega_m) &\equiv \sum_{\mathbf{k},\mathbf{q}} \Delta\tilde{\tilde{\Gamma}}^c(\mathbf{k},\mathbf{q};i\Omega_m) \\ &= \sum_{\mathbf{k}} I_0(\mathbf{k},i\Omega_m) \sum_{\mathbf{q}} 1/[1 - I_0(\mathbf{q},i\Omega_m)] \, . \end{aligned} \tag{2.12}$$

This shows that the maximally crossed ladder and the ordinary impurity ladder yield the same type of diffusion enhancement for the Coulomb vertex.[34]

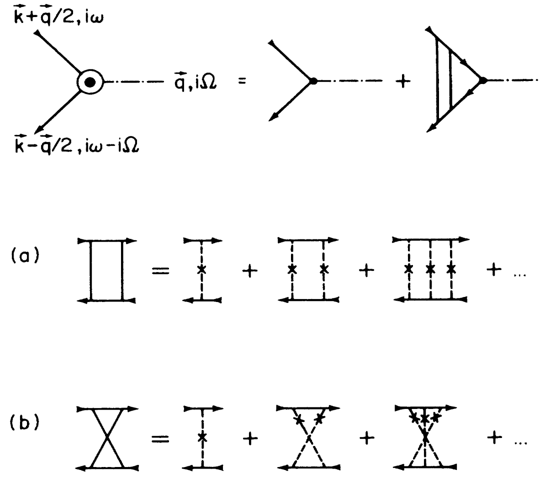

FIG. 1. (a) Renormalization of the Coulomb vertex due to impurity ladder diagrams (diffusion-propagator renormalization). (b) Maximally crossed ladder diagrams. Insertion in (a) instead of the ordinary ladder yields the Cooper-propagator renormalization of the Coulomb vertex.

C. Effective electron-phonon interaction

The problem of electron-phonon interaction in disordered metals is not straightforward, mainly because uncritical application of the usual Fröhlich model yields an incorrect result. We will use a model proposed by Blount and Tsuneto,[35] and, in a form particularly suitable for applications, by Schmid.[32] In particular, Schmid has derived a model interaction which for impure systems simply replaces the Fröhlich model, without requiring any changes in the formalism.

The result is the following effective interaction between electrons and phonons,[32,19]

$$\begin{aligned} H_{e-p}^{\text{eff}} = \frac{1}{m(\rho_i)^{1/2}} \sum_{\mathbf{q}} \frac{q}{\omega_L(\mathbf{q})} [m\tau_L(\mathbf{q}) - \tfrac{1}{3}k_F^2 d(\mathbf{q})]\phi_L(\mathbf{q}) \\ + \frac{2}{m(\rho_i)^{1/2}} \sum_{\mathbf{q}} \frac{q}{\omega_T(\mathbf{q})} m\tau_T(\mathbf{q})\phi_T(\mathbf{q}) \, . \end{aligned} \tag{2.13}$$

Here the $\tau(\mathbf{q})$ are the longitudinal (L) and transverse (T) parts of the Fourier transforms of the electron stress operator,

$$\begin{aligned} \tau_{\alpha\beta}(\mathbf{x}) = \frac{1}{4m} \sum_{\sigma} \int d\mathbf{y} (\nabla_{\mathbf{x}} - \nabla_{\mathbf{y}})_\alpha (\nabla_{\mathbf{x}} - \nabla_{\mathbf{y}})_\beta \\ \times \Psi_\sigma^\dagger(\mathbf{x})\Psi_\sigma(\mathbf{y})\delta(\mathbf{x}-\mathbf{y}) \, . \end{aligned} \tag{2.14}$$

$$d(\mathbf{q}) = \sum_{\sigma} \int d\mathbf{x}\, e^{i\mathbf{qx}} \Psi_\sigma^\dagger(\mathbf{x})\Psi_\sigma(\mathbf{x}) \, , \tag{2.15}$$

is the Fourier transform of the electronic density operator. The phonon field operators $\phi_b(\mathbf{q})$ are defined as

$$\phi_b(\mathbf{q}) = i[\omega_b(\mathbf{q})/2]^{1/2}[b_b(\mathbf{q}) + b_b^\dagger(-\mathbf{q})] \, , \tag{2.16}$$

where $\omega_b(\mathbf{q})$ is the dispersion relation for polarization branch b, the $b^\dagger, b$ are phonon creation and annihilation operators, and the $c_b = \omega_b(q \to 0)/q$ are the sound velocities. With this normalization, the bare phonon propagator

$$D_b^{(0)}(\mathbf{q},i\Omega_m)=-\int_0^{1/T} d\tau\, e^{i\Omega_m\tau}\langle T_\tau\phi_b(\mathbf{q},\tau)\phi_b^\dagger(\mathbf{q})\rangle^{(0)}=\omega_b^2(\mathbf{q})/[(i\Omega_m)^2-\omega_b^2(\mathbf{q})]\ , \tag{2.17}$$

is dimensionless. From Eq. (2.13) we read off the screened electron-phonon vertex in $\mathbf{k}$ space[19]

$$\Gamma_L(\mathbf{k},\mathbf{q})=\frac{-q}{m(\rho_i)^{1/2}\omega_L(\mathbf{q})}[(\mathbf{kq}/q)^2-k_F^2/3]\ . \tag{2.18a}$$

$$\Gamma_T(\mathbf{k},\mathbf{q})=\frac{-q}{m(\rho_i)^{1/2}\omega_T(\mathbf{q})}(\mathbf{kq}/q)\,|(\mathbf{q}/q)\times(\mathbf{k}\times\mathbf{q}/q)|\ . \tag{2.18b}$$

We will also need the corresponding expression in real space, viz.,

$$\Gamma_{\alpha\beta}(1,2,3)=-\tfrac{1}{4}\delta(\tau_1-\tau_2)\delta(\tau_2-\tau_3)\int d\mathbf{y}(\nabla_{\mathbf{x}_2}-\nabla_{\mathbf{y}})_\alpha(\nabla_{\mathbf{x}_2}-\nabla_{\mathbf{y}})_\beta\delta(\mathbf{x}_2-\mathbf{x}_1)\delta(\mathbf{y}-\mathbf{x}_3)\delta(\mathbf{y}-\mathbf{x}_2)-\tfrac{1}{3}k_F^2\delta_{\alpha\beta}\delta(1-2)\delta(2-3)\ . \tag{2.19}$$

Here we have combined space and time arguments as before, and we have defined a real-space phonon propagator

$$D_{\alpha\beta,\gamma\sigma}(1,2)\equiv\sum_{\mathbf{q},b} e^{-i\mathbf{q}(\mathbf{x}_1-\mathbf{x}_2)}\left[\frac{q^\alpha e_b^\beta(\mathbf{q})}{m(\rho_i)^{1/2}\omega_b(\mathbf{q})}D_b(\mathbf{q},\tau_1-\tau_2)\frac{q^\gamma e_b^\delta(\mathbf{q})}{m(\rho_i)^{1/2}\omega_b(\mathbf{q})}\right]\ . \tag{2.20}$$

We now consider the impurity renormalization of Γ_L in the same way as we have done for the Coulomb vertex. Figures 2(a) and 2(b) show the vertex corrections due to diffusion-propagator and Cooper-propagator renormalization, respectively. Here and in the following, we denote the electron-phonon vertex by a triangle, and the phonon propagator by a wavy line. We define quantities $\Delta\tilde{\gamma}_L(i\Omega_m)$, and $\Delta\tilde{\tilde{\gamma}}_L(i\Omega_m)$ in complete analogy to $\Delta\tilde{\gamma}_c,\Delta\tilde{\tilde{\gamma}}_c$ in Sec. II B. In terms of the integrals defined in Eq. (2.10), we obtain

$$\Delta\tilde{\gamma}_L(i\Omega_m)=\frac{k_F^2}{3m(\rho_i)^{1/2}c_L}\sum_{\mathbf{q}}[I_0(\mathbf{q},i\Omega_m)-3I_2(\mathbf{q},i\Omega_m)]/[1-I_0(\mathbf{q},i\Omega_m)]\ , \tag{2.21}$$

and

$$\Delta\tilde{\tilde{\gamma}}_L(i\Omega_m)=\frac{k_F^2}{3m(\rho_i)^{1/2}c_L}\sum_{\mathbf{k}}[I_0(\mathbf{k},i\Omega_m)-3I_2(\mathbf{k},i\Omega_m)]\sum_{\mathbf{q}}1/[1-I_0(\mathbf{q},i\Omega_m)]\ . \tag{2.22}$$

With the help of Eq. (2.11), we see that there is no diffusion enhancement of the longitudinal electron-phonon vertex due to DPR (Ref. 32) (the transverse vertex is obviously not enhanced either). However, there is a diffusion enhancement due to CPR, since in Eq. (2.22) the integrations over I_0-3I_2 and over $1/(1-I_0)$ decouple. This has interesting consequences for the sound attenuation, as has been discussed recently.[36] It is also important for the superconductivity, as we will see below. We will come back to the results expressed in Eqs. (2.9), (2.12), (2.21), and (2.22) in the next section, where we will study impurity renormalizations with a more powerful technique.

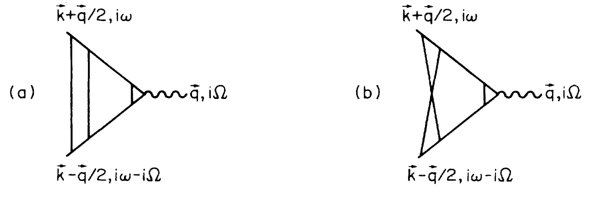

FIG. 2. Electron-phonon vertex correction due to (a) DPR and (b) CPR. Summation over $\mathbf{k}$ yields the quantities $\Delta\tilde{\gamma}$ and $\Delta\tilde{\tilde{\gamma}}$ in the text.

III. GREEN'S FUNCTIONS AND SELF-ENERGIES

A. Basic definitions

We now define Green's functions for the fully interacting system. Following Gorkov,[3] we define a normal electron Green's function

$$\mathcal{G}(1,2)\equiv\mathcal{G}(\mathbf{x}_1,\mathbf{x}_2;\tau_1,\tau_2)=-\langle T_\tau\Psi_\sigma(\mathbf{x}_1,\tau_1)\Psi_\sigma^\dagger(\mathbf{x}_2,\tau_2)\rangle\ . \tag{3.1}$$

and an anomalous one

$$\mathcal{F}(1,2)\equiv\mathcal{F}(\mathbf{x}_1,\mathbf{x}_2;\tau_1,\tau_2)=\langle T_\tau\Psi_\downarrow(\mathbf{x}_1,\tau_1)\Psi_\uparrow(\mathbf{x}_2,\tau_2)\rangle\ . \tag{3.2}$$

Here $\langle\cdots\rangle$ denotes a thermodynamic average formed with the full Hamiltonian of the system,[37] and we have not yet included an impurity average. It is useful to consider these Green's functions in the exact eigenstate basis defined in Sec. II A. We follow Ref. 19, and assume $\mathcal{G}$ and $\mathcal{F}$ to be diagonal in the exact eigenstate basis: $\mathcal{G}_{nm}\simeq\delta_{nm}\mathcal{G}_n$, $\mathcal{F}_{nm}\simeq\delta_{nm}\mathcal{F}_n$. Consequently, the self-energies are also diagonal. As in conventional strong coupling theory,[6,29] we retain two different self-energy functions formed with $\mathcal{G}$ and $\mathcal{F}$, respectively, and denote them by S and W. The generalized Dyson equations read

(from here on we omit the discrete index on Matsubara frequencies)

$$[i\omega-E_n-S_n(i\omega)]\mathscr{G}_n(i\omega)+W_n(i\omega)\mathscr{F}_n^\dagger(i\omega)=1\ , \tag{3.3a}$$

$$[-i\omega-E_n-S_n(-i\omega)]\mathscr{F}_n^\dagger(i\omega)-W_n(i\omega)\mathscr{G}_n(i\omega)=0\ . \tag{3.3b}$$

Here the self-energies S_n, W_n, are obtained from the corresponding real-space quantities. These are shown diagrammatically in Fig. 3. They have to be calculated according to the usual rules.[29] Green's functions $\mathscr{G}(\mathscr{F})$ are drawn as unidirectional (bidirectional) straight double lines. Finally, one has to perform the appropriate transformations to obtain S_n, W_n in terms of $\mathscr{G}_n, \mathscr{F}_n$. Notice that in Fig. 3 we have shown Hartree (H) contributions to the self-energie, since in our case they are nonzero as we will see. There are no Hartree contributions to W because of particle number conservation. So far we have assumed a fixed distribution of impurities. To get physical results, however, we have to average over the random impurity positions. We perform this by using the method given by Keck and Schmid[19] (see also Ref. 38). This leads to the following prescription for constructing energy-dependent, averaged quantities $f(\epsilon)$ from their exact eigenstate representation f_n:

$$f(\epsilon)=\frac{1}{N_F}\left\langle\sum_n \delta(\epsilon-E_n)f_n\right\rangle_{\text{av}}\ . \tag{3.4}$$

We now apply this average to Eqs. (3.3). Thereby we assume[19] that Green's functions and self-energies depend on n only via E_n, and that the sum may be replaced by an integral: $(1/N_F)\sum_n \simeq \int dE_n$. The latter is tantamount to assuming the exact eigenstates to be nondegenerate, and the former implies factorization of averages over products.[39] We then can solve the average Eqs. (3.3) for $\mathscr{G}$ and $\mathscr{F}$. We decompose the normal self-energy in the usual way:

$$S(\epsilon,i\omega)=i\omega[1-Z(\epsilon,i\omega)]+Y(\epsilon,i\omega)\ ,$$

where both Z and Y are even functions of the Matsubara frequency, and obtain

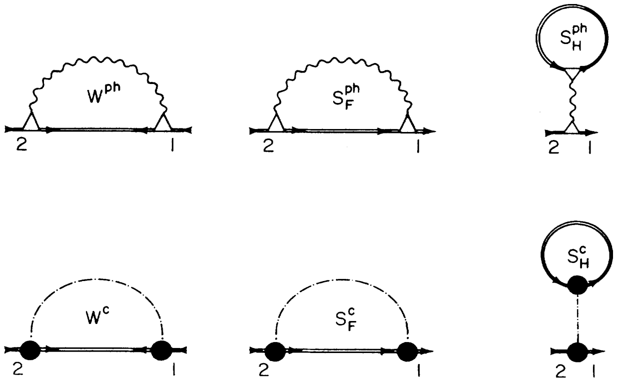

FIG. 3. All lowest-order phonon (ph) and Coulomb (c) contributions to the self energies W,S. Hartree contributions with reversed direction of electron loop are not shown.

$$\mathscr{G}(\epsilon,i\omega)=\frac{i\omega Z(\epsilon,i\omega)+[\epsilon+Y(\epsilon,i\omega)]}{[i\omega Z(\epsilon,i\omega)]^2-[\epsilon+Y(\epsilon,i\omega)]^2-W(\epsilon,i\omega)]^2}\ , \tag{3.5a}$$

$$\mathscr{F}^\dagger(\epsilon,i\omega)=\frac{-W(\epsilon,i\omega)}{[i\omega Z(\epsilon,i\omega)]^2-[\epsilon+Y(\epsilon,i\omega)]^2-W(\epsilon,i\omega)^2}\ . \tag{3.5b}$$

In the following section, we will calculate W, Z, and Y from Fig. 3 in terms of $\mathscr{G}$ and $\mathscr{F}^\dagger$ to obtain closed equations.

B. Calculation of the self-energies

1. Phonon contributions

According to the rules given in the previous section, we obtain from Fig. 3 the phonon part of the self-energy. We transform into the exact eigenstate basis and take the average by the procedure stated in Eq. (3.4). In doing so, we average separately over the phonon propagator, neglecting all cumulants as we have done in deriving Eq. (3.5). The result is

$$W^{\text{ph}}(\epsilon,i\omega)=-T\sum_{i\omega'}\sum_{\mathbf{q},b}\int d\epsilon' R_b^F(\mathbf{q},\epsilon-\epsilon')\mathscr{F}(\epsilon',i\omega')D_b(\mathbf{q},i\omega-i\omega')\ , \tag{3.6}$$

where

$$\begin{aligned}R_b^F(q,\epsilon-\epsilon')=&\sum_{\alpha,\beta}\sum_{\gamma,\delta}\frac{q^\alpha e_b^\beta(\mathbf{q})}{m(\rho_i)^{1/2}\omega_b(\mathbf{q})}\frac{q^\gamma e_b^\delta(\mathbf{q})}{m(\rho_i)^{1/2}\omega_b(\mathbf{q})}\frac{1}{N_F}\\&\times\int d\mathbf{x}_1\,d\mathbf{x}_2\,e^{-i\mathbf{q}(\mathbf{x}_1-\mathbf{x}_2)}\\&\times\int d\mathbf{y}_1[-\tfrac{1}{4}(\nabla_{\mathbf{x}_1}-\nabla_{\mathbf{y}_1})_\alpha(\nabla_{\mathbf{x}_1}-\nabla_{\mathbf{y}_1})_\beta-\tfrac{1}{3}k_F^2\delta_{\alpha\beta}]\delta(\mathbf{x}_1-\mathbf{y}_1)\\&\times\int d\mathbf{y}_2[-\tfrac{1}{4}(\nabla_{\mathbf{x}_2}-\nabla_{\mathbf{y}_2})_\gamma(\nabla_{\mathbf{x}_2}-\nabla_{\mathbf{y}_2})_\delta-\tfrac{1}{3}k_F^2\delta_{\gamma\delta}]\delta(\mathbf{x}_2-\mathbf{y}_2)\\&\times\left\langle\sum_{n,m}\delta(\epsilon-E_n)\delta(\epsilon'-E_m)\psi_n^*(\mathbf{x}_1)\psi_m^*(\mathbf{x}_2)\psi_n(\mathbf{y}_2)\psi_m(\mathbf{y}_1)\right\rangle_{\text{av}}\ .\end{aligned} \tag{3.7}$$

Here we have used the fact, explained in Appendix A, that in an excellent approximation R depends on $\epsilon-\epsilon'$ only. In Appendix A we also show that the impurity average of four wave functions occurring in Eq. (3.7) can be expressed in terms of a Kubo correlation function for noninteracting electrons. Performing a Fourier transform, we finally obtain

$$R_b^F(\mathbf{q},\epsilon-\epsilon')=\sum_{\alpha,\beta}\sum_{\gamma,\delta}\frac{q^\alpha e_b^\beta(\mathbf{q})}{m(\rho_i)^{1/2}\omega_b(\mathbf{q})}\frac{q^\gamma e_b^\delta(\mathbf{q})}{m(\rho_i)^{1/2}\omega_b(\mathbf{q})}\frac{1}{\pi N_F}\sum_{\mathbf{k},\mathbf{p}}[k_\alpha k_\beta-\tfrac{1}{3}k_F^2\delta_{\alpha\beta}]\Phi''_{\mathbf{kp}}(q,\epsilon-\epsilon')[p_\gamma p_\delta-\tfrac{1}{3}k_F^2\delta_{\gamma\delta}]\ , \tag{3.7'}$$

where $\Phi''_{\mathbf{kp}}$ is the absorptive part of the phase-space Kubo function defined in Appendix A.

Exactly the same reasoning applies to the remaining two diagrams. For the phonon exchange contribution to the normal self-energy we obtain

$$S_{\text{ph}}^F(\epsilon,i\omega)=-T\sum_{i\omega'}\sum_{\mathbf{q},b}\int d\epsilon' R_b^F(\mathbf{q},\epsilon-\epsilon')\mathscr{G}(\epsilon',i\omega')D_b(\mathbf{q},i\omega-i\omega')\ . \tag{3.8}$$

The Hartree contribution reads

$$S_{\text{ph}}^H(\epsilon,i\omega)=2T\sum_{i\omega'}\sum_{\mathbf{q},b}\int d\epsilon' R_b^H(\mathbf{q},\epsilon-\epsilon')\mathscr{G}(\epsilon',i\omega')D_b(\mathbf{q},i\Omega=0)\ , \tag{3.9}$$

with

$$R_b^H(\mathbf{q},\epsilon-\epsilon')=\sum_{\alpha,\beta}\sum_{\gamma,\delta}\frac{q^\alpha e_b^\beta(\mathbf{q})}{m(\rho_i)^{1/2}\omega_b(\mathbf{q})}\frac{q^\gamma e_b^\delta(\mathbf{q})}{m(\rho_i)^{1/2}\omega_b(\mathbf{q})}\frac{1}{\pi N_F}$$
$$\times\sum_{\mathbf{k},\mathbf{p}}(k_\alpha k_\beta-\tfrac{1}{3}k_F^2\delta_{\alpha\beta})\Phi''_{\frac{\mathbf{k}+\mathbf{p}-\mathbf{q}}{2},\frac{\mathbf{k}+\mathbf{p}+\mathbf{q}}{2}}(\mathbf{k}-\mathbf{p},\epsilon-\epsilon')(p_\gamma p_\delta-\tfrac{1}{3}k_F^2\delta_{\gamma\delta})\ . \tag{3.10}$$

2. Coulomb contributions

Calculation of the remaining three diagrams in Fig. 3, which give the Coulomb contributions to the self-energies, proceeds in exact analogy to the calculations given above. Because of our static screening approximation, the Coulomb parts of all self-energies are independent of frequency. We obtain

$$W_c(\epsilon)=-T\sum_{i\omega'}\sum_{\mathbf{q}}\int d\epsilon' R_c^F(\mathbf{q},\epsilon-\epsilon')\mathscr{F}(\epsilon',i\omega')V_c(\mathbf{q})\ , \tag{3.11}$$

$$S_c^F(\epsilon)=-T\sum_{i\omega'}\sum_{\mathbf{q}}\int d\epsilon' R_c^F(\mathbf{q},\epsilon-\epsilon')\mathscr{G}(\epsilon',i\omega')V_c(\mathbf{q})\ , \tag{3.12}$$

$$S_c^H(\epsilon)=2T\sum_{i\omega'}\sum_{\mathbf{q}}\int d\epsilon' R_c^H(\mathbf{q},\epsilon-\epsilon')\mathscr{G}(\epsilon',i\omega')V_c(\mathbf{q})\ . \tag{3.13}$$

for the Coulomb contribution to the anomalous self-energy, and the Fock and Hartree parts of the Coulomb contribution to the normal self-energy, respectively. The vertex functions read

$$R_c^F(\mathbf{q},\epsilon-\epsilon')=\frac{1}{\pi N_F}\sum_{\mathbf{k},\mathbf{p}}\Phi''_{\mathbf{kp}}(\mathbf{q},\epsilon-\epsilon') \tag{3.14}$$

$$R_c^H(\mathbf{q},\epsilon-\epsilon')=\frac{1}{\pi N_F}\sum_{\mathbf{k},\mathbf{p}}\Phi''_{\frac{\mathbf{k}+\mathbf{p}-\mathbf{q}}{2},\frac{\mathbf{k}+\mathbf{p}+\mathbf{q}}{2}}(\mathbf{k}-\mathbf{p},\epsilon-\epsilon') \tag{3.15}$$

The vertex functions $R_b^{F,H}$ and $R_c^{F,H}$ which contain the impurity renormalizations are extremely complicated, since knowledge of $\Phi''_{\mathbf{kp}}$ requires a solution of the electron-impurity problem, which cannot be obtained exactly. Furthermore, even if $\Phi''_{\mathbf{kp}}$ is explicitly known in some approximation (e.g., from kinetic theory), evaluation of the wave-vector integrals is nontrivial. In Appendix B we give a method how the latter task can be performed approximately in a systematic way.

IV. STRONG-COUPLING EQUATIONS

Basically what remains to be done is to perform the various summations other Matsubara frequencies in the expressions for the self-energies. Since the disorder does not affect the frequency dependence of the propagator, this can be done in the standard way.[29]

A. Frequency summations: Coulomb contributions

Due to our static screening approximation, the Coulomb parts of the self-energies are independent of frequency. Introducing the analytic continuations $G(\epsilon,\omega)=\mathscr{G}(\epsilon,i\omega\to\omega+i0)$, $F(\epsilon,\omega)=\mathscr{F}(\epsilon,i\omega\to\omega+i0)$, we have the spectral representation [cf. Eq. (A4)],

$$\mathscr{G}(\epsilon,i\omega)=\int\frac{d\omega}{\pi}\text{Im}G(\epsilon,\omega)/(\omega-i\omega)\ , \tag{4.1}$$

and the same way for $\mathscr{F}$. With the help of Eq. (4.1), the summations over $i\omega'$ in Eqs. (3.11)–(3.13) are elementary, and we obtain

$$W_c(\epsilon)=\int d\epsilon'\sum_{\mathbf{q}}R_c^F(\mathbf{q},\epsilon-\epsilon')V_c(\mathbf{q})$$
$$\times\int\frac{d\omega}{\pi}f(\omega)\text{Im}F(\epsilon',\omega)\ . \tag{4.2}$$

Since they are frequency independent, S_c^F and S_c^H are contributions to Y:

$$Y_c(\epsilon)=\int d\epsilon' \sum_{\mathbf{q}}[R_c^F(\mathbf{q},\epsilon-\epsilon')-2R_c^H(\mathbf{q},\epsilon-\epsilon')]V_c(\mathbf{q})\int\frac{d\omega}{\pi}f(\omega)\mathrm{Im}G(\epsilon',\omega)\ . \tag{4.3}$$

There is no Coulomb contribution to Z.

B. Frequency summations continued: Phonon contributions

For the phonon propagator, one introduces the spectral function $B_b(\mathbf{q},\nu)$, which allows for the spectral representation

$$D_b(\mathbf{q},i\Omega)=2\int d\nu\frac{\nu}{(i\Omega)^2-\nu^2}B_b(\mathbf{q},\nu)\ . \tag{4.4}$$

From Eq. (3.6) we obtain

$$\begin{aligned}W_{\mathrm{ph}}(\epsilon,i\omega)=\int d\epsilon'\int d\nu\,\alpha^2F^F(\epsilon-\epsilon',\nu)\Bigg[&n(\nu)[\mathcal{F}(\epsilon',i\omega+\nu)+\mathcal{F}(\epsilon',i\omega-\nu)]\\&+\int\frac{dx}{\pi}\mathrm{Im}F(\epsilon',x)\left[\frac{f(x)}{x-i\omega-\nu}+\frac{1-f(x)}{x-i\omega+\nu}\right]\Bigg]\ ,\end{aligned} \tag{4.5}$$

where $n(\omega)=[\exp(\omega/T)-1]^{-1}$ is a phonon occupation number, and

$$\alpha^2F^F(\epsilon,\nu)=\sum_{\mathbf{q},b}R_b^F(\mathbf{q},\epsilon)B_b(\mathbf{q},\nu)\ . \tag{4.6}$$

The exchange contribution to the normal self-energy,

$$\begin{aligned}S_{\mathrm{ph}}^F(\epsilon,i\omega)=\int d\epsilon'\int d\nu\,\alpha^2F^F(\epsilon-\epsilon',\nu)\Bigg[&n(\nu)[\mathcal{G}(\epsilon',i\omega+\nu)+\mathcal{G}(\epsilon',i\omega-\nu)]\\&+\int\frac{dx}{\pi}\mathrm{Im}G(\epsilon',x)\left[\frac{f(x)}{x-i\omega-\nu}+\frac{1-f(x)}{x-i\omega+\nu}\right]\Bigg]\ ,\end{aligned} \tag{4.7}$$

has to be separated in its contributions to Z and Y. This is easily accomplished by dividing the rhs into terms which are odd and even, respectively, in the Matsubara frequency $i\omega$. We find

$$\begin{aligned}i\omega[1-Z(\epsilon,i\omega)]=\int d\epsilon'\int d\nu\,\alpha^2F^F(\epsilon-\epsilon',\nu)\Bigg[&n(\nu)[\mathcal{G}(\epsilon',i\omega+\nu)+\mathcal{G}(\epsilon',i\omega-\nu)]\\&+i\omega\int\frac{dx}{\pi}\mathrm{Im}G(\epsilon',x)\left[\frac{f(x)}{(x-\nu)^2-(i\omega^2)}+\frac{1-f(x)}{(x+\nu)^2-(i\omega)^2}\right]\Bigg]\ ,\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}Y_{\mathrm{ph}}^F(\epsilon,i\omega)=\int d\epsilon'\int d\nu\,\alpha^2F^F(\epsilon-\epsilon',\nu)\Bigg[&n(\nu)[\mathcal{G}_+(\epsilon',i\omega+\nu)+\mathcal{G}_+(\epsilon',i\omega-\nu)]\\&+\int\frac{dx}{\pi}\mathrm{Im}G(\epsilon',x)\left[f(x)\frac{x-\nu}{(x-\nu)^2-(i\omega)^2}+(1-f(x))\frac{x+\nu}{(x+\nu)^2-(i\omega)^2}\right]\Bigg]\ .\end{aligned} \tag{4.9}$$

Here $\mathcal{G}_\pm(\epsilon,i\omega)=\frac{1}{2}[\mathcal{G}(\epsilon,i\omega)\pm\mathcal{G}(\epsilon,-i\omega)]$, and we have anticipated that there is no other contribution to Z. Indeed, the last remaining term, Eq. (3.9), is independent of $i\omega$, and is therefore a contribution to Y:

$$Y_{\mathrm{ph}}^H(\epsilon,i\omega)=4\int d\epsilon'\int\frac{d\nu}{\nu}\alpha^2F^H(\epsilon-\epsilon',\nu)\int\frac{dx}{\pi}f(x)\mathrm{Im}G(\epsilon',x)\ , \tag{4.10}$$

where

$$\alpha^2F^H(\epsilon,\nu)=\sum_{\mathbf{q},b}R_b^H(\mathbf{q},\epsilon)B_b(\mathbf{q},\nu)\ . \tag{4.11}$$

We have now completed all simplifications of Eqs. (3.6)–(3.9), and (3.11)–(3.13), which can be carried out in complete generality, without further assumptions or approximations.

C. The strong-coupling equations

We may now combine Eqs. (4.2) and (4.5), (4.8), and (4.3), (4.9), and (4.10) to write down our strong-coupling equations for the retarded functions $W(\epsilon,\omega)=W(\epsilon,i\omega\rightarrow\omega+i0)$, etc.,

$$\begin{aligned} W(\epsilon,\omega) = \int d\epsilon' \int d\nu\, \alpha^2 F^F(\epsilon-\epsilon',\nu) & \left[n(\nu)[F(\epsilon',\omega+\nu)+F(\epsilon',\omega-\nu)] + \int \frac{dx}{\pi} \mathrm{Im} F(\epsilon',x) \left[\frac{-f(x)}{x-\omega-\nu} + \frac{1-f(x)}{x-\omega+\nu} \right] \right] \\ & + \int d\epsilon' U_c^W(\epsilon-\epsilon') \int \frac{dx}{\pi} f(x) \mathrm{Im} F(\epsilon',x) \ , \end{aligned} \tag{4.12}$$

$$\begin{aligned} \omega[1-Z(\epsilon,\omega)] = \int d\epsilon' \int d\nu\, \alpha^2 F^F(\epsilon-\epsilon',\nu) & \left[n(\nu)[G_-(\epsilon',\omega+\nu)+G_-(\epsilon',\omega-\nu)] \right. \\ & \left. + \omega \int \frac{dx}{\pi} \mathrm{Im} G(\epsilon',x) \left[\frac{f(x)}{(x-\nu)^2-\omega^2} + \frac{1-f(x)}{(x+\nu)^2-\omega^2} \right] \right] \ , \end{aligned} \tag{4.13}$$

$$\begin{aligned} Y(\epsilon,\omega) = \int d\epsilon' \int d\nu\, \alpha^2 F^F(\epsilon-\epsilon,\nu) & \left[n(\nu)[G_+(\epsilon',\omega+\nu)+G_+(\epsilon',\omega-\nu)] \right. \\ & \left. + \int \frac{dx}{\pi} \mathrm{Im} G(\epsilon',x) \left[f(x) \frac{x-\nu}{(x-\nu)^2-\omega^2} + (1-f(x)) \frac{x+\nu}{(x+\nu)^2-\omega^2} \right] \right] \\ & + \int d\epsilon' \int d\nu \left[\frac{4}{\nu} \alpha^2 F^H(\epsilon-\epsilon',\nu) + U_c^Y(\epsilon-\epsilon')\delta(\nu) \right] \int \frac{dx}{\pi} f(x) \mathrm{Im} G(\epsilon',x) \ . \end{aligned} \tag{4.14}$$

α^2F^F and α^2F^H have been given in Eqs. (4.6) and (4.11), respectively, and the Coulomb kernels are defined as

$$U_c^W(\epsilon) = \sum_{\mathbf{q}} R_c^F(\mathbf{q},\epsilon) V_c(\mathbf{q}) \ , \tag{4.15}$$

$$U_c^Y(\epsilon) = \sum_{\mathbf{q}} [R_c^F(\mathbf{q},\epsilon) - 2R_c^H(\mathbf{q},\epsilon)] V_c(\mathbf{q}) \ . \tag{4.16}$$

Finally, the retarded Green's functions are obtained by replacing $i\omega \to \omega + i0$ in Eqs. (3.5). G_- (G_+) are obtain by retaining in the numerator of Eq. (3.5a) the term $\omega Z([\epsilon+Y])$ only. According to the definitions of Z and Y, the symmetry requirements are then fulfilled. As in standard strong coupling theory one does not have to distinguish between $\mathcal{F}^\dagger$ and $\mathcal{F}$. Eqs. (4.12)–(4.16), together with Eq. (3.5), form a closed set of integral equations in two variables for the self-energies W, Z, and Y. They represent the central result of the present paper. We will make no attempt to solve these equations, but rather give a general discussion of their content in the next sections. An approximate solution for T_c of McMillan type will be given in a separate publication.[28]

V. DISCUSSION

A. Comparison with previous theories

We start the discussion of Eqs. (4.12)–(4.14) by comparing them with previous work. First we show that in the weak disorder limit, the equations correctly reduce to standard results. In this limit, the scale for frequency variations of the correlation functions entering $\alpha^2F^{F,H}$ and the Coulomb kernels $U_c^{W,Y}$ is given by the Fermi energy. This follows from using Eq. (B7) in Eqs. (3.7′), (3.10), (3.14), and (3.15). Consequently, the phonon self-energies are energy dependent. This is not strictly true for the Coulomb self-energies, since there the integration range is also of the order of the Fermi energy. For our purposes we neglect this, and follow Morel and Anderson[5] in assuming an energy independent Coulomb kernel. Then the Coulomb self-energy is energy independent as well. It should be mentioned, however, that in the absence of large disorder this point has been improved upon.[40] If we assume that Y is frequency independent as well, we find $\int d\epsilon' G_+(\epsilon',\omega) = 0$, and $\mathrm{Im}G(\epsilon,-\omega) = \mathrm{Im}G(\epsilon,\omega)$. Therefore the first two contributions in Eq. (4.14) vanish, and Y is indeed a constant which just renormalizeds the chemical potential. Thus Y can be neglected.[2] As for the remaining terms, with Eq. (B7) in Eq. (3.14), we find for the Coulomb potential, Eq. (4.15),

$$U_c^W(\epsilon)^{(0)} \equiv \mu = \tfrac{1}{2}(\kappa/2k_F)^2 \ln[1+(2k_F/\kappa)^2] \ , \tag{5.1}$$

with the screening wave number κ from Eq. (2.6). This is the result of Morel and Anderson.[5] For the same reason, α^2F^F is independent of energy. With energy-independent kernels, the integrations over ϵ' in Eqs. (4.12), (4.13) are elementary, and we obtain the standard strong-coupling equations.[6,29] Working out the stress correlation function for a jellium model, one finds that there is no contribution from transverse phonons,[19] while the longitudinal phonons yield[19] (in the long-wavelength limit)

$$\alpha^2F^F(\epsilon,\nu)^{(0)} \equiv \alpha^2F(\nu) = g_L^2 N_F \int_0^{2k_F} \frac{dq\, q}{2k_F^2} B_L(q,\nu) \ , \tag{5.2}$$

where $g_b = k_F^2/3m(\rho_i)^{1/2}c_b$.

We now turn to the disorder corrections. Let us disregard for a while all Coulomb contributions to the self-energies, as well as Y_{ph}. Then the only vertex correction we have to deal with is the one in α^2F^F. Adopting the scheme developed in Appendix B, and using free Fermion static correlation functions in Eq. (B6), we find that the density mode $A_0(\mathbf{q}) = d(\mathbf{q})$ does not yield a contribution if Eq. (4.6) is rewritten in the form of Eq. (B8a). This reflects the fact that there is no diffusion enhancement of the electron-phonon vertex due to diffusion (i.e., density fluctuations). This was shown diagrammatically in Ref. 32, and we have repeated the argument in Sec. II C [cf. Eq. (2.21)]. For the time being, we ignore the possibility of CPR [Eq. (2.22)], to which we will come back later, and go on with the approximation scheme of Appendix B. The current mode does not contribute either (because of symmetry under time reversal), and the first nonzero contribution stems from the longitudinal stress operator

$$s_L(q)=\sum_{\mathbf{k}}(k_z^2-k^2/3)f_{\mathbf{k}}(\mathbf{q})$$

($\mathbf{q}$ in the z direction). Similarly, the transverse stress operator $s_T(q)=\sum_{\mathbf{k}} k_z k_x f_{\mathbf{k}}(\mathbf{q})$ yields the first nonvanishing contribution in the transverse channel. The correlation functions

$$\Phi_b''(q,0)=\lim_{\omega\to 0}[s_b(\mathbf{q})\,|\,\pi\delta(\omega-\mathscr{L})\,|\,s_b(\mathbf{q})]$$

can be calculated from a Boltzmann equation in the collision-time approximation. In the long-wavelength limit, the result is

$$\Phi_b''(q,0)=\frac{\pi^3 k_F^2 N_F}{9q}\Phi_b(ql)\ , \tag{5.3}$$

where $l=v_F\tau$ is the electronic mean free path, and

$$\Phi_L(x)=\frac{2}{\pi}\left[\frac{x\arctan x}{x-\arctan x}-\frac{3}{x}\right]\ , \tag{5.4a}$$

$$\Phi_T(x)=\frac{3}{\pi x^4}[2x^3+3x-3(x^2+1)\arctan x]\ . \tag{5.4b}$$

Insertion into Eq. (4.6) yields the result of Keck and Schmid.[19] There is an enhancement of T_c due to disorder, since the increase of the transverse phonon contribution , [Eq. (5.4b)], overcompensates the decrease of the longitudinal one. One has to keep in mind, however, that this result has been derived by neglecting Y and all Coulomb contributions as well as the Cooperon renormalization of α^2F.

To understand the work of Refs. 20, 22 within the present frame, we ignore the Keck-Schmid mechanism discussed above, taking Eq. (5.2) for α^2F. We also ignore Y, and all disorder corrections to Z, concentrating on the Coulomb contribution to the gap parameter, $W_c(\epsilon)$, Eq. (4.2). From Eq. (3.14) we see that, contrary to the case of the electron-phonon vertex, here the density mode does yield a contribution. With the diffusion-pole approximation for the density propagator,

$$\delta\Phi''(\mathbf{q},\epsilon)=\sum_{\mathbf{k},\mathbf{p}}\delta\Phi_{\mathbf{kp}}''(\mathbf{q},\epsilon)\simeq\frac{Dq^2g(q)}{\epsilon^2+D^2q^4}\ , \tag{5.5}$$

where D is the diffusion constant, and $g(q)=[d(\mathbf{q})\,|\,d(\mathbf{q})]$ is the q-dependent compressibility, we have

$$U_c^W(\epsilon)=\mu+\frac{1}{\pi N_F}\sum_{\mathbf{q}}\frac{Dq^2g(q)}{\epsilon^2+D^2q^4}V_c(q)\ . \tag{5.6}$$

This is an alternative derivation of the enhancement due to diffusion, cf. Ref. 27 and Sec. II B. This enhancement alone does not yield a large effect, since in the Coulomb pseudopotential,[5] U_c^W enters in the denominator as well as in the numerator. Anderson *et al.*[20] therefore considered a scale-dependent diffusion constant, borrowed from weak-localization theory,[21] in Eq. (5.6). For systems sufficiently close to an Anderson transition, this results in a strong decrease of T_c. We note that contrary to the CPR of the vertex, Eq. (2.12) (which has been neglected in Ref. 20), this procedure effects U_c in an indirect way, viz., via critical slowing down effects affecting D. This should be important in the critical regime of an Anderson transition only. In an attempt to describe less extremely disordered materials, Entin-Wohlman *et al.*[22] concentrated on systems with quasi-one-dimensional structure, such as $A-15$'s. Accordingly, they replace the q sum in Eq. (5.6) by a strongly anisotropic integral. Though this strongly enhances the diffusion effect on U_c^W, the authors conclude that this effect for itself is not sufficient to explain the observed T_c degradation with reasonable parameters. This is perhaps not too surprising. As we have seen, both Refs. 20 and 22(b) have singled out one mechanism, and neglected several others whose effects are comparable. For instance, Leavens[41] found that inclusion of the critical slowing down effects of Ref. 20 on the renormalization function Z (but again neglecting Y and the renormalization of the phonon contributions) actually yields an increase of T_c in the critical region, contrary to what is observed in experiment.

The work of Wysokinski and Kuzemsky[23] contains both the effects discussed in Refs. 19 and 20, and in Ref. 22(b), respectively, albeit for a different model. The diffusion enhancement of the Coulomb kernel, our Eqs. (4.15) and (5.6), is implicitly contained in their effective Coulomb potential U_{eff}.

Up to now we have shown that previous work on dirty strong-coupling superconductors is contained in the present theory as special cases of Eqs. (4.12)–(4.16). Comparison with the work on BCS and Ginzburg-Landau models mentioned in the introduction is more difficult. Let us only mention that the effects discussed in Ref. 26 have been, in the present language, entirely due to Y. Both Fock and Hartree contributions with DPR as well as CPR have been taken into account. However, in the BCS approach the disorder dependences of W and Z have been missing. We will give a more detailed comparison in Ref. 28, where we will solve the present equations approximately for T_c.

Finally, let us tentatively apply Eqs. (4.12)–(4.14) to an Anderson insulator. The nonergodic properties of the insulating phase are reflected by a contribution proportional to $\delta(\epsilon)$ to the phase-space Kubo function $\Phi_{\mathbf{kp}}''(\mathbf{q},\epsilon)$.[42] This just leads to an energy-independent contribution to all self-energies, and no obvious accident happens. Though the question of T_c remains open without solving the equations and a detailed study of the parameters, this seems to imply the possibility of superconductivity on the insulating side of an Anderson transition, in accord with the works quoted under Ref. 13. We emphasize, however, that because of our crude approximation for the Coulomb potential, we believe such a procedure to be grossly inadequate (cf. the remarks in Refs. 13 and 31).

B. General discussion of the integral equations

Finally we give a brief general discussion of Eqs. (4.12)–(4.14), with emphasis on the new aspects. The two most important features are as follows: The energy dependence of the self-energies, and the appearance of an additional renormalization function Y. We start with a discussion of the former.

The energy dependence of the self-energies has to be seen in analogy to their wave-vector dependence in the clean case. In a disordered system the momentum is no longer a good quantum number, and one has to classify in terms of energy instead. This point has been taken care of in the definition of the ensemble average in Eq. (3.4). The energy dependence then appears technically in the form of the frequency argument of the Kubo functions entering the various vertices. Thus in the weak disorder limit it disappears in a natural way, at least for the phonon parts, as has been discussed above. This corresponds to the fact that in standard clean limit theory the wave-vector integrals can be performed in an excellent approximation.[2,6] With increasing disorder, the Kubo functions develop frequency dependence on a scale smaller than the Fermi energy, and so the kernels become energy dependent. Let us consider the case of strongest variation, which is that of a diffusion-enhanced vertex, Eq. (5.6). The Coulomb potential can be taken out of the integral at some average wave number, so basically we are interested in the ϵ dependence of the function

$$y(\epsilon)=\int_0^{2k_F} dq\, q^2 Dq^2/(\epsilon^2+D^2q^4)\ . \tag{5.7}$$

Here we have used the fact that the compressibility $g(q)$ effectively cut off the integral at $2k_F$ (cf. the discussion in Ref. 26). $y(\epsilon\to 0)$ behaves nonanalytically because of the diffusion pole,[27] and one obtains

$$y(\epsilon)=y(0)\left[1-\frac{\pi}{4k_F}\sqrt{\epsilon/2D}+O(\epsilon)\right]\ . \tag{5.7$'$}$$

The energy range we are interested in Eq. (4.12) is a shell of width $2\omega_D$ around the Fermi level. Putting $\epsilon=\omega_D$, we obtain

$$y(\omega_D)=y(0)\left[1-\frac{\pi\sqrt{3}}{8}(\rho/\rho_M)^{1/2}(\omega_D/\epsilon_F)^{1/2}\right.$$
$$\left.+O(\pi\omega_D/\rho_M\epsilon_F)\right]\ , \tag{5.8}$$

where $\rho=1/2N_FD$ is the resistivity, and ρ_M is the resistivity scale defined in Appendix A. We conclude that in the regime $\rho \ll \rho_M$, where the present theory is valid, the energy dependence of W is extremely small, and can safely be neglected. This is not quite true for Z and Y, since contrary to F, $G(\epsilon,\nu)$ is not small for $|\epsilon|>\omega_D$. To estimate the maximum error, we have to replace $(\omega_D/\epsilon_F)^{1/2}$ in Eq. (5.8) by one. For not too large a resistivity it then still is a reasonable approximation to neglect the ϵ dependence of Z, which greatly simplifies Eqs. (4.12) and (4.13). The energy dependence of Y must not be ignored as we will discuss below.

The renormalization function $Y(\epsilon,\omega)$ has been defined in Sec. III A as that part of the normal self-energy which is an even function of frequency. Of course, such a contribution always exists, but in standard theory it can be neglected as discussed above. Even for strong disorder, the main contribution to Y, that is the last term in Eq. (4.14),[43] is still frequency independent. For this reason one must not ignore the energy dependence of Y lest one loses Y entirely and therewith a qualitative effect. It is clear from the structure of Eqs. (3.5) and known from experience within the BCS model,[26] that Y will be adverse to superconductivity. We conclude that frequency independence of a contribution to the normal self-energy is no sufficient justification for its neglect if its energy-dependent part is strongly disorder dependent. In Ref. 26 the physics which is here described by Y has been interpreted as Altshuler-Aronov corrections[27] to the Green's function which make up the Cooper propagator. This interpretation is confirmed by Eqs. (3.5b) and (4.14). In particular the present microscopic derivation confirms the existence of a phonon contribution of Hartree type to this one-particle renormalization. Such a contribution has first been found within a BCS model by Fukuyama *et al.*,[24] but was subsequently neglected. It was found to be of importance in connection with the T_c degradation effect in Ref. 26. In agreement with Ref. 26, we find from Eq. (4.14) that Y is larger the larger the electron-phonon coupling. Since Y acts strongly adverse to superconductivity, we conclude that the T_c-decreasing effects should be strongest for strong-coupling superconductors. On the other hand, at weak coupling Y will be less dominant, and the T_c-enhancing effects inherent in W, Eq. (4.12), will gain importance. We conclude that the theory has an inherent tendency to favor T_c-degradation at strong coupling, and a rise of T_c for weak coupling. This is in agreement with the experimentally observed correlation.[15,18]

We finally readdress the question of diffusion enhancement of the vertices. In Sec. II B we have seen diagrammatically that both DPR and CPR of the Coulomb vertex yield a diffusion-enhanced result. The former we have obtained with the correlation function technique also, by evaluating Eq. (4.15) with the help of a diffusion propagator. The result, Eq. (5.6), agrees with the diagrammatic one, Eq. (2.9). Since the correlation function technique is formally exact, the CPR must also be contained in Eq. (4.15). To see this, we recall a peculiar symmetry of the phase space Kubo function. This symmetry holds for noninteracting electrons in the presence of time reversal invariance. It is expressed by the identity[33]

$$\Phi''_{\mathbf{kp}}(\mathbf{q},\omega)=\Phi_{\frac{\mathbf{k}-\mathbf{p}+\mathbf{q}}{2},\frac{\mathbf{p}-\mathbf{k}+\mathbf{q}}{2}}(\mathbf{k}+\mathbf{p},\omega)\ , \tag{5.9}$$

which, under the mentioned conditions, is easily proven from the Green's function representation of the Kubo function, Eq. (A7). Clearly, this symmetry has been lost by writing the diffusion-pole approximation in Eq. (5.5). According to Eq. (5.9), the phase-space Kubo function shows a diffusion pole at small $\mathbf{k}+\mathbf{p}$ and ω as well as at small $\mathbf{q}$ and ω. Accordingly, there is another dangerous region in phase space which has to be taken into account in the evaluation of Eq. (4.15). This is easily accomplished[44] by using the identity, Eq. (5.9), in Eq. (4.15) *before* projecting onto the density according to the lowest-order approximation in the scheme given in Appendix B. Of course, we must not count the clean contribution twice. We then obtain, instead of Eq. (5.6),

$$U_c^W(\epsilon)=\mu+\frac{1}{\pi N_F}\sum_{\mathbf{q}}\frac{Dq^2g(q)}{\epsilon^2+D^2q^4}\left[V_c(\mathbf{q})+\frac{1}{g^2(q)}\sum_{\mathbf{k},\mathbf{p}}[f_{\mathbf{k}}(\mathbf{q})\,|\,d(\mathbf{q})]V_c(\mathbf{k}+\mathbf{p})[d(\mathbf{q})\,|\,f_{\mathbf{p}}(\mathbf{q})]\right] . \tag{5.10}$$

The second term in square brackets corresponds to the one obtained diagrammatically in Eq. (2.12).

For the phonon vertex, the situation is slightly different. The evaluation of α^2F^F in Sec. V A following Keck and Schmid[19] corresponds to the diagrammatic DPR. The result was absence of diffusion enhancement, in agreement with Eq. (2.21). However, if we use Eq. (5.9) in Eq. (4.6) and project onto the density, we do obtain a diffusion-enhanced vertex with a rather complicated vertex function. Again, this is in agreement with the diagrammatic analysis, Eq. (2.22). In the above arguments, we have used only density contributions treated in diffusion pole approximation to demonstrate by comparison with the diagrammatic analysis, that the correlation function technique correctly yields CPR as well as DPR effects. We would like to emphasize, however, that the correlation function method is potentially much more powerful than the standard diagrammatic technique. For instance, it easily yields nonsingular contributions like the one to α^2F discussed by Keck and Schmid, which would be hard to obtain by diagrammatic means.

Neither the CPR's, nor the various Hartree contributions, nor any contributions to Y have been kept before in a strong-coupling theory for dirty superconductors. This shows that the physical content of Eq. (4.12)—(4.14) goes considerably beyond a unification of what has been achieved before. Since all renormalizations leading to diffusion-enhanced vertices are of comparable magnitude, we conclude that all previous theories have neglected contributions comparable to those which have been kept. With the help of the scheme given in Appendix B, it is possible to consider all disorder effects belonging to a given class (e.g., diffusion enhancement), which will be necessary for a convincing comparison with experiment. Unfortunately, solving the integral equations is also considerable more difficult than for the standard strong coupling theory. As a first step towards a solution, we have solved for T_c by adapting the well-known McMillan procedure.[7] These results will be presented in a separate publication.[28]

Note added in proof. After the present work was submitted, I learned that D. A. Browne, K. Levin, and K. A. Muttalib [Phys. Rev. Lett. (to be published)] had independently also realized the importance of the normal self-energy part Y. I thank Dr. K. Levin for sending me a preprint, and Dr. D. A. Browne for an interesting discussion.

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APPENDIX A: VERTEX RENORMALIZATIONS AND KUBO FUNCTIONS

In this appendix we show that the vertex renormalizations due to impurities encountered in Sec. III B can be expressed in terms of correlation functions for noninteracting electrons. In any space of dynamical variables $A,B,\ldots$ one can define a scalar product, the Kubo product, by the definition[45]

$$[A\,|\,B]=\int_0^{1/T}d\lambda(\langle\langle A^\dagger B(i\lambda)\rangle\rangle_{\text{av}}-\langle\langle A^\dagger\rangle\rangle_{\text{av}}\langle\langle B\rangle\rangle_{\text{av}}) . \tag{A1}$$

The time evolution of the variables is given by $A(t)=exp(i\mathscr{L}t)A$, with the Liouvillian $\mathscr{L}$ defined by

$$\mathscr{L}A=[H,A] . \tag{A2}$$

The Kubo correlation function of any two variables for complex frequency z is defined as the resolvent matrix element of the Louivillian,

$$\Phi_{AB}(z)=\left[A\left|\frac{1}{\mathscr{L}-z}\right|B\right] . \tag{A3}$$

It allows for the spectral representation

$$\Phi_{AB}(z)=\int\frac{d\omega}{\pi}\Phi''_{AB}(\omega)/(\omega-z) , \tag{A4}$$

where the spectral function is given by

$$\begin{aligned}\Phi''_{AB}(\omega)&=[\Phi_{AB}(\omega+i0)-\Phi_{AB}(\omega-i0)]/2i\\&=\pi[A\,|\,\delta(\omega-\mathscr{L})\,|\,B] .\end{aligned} \tag{A5}$$

We now apply this formalism to the electron impurity problem defined in Sec. II A, choosing the dynamical variables to be products $\Psi^\dagger\Psi$ of the electron field operators. Then

$$\Phi^{\xi''}_{\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4}(\omega)=[\Psi^\dagger(\mathbf{x}_1)\Psi(\mathbf{x}_2)\,|\,\pi\delta(\omega-\mathscr{L})\,|\,\Psi^\dagger(\mathbf{x}_3)\Psi(\mathbf{x}_4)] \tag{A6}$$

is the most general four-point Kubo function for a system with chemical potential ξ. Since we are dealing with a bilinear Hamiltonian, the thermodynamic average in the definition of the Kubo product, Eq. (A1), factorizes, and it is easy to show

$$\Phi^{\xi''}_{\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4}(\omega)=\pi\int\frac{d\epsilon}{\pi}\int\frac{d\epsilon'}{\pi}\delta(\omega-\epsilon'+\epsilon)\frac{f(\epsilon)-f(\epsilon')}{\epsilon-\epsilon'}\langle G''(\mathbf{x}_1,\mathbf{x}_3;\epsilon')G''(\mathbf{x}_4,\mathbf{x}_2;\epsilon)\rangle_{\text{av}} . \tag{A7}$$

Here G'' is the spectral function for the Matsubara function defined in Eq. (2.2) taken without the impurity average, and $f(\epsilon)=\{\exp[(\epsilon-\xi)/T]+1\}^{-1}$ is the Fermi function. Finally, we write G'' in the exact eigenstate representation to obtain

$$\Phi^{\xi''}_{\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4}(\omega)=\frac{\pi}{\omega}\int d\epsilon[f(\epsilon)-f(\epsilon+\omega)]F^{\epsilon}_{\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4}(\omega)\ , \tag{A8a}$$

where

$$F^{\epsilon}_{\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4}(\omega)=\left\langle\sum_{n,m}\delta(\epsilon+\omega-E_n)\delta(\epsilon-E_m)\psi_n^*(\mathbf{x}_1)\psi_m^*(\mathbf{x}_4)\psi_n(\mathbf{x}_3)\psi_m(\mathbf{x}_2)\right\rangle_{\text{av}}\ , \tag{A8b}$$

is the function which appears in the vertex renormalization. The temperatures we are interested in are very small compared with the chemical potential, so in an excellent approximation we can replace the $f(\epsilon)$ by step functions, and ξ by ϵ_F. One then sees that as long as $F^{\epsilon}(\omega)\approx F^{\epsilon_E}(\omega)$ for $\epsilon_F-\omega\leq\epsilon\leq\epsilon_F$, we have the approximate relation[46]

$$\Phi^{\epsilon_F''}_{\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4}(\omega)\simeq\pi F^{\epsilon_F}_{\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4}(\omega)\ . \tag{A9}$$

Finally, we define the Fourier transform of Φ to obtain the phase space Kubo function

$$\Phi''_{\mathbf{kp}}(\mathbf{q},\omega)=[c^{\dagger}_{\mathbf{k}-\mathbf{q}/2}c_{\mathbf{k}+\mathbf{q}/2}\,|\,\pi\delta(\omega-\mathscr{L})\,|\,c^{\dagger}_{\mathbf{p}-\mathbf{q}/2}c_{\mathbf{p}+\mathbf{q}/2}]\ , \tag{A10}$$

where $c_{\mathbf{k}}=\int d\mathbf{x}\exp(i\mathbf{kx})\Psi(\mathbf{x})$. This completes the derivation of Eq. (3.7′) from Eq. (3.7).

We add a remark concerning the validity of the approximation involved in Eq. (A9). In the phonon contributions to the self-energy, the frequency variation is effectively restricted to a layer of width $2\omega_D$ around the Fermi energy, and Eq. (A9) is an excellent approximation. For the Coulomb contributions this is less clear since the energy scale is of the order of ϵ_F. However, the approximation will fail only for systems whose Fermi energy is close to a mobility edge ϵ_c, as can be seen from the following argument. Near a mobility edge, the resistivity scales like[21]

$$\rho=\rho_M[(\epsilon_F-\epsilon_c)/\epsilon_c]^{-s}\ , \tag{A11}$$

where $\rho_M=(\hbar/e^2)3\pi^2/k_F$ is Mott's resistivity, which serves as resistivity scale for systems approaching the Anderson transition, and $s=O(1)$. Therefore, as long as $\rho/\rho_M\ll 1$, we have $\epsilon_F/\epsilon_c\gg 1$, and the approximation leading to Eq. (A9) fails only for a fraction $\epsilon_c/\epsilon_F\ll 1$ of the integration interval. Once agains we conclude that our procedure for treating the Coulomb potential is only reliable for systems not too close to an Anderson transition. This is quantified by the condition $\rho/\rho_M\ll 1$.

APPENDIX B: A SYSTEMATIC METHOD FOR CALCULATING VERTEX FUNCTIONS

In this appendix we give an exact reformulation of the vertex renormalization encountered in Sec. III in order to cast them into a form which facilitates explicit calculations. We first notice that we always have to deal with expressions of the type

$$\sum_{\mathbf{q}}K(\mathbf{q},\epsilon-\epsilon')P(\mathbf{q},i\Omega)\ , \tag{B1}$$

where K stands for $R_b^{F,H}$ of $R_c^{R,H}$, and P can be D_b of V_c. For the exchange contributions, K has the structure

$$K^F(\mathbf{q},\epsilon-\epsilon')=\sum_{\mathbf{k},\mathbf{p}}v(\mathbf{k})\Phi''_{\mathbf{kp}}(\mathbf{q},\epsilon-\epsilon')v(\mathbf{p})\ , \tag{B2a}$$

while for the direct one it reads

$$K^H(\mathbf{q},\epsilon-\epsilon')=\sum_{\mathbf{k},\mathbf{p}}v(\mathbf{k})\Phi''_{\frac{\mathbf{k}+\mathbf{p}-\mathbf{q}}{2},\frac{\mathbf{k}+\mathbf{p}+\mathbf{q}}{2}}(\mathbf{k}-\mathbf{p},\epsilon-\epsilon')v(\mathbf{p})\ . \tag{B2b}$$

The function $v(\mathbf{k})$ is a second rank tensor for electron-phonon coupling, and one for Coulomb coupling. According to Eq. (A10), we can write Kubo functions as a matrix element of $\delta(\epsilon-\epsilon'-\mathscr{L})$ formed with the phase-space density $f_{\mathbf{k}}(\mathbf{q})=c^{\dagger}_{\mathbf{k}-\mathbf{q}/2}c_{\mathbf{k}+\mathbf{q}/2}$. Hence, we can write the expression (B1) in the form

$$\sum_{\mathbf{q}}K(\mathbf{q},\epsilon-\epsilon')P(\mathbf{q},i\Omega)=\sum_{\mathbf{k},\mathbf{p},\mathbf{q}}v(\mathbf{K})v(\mathbf{P})P(\mathbf{Q},i\Omega)[f_{\mathbf{k}}(\mathbf{q})\,|\,\pi\delta(\epsilon-\epsilon'-\mathscr{L})\,|\,f_{\mathbf{p}}(\mathbf{q})]\ . \tag{B3}$$

Here $(\mathbf{K},\mathbf{P},\mathbf{Q})=(\mathbf{k},\mathbf{p},\mathbf{q})\{[(\mathbf{k}+\mathbf{p}-\mathbf{q})/2,(\mathbf{k}+\mathbf{p}+\mathbf{q})/2,\mathbf{k}-\mathbf{p}]\}$ for Fock {Hartree} contributions. We now define a complete set of orthogonalized modes

$$A_n(\mathbf{q})=\sum_{\mathbf{k}}a_n^{\mathbf{q}}(\mathbf{k})f_{\mathbf{k}}(\mathbf{q})\ , \tag{B4a}$$

$$[A_n(\mathbf{q})\,|\,A_m(\mathbf{q})]=\delta_{nm}g_n(q) \tag{B4b}$$

and rewrite Eq. (B3) in the form

$$\sum_{\mathbf{q}}K(\mathbf{q},\epsilon-\epsilon')P(\mathbf{q},i\Omega)=\sum_{\mathbf{k},\mathbf{p},\mathbf{q}}v(\mathbf{K})v(\mathbf{P})P(\mathbf{Q},i\Omega)\sum_{n,m}W^{\mathbf{kp}}_{nm}(\mathbf{q})\Phi''_{nm}(\mathbf{q},\epsilon-\epsilon')\ . \tag{B5}$$

Here $\Phi''_{nm}(\mathbf{q},\epsilon)=[A_n(\mathbf{q})\,|\,\pi\delta(\epsilon-\mathscr{L})\,|\,A_m(\mathbf{q})]$ is the Kubo function in the new basis, and

$$W^{\mathbf{kp}}_{nm}(\mathbf{q})=[f_{\mathbf{k}}(\mathbf{q})\,|\,A_n(\mathbf{q})]\frac{1}{g_n(\mathbf{q})g_m(\mathbf{q})}[A_m(\mathbf{q})\,|\,f_{\mathbf{p}}(\mathbf{q})]\ . \tag{B6}$$

Often it is advantageous to split off the clean contribution, formed with the free-Fermi-gas correlation function

$$(\Phi_{\mathbf{kp}}^{(0)})''(\mathbf{q},\omega)=\delta_{\mathbf{kp}}(\pi/\omega)\delta(\omega+\mathbf{kq}/m)[f(\mathbf{k}-\mathbf{q}/2)-f(\mathbf{k}+\mathbf{q}/2)] \ . \tag{B7}$$

Denoting the vertex function in the clean limit by $K^{(0)}$, we then obtain our final result

$$\begin{aligned}\sum_{\mathbf{q}} K(\mathbf{q},\epsilon-\epsilon')P(\mathbf{q},i\Omega)=&\sum_{\mathbf{q}} K^{(0)}(\mathbf{q},\epsilon-\epsilon')P(\mathbf{q},i\Omega)\\ &+\sum_{\mathbf{k},\mathbf{p},\mathbf{q}} v(\mathbf{K})v(\mathbf{P})P(\mathbf{Q},i\Omega)\sum_{n,m} W_{nm}^{\mathbf{kp}}(\mathbf{q})\delta\Phi_{nm}''(\mathbf{q},\epsilon-\epsilon') \ ,\end{aligned} \tag{B8a}$$

with

$$\delta\Phi_{nm}''(\mathbf{q},\epsilon)=\Phi_{nm}''(\mathbf{q},\epsilon)-(\Phi_{nm}^{(0)})''(\mathbf{q},\epsilon) \ . \tag{B8b}$$

With a suitable choice for the set of modes $A_n(\mathbf{q})$, the sum over n,m in Eq. (B8a) should converge rapidly. It will then be possible to get reasonably accurate results from calculating only a few dynamic [Eq. (B8a)] and static [Eq. (B6)] correlation functions. A natural choice for the longitudinal modes would be $a_0^{\mathbf{q}}(\mathbf{k})=1$, $a_1^{\mathbf{q}}(\mathbf{k})=\mathbf{kq}/q$, $a_2^{\mathbf{q}}(\mathbf{k})=(\mathbf{kq}/q)^2-k^2/3$, etc., thus taking density, current, and stress as the first few modes. For the corresponding correlation functions, a number of exact properties are known, and approximation schemes exist even for strong disorder.[47]

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[39] This assumption will probably fail in the critical region near a metal-insulator transition. However, its range of validity is certainly larger than that of the static screening approximation, Ref. 31.

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[43] The other terms on the rhs of Eq. (4.14) are nonzero only due to the ϵ dependence of Y. If the disorder is not too large, Y will be only weakly ϵ dependent, and one expects the last term to be the principal one.

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[46] This relation is also used in the derivation of the standard dirty limit theory given by P. G. De Gennes, *Superconductivity of Metals and Alloys* (Benjamin, New York, 1966).

[47] See, e.g., W. Götze, in *Localization, Interaction, and Transport Phenomena*, Ref. 21.