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# Transport properties of a multicomponent Fermi liquid

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Starting from the Landau kinetic equation, we derive exact expressions for transport coefficients of a multicomponent Fermi liquid. We consider arbitrary ratios of Fermi momenta and effective masses, and completely general scattering rates. As specific examples we evaluate the thermal conductivity, shear viscosity, and diffusion coefficients for the multicomponent system, and give further details for the two-component case.

## I. INTRODUCTION

The transport properties of one-component Fermi liquids at low temperatures may be calculated exactly using the techniques first developed by Brooker and Sykes[1] and by Højgaard Jensen, Smith, and Wilkins.[2] In recent years, multicomponent Fermi liquids have become important in a number of different contexts, for example, asymmetric nuclear matter, the coupled electron, proton, neutron, and neutrino Fermi liquids found in stellar collapse and in the cores of neutron stars, spin-polarized $^3$He, either pure or dissolved in liquid $^4$He, liquid metallic hydrogen and electron-hole droplets in semiconductors. Transport properties of the coupled electron-proton-neutron system have been discussed by Flowers and Itoh,[3] and transport of spin-polarized $^3$He has been considered by a number of people.[4,5] Flowers and Itoh[3] sketched how this could be done, but obtained explicit results only for some special cases. Meyerovich[4] considered only specific forms of the scattering amplitudes and neglected the energy dependence of the deviation functions. Mullin and Miyake[5] obtained the exact solution of the transport equation for the case when scattering of quasiparticles by other quasiparticles from the same component could be neglected. Exact expressions for transport coefficients of liquid metallic hydrogen, for which the electron and proton Fermi momenta are the same, have been found by Oliva and Ashcroft.[6]

In this paper we shall show how to generalize the calculations of Refs. 1 and 2 to the multicomponent case and shall derive exact expressions for the transport coefficients. We consider arbitrary ratios of the Fermi momenta and effective masses of the components, and completely general scattering rates, and we shall express our results in terms of the functions that occur in the expressions for the transport coefficients of single-component Fermi liquids.

## II. TRANSPORT EQUATIONS

The standard Landau kinetic equation has the form

$$\frac{\partial n_{\mathbf{p}i}}{\partial t}+\nabla n_{\mathbf{p}i}\cdot\nabla_{\mathbf{p}}\epsilon_{\mathbf{p}i}-\nabla_{\mathbf{p}}n_{\mathbf{p}i}\cdot\nabla\epsilon_{\mathbf{p}i}=I_i\ , \tag{1}$$

where $\mathbf{p}$ is the momentum of a quasiparticle and $\epsilon_{\mathbf{p}}$ is its energy, including nonequilibrium contributions. The index $i$, which labels the components, includes information both about the species of a quasiparticle and its spin. $I_i$ is the collision integral which will depend on the quasiparticle distribution for other indices than $i$, and it is given by

$$I_{\mathbf{p}_1 i}=-\frac{1}{V^2}\sum_{\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4}{}'\sum_{j,k,l}W_{ijkl}(1,2;3,4)\delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_3+\mathbf{p}_4}\times[n_{1i}n_{2j}(1-n_{3k})(1-n_{4l})-(1-n_{1i})(1-n_{2j})n_{3k}n_{4l}]\delta(\epsilon_{1i}+\epsilon_{2j}-\epsilon_{3k}-\epsilon_{4l})\ , \tag{2}$$

where the transition probability $W$ is given by

$$\frac{2\pi}{\hbar}\,|\langle 3,k;4,l\,|\,t\,|\,1,i;2,j\rangle|^2=\frac{W_{ijkl}(1,2;3,4)}{V^2}\delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_3+\mathbf{p}_4}\ , \tag{3}$$

where $\langle 3,k;4,l\,|\,t\,|\,1,i;2,j\rangle$ is the $T$ matrix for scattering of quasiparticles from components $i$ and $j$, with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively, to a state with quasiparticles from components $k$ and $l$, with momenta $\mathbf{p}_3$ and $\mathbf{p}_4$. Here we have adopted a shorthand notation, and have replaced $p_\alpha$ by $\alpha$. The prime in the sum in Eq. (2) indicates that if $k$ and $l$ are the same, the sum must be taken only over distinguishable states.

As usual, one linearizes the collision term by writing

$$n_{\mathbf{p}i}=n_i^0(\epsilon_{\mathbf{p}i})-T\frac{\partial n_{\mathbf{p}i}^0}{\partial\epsilon_{\mathbf{p}i}}\Phi_{\mathbf{p}i}\ , \tag{4}$$

where $n_i^0$ is the Fermi function. The collision integral then becomes

$$I_{1i}=\sum_{2,3,4}{}' n_{1i}^0 n_{2j}^0(1-n_{3k}^0)(1-n_{4l}^0)W_{ij,kl}(1,2;3,4)\delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_3+\mathbf{p}_4}\delta(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)(\overline{\Phi}_{1i}+\overline{\Phi}_{2j}-\overline{\Phi}_{3k}-\overline{\Phi}_{4l}) . \quad (5)$$

Rather than solve the equations in complete generality which is rather cumbersome, we shall discuss a number of specific examples of physical interest. However, we shall obtain, in quite general terms, the response of the multicomponent systems to gradients of the variables determining the local equilibrium of the system.

## III. SOLUTIONS FOR MULTICOMPONENT SYSTEMS

In this paper we shall investigate examples where the number of quasiparticles of each component is conserved. This is the case in spin-polarized liquid $^3$He if the number of quasiparticles of each spin is conserved separately, which is true if the dipole-dipole interaction between nuclear spins may be neglected. This is reasonable unless non-spin-conserving processes are essential, as in calculations of the longitudinal spin relaxation time.[7] For asymmetric nuclear matter, the numbers of neutrons and protons are separately conserved if beta decay processes are neglected, and to the extent that spin polarization is absent, the system may be treated as a two-component, rather than a four-component system.

At low temperatures, the momenta of all thermally excited quasiparticles (and quasiholes) are confined to lie in the vicinity of their respective Fermi surfaces, and therefore one may decouple the angular and energy variables in the manner first introduced by Abrikosov and Khalatnikov.[8] In the two-body scattering problem one may use the angles $\theta$ and $\phi$ to describe the relative orientations of the quasiparticle momenta, where $\theta$ is the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$ and $\phi$ is the angle between the plane containing $\mathbf{p}_1$ and $\mathbf{p}_2$, and the one containing $\mathbf{p}_3$ and $\mathbf{p}_4$. As usual we shall adopt the convention that quasiparticle 3 is from the same component as quasiparticle 1, and hence that quasiparticle 4 belongs to the same component as quasiparticle 2. We therefore simplify the notation for this case by writing

$$W(1i,2j;3i,4j)=W_{ij}(\theta,\phi) . \quad (6)$$

The phase space sums are converted into angular and energy integrals by using the result[8]

$$\frac{1}{V^2}\sum_{\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4}{}' \delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_3+\mathbf{p}_4} \rightarrow \frac{m_i^* m_j^{*2}}{1+\delta_{ij}}\int \frac{d\Omega}{4\pi}\frac{d\phi_2}{2\pi}\beta_{ij}d\epsilon_2 d\epsilon_3 d\epsilon_4 , \quad (7)$$

where

$$\beta_{ij}=p_j/P_{ij} , \quad (8)$$

with

$$P_{ij}=(p_i^2+p_j^2+2p_ip_j\cos\theta)^{1/2} . \quad (9)$$

The factor $(1+\delta_{ij})^{-1}$ occurs because of the prime on the sum over momentum states which was introduced to avoid overcounting the final states when the two final quasiparticles are identical; $p_i$ and $p_j$ are the Fermi momenta of species $i$ and $j$, respectively, $d\Omega=\sin\theta\, d\theta\, d\phi/4\pi$, and $\phi_2$ is the azimuthal angle of $\mathbf{p}_2$ with respect to $\mathbf{p}_1$. The collision integral therefore becomes

$$I_{1i}=-\frac{m_i^* T^2}{8\pi^4\hbar^6}\int\int dx_2 dx_3 n^0(x_1)n^0(x_2)[1-n^0(x_3)][1-n^0(x_1+x_2-x_3)]$$
$$\times\left[\sum_j m_j^{*2}\int\int\frac{d\Omega}{4\pi}\frac{d\phi_2}{2\pi}\frac{W_{ij}\beta_{ij}}{1+\delta_{ij}}[\overline{\Phi}_i(\mathbf{p}_1)+\overline{\Phi}_j(\mathbf{p}_2)-\overline{\Phi}_i(\mathbf{p}_3)-\overline{\Phi}_j(\mathbf{p}_4)]\right] , \quad (10)$$

with

$$x_\alpha=(\epsilon_\alpha-\mu_\alpha)/T . \quad (11)$$

For $i=j$, $\beta_{ij}$ reduces to the well-known result $[2\cos(\theta/2)]^{-1}$.

It is convenient to introduce a quasiparticle lifetime for the state of momentum $\mathbf{p}$, which is defined by evaluating the collision integral (2) with $n_{\mathbf{p}i}=n_i^0(\epsilon_{\mathbf{p}i})$ for all quasiparticle states except that with momentum $\mathbf{p}$ for the component of interest,[9] in which case

$$I_{\mathbf{p}i}=-\frac{n_{\mathbf{p}i}-n_i^0(\epsilon_{\mathbf{p}i})}{\tau_i(x)} . \quad (12)$$

On performing the phase-space manipulations described above and on making use of a standard integral,[9] one finds

$$\frac{1}{\tau_i(x)}=\frac{m_i^* T^2}{8\pi^4\hbar^6}\frac{x^2+\pi^2}{2}\sum_j m_j^{*2}\int\frac{d\Omega}{4\pi}\frac{W_{ij}}{1+\delta_{ij}}\beta_{ij} . \quad (13)$$

In the hydrodynamic limit, in which we are interested here, we may, following the standard procedure due to Chapman and Enskog, replace the distribution functions on the left-hand sides of the transport equations by the local equilibrium ones, and eliminate the time derivatives by employing the conservation laws. The transport equation then has the form

$$-\frac{\partial n_i^0}{\partial x_1}X_i(x_1,\hat{\mathbf{p}}_1)=I_{1i} . \quad (14)$$

As is clear from the form of the transport equation, the distribution functions of the various components are coupled through the collision term, since the scattering rate for one component depends on the distribution functions of the other components.

In general, the driving term in the Boltzmann equation may be written as a sum of terms of the type

$$X_i(x_1,\hat{\mathbf{p}})=F_iX_Q(x_1)D_Q(\hat{\mathbf{p}})Q\ , \tag{15}$$

where $Q$ is related to the gradients of the variables specifying local equilibrium of the system, $D_Q(\hat{\mathbf{p}})$ is an irreducible tensor quantity (for example, a spherical harmonic) which depends on the direction $\hat{\mathbf{p}}$ on the Fermi surface, $X_Q(x_1)$ is a function only of the energy variable $x_1$, and $F_i$ is a scalar function which can depend on Fermi velocities, temperature, etc. For example, for thermal conduction, $X_i(x_1,\hat{\mathbf{p}})=x_1v_i\hat{\mathbf{p}}\cdot(\nabla T/T)$, and $F_i=v_i/T$, $X_Q(x_1)=x_1$, $D_Q(\hat{\mathbf{p}})=\hat{\mathbf{p}}\cdot\hat{\boldsymbol{\rho}}$, and $Q=\nabla_\rho T$, where $\rho$ is a spatial index. Here $v_i=p_i/m_i^*$ is the Fermi velocity. Since we shall work with the linearized transport equation, it is enough to consider terms of the type (15) one at a time. Also, since the kernel in collision integral is invariant under changing the signs of all $x_\alpha$ simultaneously, we may confine our attention to driving terms which are either odd or even functions of $x_1$. In this case, the deviation function $\overline{\Phi}_i(x_\alpha)$ are even or odd functions of $x_\alpha$, according to the symmetry of the driving term.

Since in the collision integral the energy and angular integrals are not coupled, the solution to the transport equation may be written in the form

$$\overline{\Phi}_i(x_\alpha,\hat{\mathbf{p}}_\alpha)=\gamma_i(x_\alpha)D_Q(\hat{\mathbf{p}}_\alpha)Q\ , \tag{16}$$

where $\gamma_i(x_\alpha)$ is a function of the energy variable $x_\alpha$ and $Q$ and $D_Q(\hat{\mathbf{p}}_\alpha)$ are defined above. The transport equation then becomes

$$\frac{F_iX^P_{Q,i}(x_1)}{4\cosh^2(x_1/2)}=-\int\frac{m_i^*T^2}{8\pi^4\hbar^6}\int dx_2L(x_1,x_2)\sum_j m_j^{*2}\int\int\frac{d\Omega}{4\pi}\frac{W_{ij}\beta_{ij}}{1+\delta_{ij}}$$
$$\times[\gamma_i(x_1)+P\gamma_j(x_2)P_l(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_2)-\gamma_i(x_2)P_l(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_3)$$
$$-\gamma_j(x_2)P_l(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_4)]\ , \tag{17}$$

where

$$L(x_1,x_2)=\frac{1}{8\cosh(x_1/2)\cosh(x_2/2)}\frac{x_1-x_2}{\sinh[(x_1-x_2)/2]}\ , \tag{18}$$

and $P$ denotes the evenness or oddness of the driving term as a function of $x_1$.

By inspecting Fig. 1, one may easily evaluate the angles between the various momentum vectors, and one finds

$$\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_2=\cos\theta\ ,$$
$$\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_3=1-\frac{p_j^2}{P_{ij}^2}\sin^2\theta(1-\cos\phi)\ , \tag{19}$$

and

$$\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_4=\cos\theta+\frac{p_ip_j}{P_{ij}^2}\sin^2\theta(1-\cos\phi)\ .$$

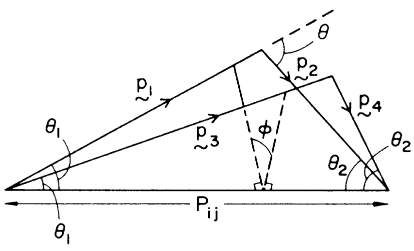

FIG. 1. Geometry of momenta in a two-quasiparticle collision. In deriving Eqs. (19), we use the fact that $\sin\theta_1/p_2=\sin\theta_2/p_1=\sin\theta/P_{ij}$. By convention, $p_1=p_3=p_i$ and $p_2=p_4=p_j$. We neglect deviations of the quasiparticle momenta from the respective Fermi momenta.

On introducing new variables

$$\Xi_i^P(x_\alpha)=-[m_i^*p_i\tau_i(0)]^{1/2}\frac{X^P_{Q,i}(x_\alpha)}{2\cosh(x_\alpha/2)}F_i \tag{20}$$

and

$$\Gamma_i(x_\alpha)=\left[\frac{m_i^*p_i}{\tau_i(0)}\right]^{1/2}\frac{\gamma_i(x_\alpha)}{2\cosh(x_\alpha/2)}\ , \tag{21}$$

the transport equation (17) reduces to the simple form

$$\Xi^P_{1i}(x_1)=\left[1+\frac{x_1^2}{\pi^2}\right]\Gamma_i(x_1)$$
$$-\sum_j\lambda_{ij}\int_{-\infty}^{\infty}\frac{dx_2}{\pi^2}\frac{x_1-x_2}{\sinh[(x_1-x_2)/2]}\Gamma_j(x_2) \tag{22}$$

where

$$\lambda_{ii}=\frac{\tau_i(0)T^2}{16\pi^2\hbar^6}m_i^*\sum_j\frac{m_j^{*2}}{1+\delta_{ij}}\int\frac{d\Omega}{4\pi}W_{ij}\frac{p_j}{P_{ij}}\xi^i_{ij} \tag{23}$$

$$=\frac{\displaystyle\sum_j\left[\frac{m_j^*}{m_i^*}\right]^2\int\frac{d\Omega}{4\pi}\frac{W_{ij}}{1+\delta_{ij}}\frac{p_j}{P_{ij}}\xi^i_{ij}}{\overline{W}_i}\ , \tag{24}$$

$$\lambda_{ij}=\frac{\sqrt{\tau_i(0)\tau_j(0)}T^2}{16\pi^2\hbar^6}(m_i^* m_j^*)^{3/2}\int\frac{d\Omega}{4\pi}W_{ij}\frac{\sqrt{p_ip_j}}{P_{ij}}\xi_{ij}^j \tag{25}$$

$$=\frac{\displaystyle\int\frac{d\Omega}{4\pi}W_{ij}\frac{\sqrt{p_ip_j}}{P_{ij}}\xi_{ij}^j}{\sqrt{\overline{W}_i\overline{W}_j}}\quad\text{for } i\neq j\ , \tag{26}$$

where

$$\overline{W}_i=\sum_j\left[\frac{m_j^*}{m_i^*}\right]^2\int\frac{d\Omega}{4\pi}\frac{W_{ij}}{1+\delta_{ij}}\frac{p_j}{P_{ij}}\ , \tag{27}$$

$$\xi_{ii}^i=-PP_l(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_2)+P_l(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_3)+P_l(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_4)\ , \tag{28}$$

$$\xi_{ij}^i=P_l(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_3)\quad(i\neq j)\ , \tag{29}$$

and

$$\xi_{ij}^j=-PP_l(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_2)+P_l(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_4)\quad(i\neq j)\ . \tag{30}$$

Equation (22) resembles the standard form of the transport equation for degenerate fermions, except that $\lambda_{ij}$ is a matrix here. Since $\hat{\mathbf{p}}_2\cdot\hat{\mathbf{p}}_3=\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_4$, it can be easily verified that $\xi_{ij}^j=\xi_{ji}^i$. Thus $\lambda_{ij}$ is symmetrical in the indices $i$ and $j$, and we may diagonalize the set of equations by an orthogonal transformation $S$ such that

$$\widetilde{S}_{ij}\lambda_{jl}S_{lk}=\lambda_i\delta_{ik}\ , \tag{31}$$

where $\lambda_i$ are the eigenvalues. The transformation matrices $S$ and $\widetilde{S}$, and the eigenvalues $\lambda_i$ for the two-component case are given in Appendix A. The transport equation then becomes a decoupled set of equations similar in form to those for single-component Fermi liquids:

$$\overline{\Xi}_i^P(x_1)=\left[1+\frac{x_1^2}{\pi^2}\right]\overline{\Gamma}_i(x_1)-\lambda_i\int_{-\infty}^{\infty}\frac{dx_2}{\pi^2}\frac{x_1-x_2}{\sinh[(x_1-x_2)/2]}\overline{\Gamma}_i(x_2)\ , \tag{32}$$

where

$$\overline{\Xi}_i^P(x_\alpha)=\widetilde{S}_{ij}\Xi_j^P(x_\alpha) \tag{33}$$

and

$$\overline{\Gamma}_i(x_\alpha)=\widetilde{S}_{ij}\Gamma_j(x_\alpha)\ . \tag{34}$$

Equation (32) may be solved by expanding $\overline{\Gamma}$ in terms of the eigenfunctions, $\psi_\nu$ of the equation

$$\int_{-\infty}^{\infty}\frac{dx_2}{\pi^2}\frac{x-x_2}{\sinh\left[\dfrac{x-x_2}{2}\right]}\psi_\nu(x_2)=\frac{1}{\alpha_\nu}\left[1+\frac{x^2}{\pi^2}\right]\psi_\nu(x)\ . \tag{35}$$

The eigenvalues are

$$\alpha_\nu=1/2\nu(\nu+1),\quad \nu=1,2,3,\ldots \tag{36}$$

and the eigenfunctions are given in Ref. 9. Because we are working in terms of the time $\tau(0)$, rather than $\tau=\pi^2\tau(0)/2$, it is convenient to choose a normalization condition

$$\int_{-\infty}^{\infty}dx\left[1+\frac{x^2}{\pi^2}\right]\psi_\nu^*(x)\psi_\mu(x)=\delta_{\nu,\mu} \tag{37}$$

which differs from the convention used earlier. In the notation of Refs. 1 and 8, $\psi_\nu=\pi\phi_\nu$. The solution to Eq. (32) is

$$\overline{\Gamma}_i(x)=\sum_\nu\frac{\psi_\nu(x)}{\left[1-\dfrac{\lambda_i}{\alpha_\nu}\right]}\int_{-\infty}^{\infty}dx'\,\psi_\nu^*(x')\overline{\Xi}_i^P(x')\ . \tag{38}$$

The final step is to evaluate the response of interest, which we write quite generally in the form

$$\begin{aligned}J&=\sum_i\sum_{\mathbf{p}}O_{\mathbf{p}i}[n_{\mathbf{p}i}-n_i^0(\epsilon_{\mathbf{p}i})]\\&=T\sum_i\sum_{\mathbf{p}}O_{\mathbf{p}i}\frac{\partial n_{\mathbf{p}i}^0}{\partial x_i}\overline{\Phi}_{\mathbf{p}i}\ ,\end{aligned} \tag{39}$$

where $O_{\mathbf{p}i}$ may be written in a form analogous to (15),

$$O_{\alpha i}=G_iX_0(x_\alpha)D_0(\hat{\mathbf{p}}_\alpha)\ . \tag{40}$$

where $G_i$ is a scalar function like $F_i$. Equation (39) may be rewritten in terms of $\overline{\Gamma}$ by use of Eqs. (16), (21), and (34):

$$J=-QT\sum_i N_i(0)G_i\int\frac{d\Omega_{\hat{\mathbf{p}}}}{4\pi}D_0(\hat{\mathbf{p}})D_Q(\hat{\mathbf{p}})\int dx\frac{X_0(x)}{2\cosh(x/2)}\left[\frac{\tau_i(0)}{m_i^*p_i}\right]^{1/2}S_{ij}\overline{\Gamma}_j\ . \tag{41}$$

Here $N_i(0)$ is the density of states at the Fermi surface for species $i$, and one must be careful to include a factor for possible spin or isospin degeneracy of the component. Finally, inserting the solution (38) into (41), and using Eqs. (20) and (33), we can write the transport coefficient in the compact form

$$\frac{J}{Q}=-T\int d\Omega_{\hat{\mathbf{p}}}D_0(\hat{\mathbf{p}})D_Q(\hat{\mathbf{p}})\sum_{i,j,k}\sqrt{N_i(0)\tau_i(0)N_k(0)\tau_k(0)}G_iS_{ij}\sum_\nu\langle X_0\,|\,\nu\rangle\frac{1}{\left[1-\dfrac{\lambda_j}{\alpha_\nu}\right]}\langle\nu\,|\,X_Q\rangle\widetilde{S}_{jk}F_k\ , \tag{42}$$

where

$$\langle X_0\,|\,\nu\rangle=\int_{-\infty}^{\infty}\frac{dx\,X_0(x)\psi_\nu(x)}{2\cosh(x/2)}\ . \tag{43}$$

In writing Eq. (42), we have assumed that the spin or isospin degeneracy of all components is the same. The sum over $\nu$ in this equation is essentially the transport coefficient for a single-component system. The sum over $j$

TABLE I. Quantities specifying the driving terms $X_i(x,\hat{\mathbf{p}})$ [Eq. (15)], the deviation functions $\Phi_i(x,\hat{\mathbf{p}})$ [Eq. (16)], and the current operator $O_{pi}$ [Eq. (40)], for thermal conductivity $\kappa$, viscosity $\eta$, and diffusion $D$. $v_i$ ($p_i$) are the Fermi velocities (momenta), $x=(\epsilon-\mu)/T$ is the energy variable, $T$ the temperature, $m=n_1-n_2$ the asymmetry in the number densities with $\sigma_i$ taking on the values $+1$ or $-1$, and $\hat{\mathbf{n}}$ is the unit vector along the relevant gradient.

| | $\kappa$ | $\eta$ | $D$ |
|---|---|---|---|
| $F_i$ | $v_i/T$ | $p_iv_i/T$ | $\left[\sigma_i-\dfrac{m}{n}\right]v_i\Big/T$ |
| $X_Q=X_0$ | $x$ | 1 | 1 |
| $D_Q(\hat{\mathbf{p}})=D_0(\hat{\mathbf{p}})$ | $\hat{\mathbf{p}}\cdot\hat{\mathbf{n}}$ | $(\hat{\mathbf{p}}\cdot\hat{\mathbf{x}})(\hat{\mathbf{p}}\cdot\hat{\mathbf{y}})$ | $\hat{\mathbf{p}}\cdot\hat{\mathbf{n}}$ |
| $Q$ | $\nabla T$ | $\dfrac{\partial u_x}{\partial y}$ | $\nabla(\delta\mu)$ |
| $G_i$ | $Tv_i$ | $p_iv_i$ | $\left[\sigma_i-\dfrac{m}{n}\right]v_i$ |

picks up the eigenvalues $\lambda_j$. For each $i$ and $k$, the matrix $\tilde{S}_{jk}$ transforms from the initial basis to the eigenstates of the collision operator, while $S_{ij}$ transforms back to the initial one.

We now turn to specific examples. The various quantities specifying the driving terms $X_i(x,\hat{\mathbf{p}})$, the deviation functions $\Phi_i(x,\hat{\mathbf{p}})$, and the current operator $O_{pi}$ in the examples we consider are tabulated in Table I.

## IV. EVALUATION OF TRANSPORT COEFFICIENTS

### A. Thermal conductivity

As shown on Table I, for thermal conduction the driving term at low temperatures is $X_i(x,\hat{\mathbf{p}})=x_1v_i\hat{\mathbf{p}}\cdot\nabla T/T$, or $F_i=v_i/T$, $X_Q(x)=x$, $Q=\nabla T$, and $D_Q(\hat{\mathbf{p}})=\hat{\mathbf{p}}\cdot\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector in the direction of $\nabla T$. The current of interest is the heat current, for which at low temperatures $\mathbf{O}_{pi}=(\epsilon_{pi}-\mu_i)v_i\hat{\mathbf{p}}$. Its only non-vanishing component is along $\nabla T$, and therefore $G_i=Tv_i$, $X_0=x$, and $D_0(\hat{\mathbf{p}})=\hat{\mathbf{p}}\cdot\hat{\mathbf{n}}$. The driving term is an odd function of $x$ and has $l=1$ angular symmetry, and therefore in the expression for $\xi_{ij}^k$ in Eqs. (28)–(30), $P$ is $-1$, and $l=1$.

The integral (43) is given by

$$\int_{-\infty}^{\infty}dx\frac{x\psi_\nu(x)}{2\cosh(x/2)}=-\frac{i\pi\sqrt{2\nu+1}}{\nu(\nu+1)},\quad \nu\text{ even} \tag{44}$$

and the other one is its complex conjugate. The angular integral gives a factor $\frac{1}{3}$, and we find for the thermal conductivity $\kappa$,

$$\kappa=\left[\frac{\pi}{3}\right]^2 T\sum_{i,j,k}\sqrt{N_i(0)\tau_i(0)N_k(0)\tau_k(0)}\times v_iv_kS_{ij}R^-(\lambda_j)\tilde{S}_{jk}\ , \tag{45}$$

where

$$R^-(\lambda)=3\sum_{\nu=2,4,6,\ldots}\frac{2\nu+1}{\nu(\nu+1)[\nu(\nu+1)-2\lambda]} \tag{46}$$

is the ratio of the relaxation time for thermal conductivity compared with the quasiparticle lifetime at the Fermi surface for a single-component system, with $\lambda$ the measure of the relative importance of the integral term in the Boltzmann equation. The minus sign on $R$ symbolizes that it is for a driving term odd in $x$, and therefore only terms with even $\nu$ contribute. If $\lambda$ approaches 3, the simplest variational solution[10] of the Boltzmann equation becomes exact; the sum in (46) is dominated by the first term, $\frac{5}{12}/(1-\lambda/3)$. A plot of $R^-(\lambda)$ is given in Fig. 2(a). We also give, in Fig. 2(b), a plot of $R^-(\lambda)$ divided by the simplest variational result. For the single-component case Eq. (45) reduces to the familiar result

$$\kappa=\frac{\pi^2}{3}N(0)T\frac{v_F^2}{3}\tau_\kappa=C_v\frac{v_F^2}{3}\tau_\kappa\ , \tag{47}$$

where $\tau_\kappa=R^-(\lambda)\tau(0)$, $v_F$ is the Fermi velocity, and $C_v$ is the specific heat per unit volume.

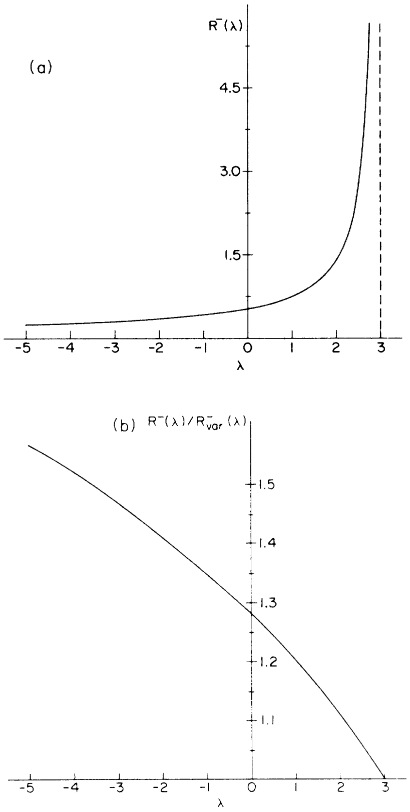

FIG. 2. Plots showing the behavior of (a) $R^-(\lambda)$ [Eq. (46)] as a function of $\lambda$; (b) $R^-(\lambda)/R^-_{\text{var}}(\lambda)$ as a function of $\lambda$. $R^-(\lambda)$ enters into thermal conductivity [Eq. (45)] as the ratio $\tau_\kappa/\tau(0)$, while $\lambda$ is the measure of the relative importance of the integral term in the Boltzmann equation. $R^-_{\text{var}}(\lambda)$ is the simplest variational result.

### B. Shear viscosity

Since the system has rotational invariance, we may evaluate the viscosity by considering a shear, in which the fluid velocity $\mathbf{u}$ is in the $x$ direction and varies only in the $y$ direction. The driving term (see Table I) is

$$X_i(x,\hat{\mathbf{p}})=\frac{p_{ix}v_{iy}}{T}\frac{\partial u_x}{\partial y} ,$$

and therefore $F_i=p_iv_i/T$, $X_Q=1$, $D_Q(\hat{\mathbf{p}})=(\hat{\mathbf{p}}\cdot\hat{\mathbf{x}})(\hat{\mathbf{p}}\cdot\hat{\mathbf{y}})$, and $Q=\partial u_x/\partial y$. It is an even function of $x$ and has $l=2$ rotational symmetry, so in Eqs. (28)—(30), $P$ is $+1$ and $l$ is 2. To evaluate the viscosity, we calculate the momentum current, for which $O_{\mathbf{p}i}=p_{ix}v_{iy}$, and therefore $G_i=p_iv_i$, $X_0=1$, and $D_0(\hat{\mathbf{p}})=D_Q(\hat{\mathbf{p}})$. The integral (43) for this case is

$$\int_{-\infty}^{\infty} dx\frac{\psi_\nu(x)}{2\cosh(x/2)}=-\frac{\sqrt{2\nu+1}}{\nu(\nu+1)}, \quad \nu \text{ odd} . \tag{48}$$

The viscosity is therefore given by

$$\eta=\frac{1}{15}\sum_{i,j,k}\sqrt{N_i(0)\tau_i(0)N_k(0)\tau_k(0)} \times p_iv_ip_kv_kS_{ij}R^{+}(\lambda_j)\widetilde{S}_{jk} , \tag{49}$$

where

$$R^{+}(\lambda)=\sum_{\nu=1,3,5,\ldots}\frac{2\nu+1}{\nu(\nu+1)[\nu(\nu+1)-2\lambda]} \tag{50}$$

is the ratio of the relaxation time for viscosity compared with the quasiparticle lifetime at the Fermi surface for a single-component system. When $\lambda$ approaches 1, the simplest variational solution of the transport equation is exact, and $R^{+}(\lambda)$ is dominated by the first term in (50), $\frac{3}{4}/(1-\lambda)$. $R^{+}(\lambda)$ is plotted in Fig. 3(a), and in Fig. 3(b) we give a plot of $R^{+}(\lambda)$ divided by the simplest variational result. For a single-component system, $\eta$ reduces to the standard result

$$\eta=\frac{1}{15}N(0)p_F^2v_F^2\tau_\eta , \tag{51}$$

where $\tau_\eta=R^{+}(\lambda)\tau(0)$.

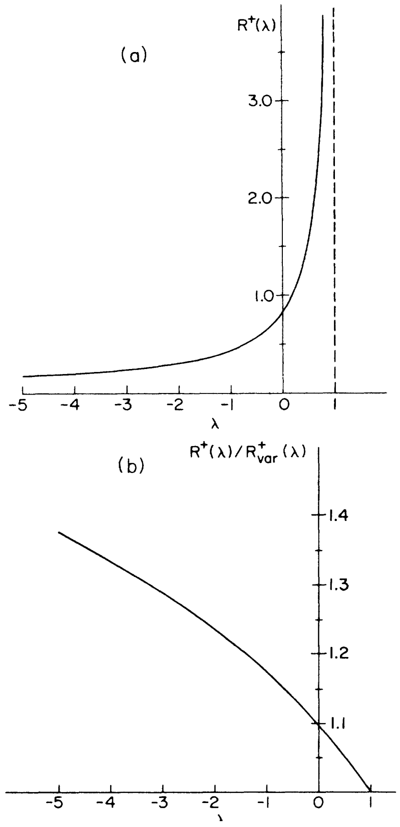

FIG. 3. Plots showing the behavior of (a) $R^{+}(\lambda)$ [Eq. (50)] as a function of $\lambda$; (b) $R^{+}(\lambda)/R^{+}_{\text{var}}(\lambda)$ as a function of $\lambda$. $R^{+}(\lambda)$ enters into viscosity and diffusion [Eqs. (49), (57), and (61)] as $\tau_\eta/\tau(0)$ and $\tau_D/\tau(0)$, respectively. $\lambda$ signifies the relative importance of the integral term in the Boltzmann equation. $R^{+}_{\text{var}}(\lambda)$ is the simplest variational result.

### C. Diffusion

For clarity we shall concern ourselves only with the two-component case here. One example of this is the diffusion of neutrons relative to protons in asymmetric nuclear matter, where the two components are the two types of nucleons and the spin variable plays no role as far as diffusion is concerned. Another is spin diffusion in polarized $^3$He, where the two components are now two different spin populations. Yet another is spin diffusion in liquid metallic hydrogen.[6]

We begin by considering situations similar to the first two. For relative diffusion of neutrons and protons, the component labels $i$ and $j$ refer to neutrons and protons, and spins need not be referred to explicitly, while for spin-polarized $^3$He, the labels refer to up- and down-spin quasiparticles. Spin diffusion in liquid metallic hydrogen is somewhat different from the first two cases, and we shall discuss it later.

The gradients of the densities of the components give rise to chemical potential gradients which cause counterflow of the components. Since we are concerned with diffusion at constant pressure $P$ and temperature, the Gibbs-Duhem relation is

$$\nabla P-S\nabla T=n_1\nabla\mu_1+n_2\nabla\mu_2=0 , \tag{52}$$

where $S$ is the entropy per unit volume. Therefore we may write

$$\nabla\mu_i=(\sigma_i-m/n)\nabla(\delta\mu) , \tag{53}$$

where $\mu_i$ is the chemical potential of the $i$th component, $\delta\mu=(\delta\mu_1-\delta\mu_2)/2$ is half the difference in the chemical potentials of the components, $m=n_1-n_2$ is the difference in the number densities of the components (proportional to magnetization in the case of spin-polarized $^3$He), $n=n_1+n_2$ is the total particle density, and $\sigma_i=+1$ for $i=1$, and $-1$ for $i=2$. Then the driving term (see Table I) is

$$X_i(x,\hat{\mathbf{p}})=v_i\hat{\mathbf{p}}\cdot\nabla\mu_i/T=\left[\sigma_i-\frac{m}{n}\right]v_i\hat{\mathbf{p}}\cdot\nabla(\delta\mu)/T ;$$

therefore $F_i=(\sigma_i-m/n)v_i/T$, $X_Q(x)=1$, $Q=\nabla(\delta\mu)$, and

$D_Q(\hat{\mathbf{p}})=\hat{\mathbf{p}}\cdot\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is a unit vector along $\nabla(\delta\mu)$. It is an even function of $x$ and has $l=1$ symmetry; so in Eqs. (28)–(30), $P$ is $+1$ and $l=1$. The flux of interest is the number current, $\mathbf{J}_i=\sum_{\mathbf{p}}\mathbf{v}_{\mathbf{p}i}[n_{\mathbf{p}i}-n^0(\epsilon_{\mathbf{p}i})]$, which corresponds to $\mathbf{O}_{\mathbf{p}i}=v_i\hat{\mathbf{p}}$, therefore one can determine these by the standard procedure.

To evaluate diffusion coefficients it is important to take into account that the particle flux will generally have two contributions, one due to diffusion, and the other due to the bulk motion. Consequently to determine the current due to diffusion one must subtract the contribution from bulk motion. The current for each component can be written as

$$\mathbf{J}_i=(\mathbf{J}_i)_{\text{diff}}+\bar{\mathbf{v}}n_i\ , \tag{54}$$

where $(\mathbf{J}_i)_{\text{diff}}$ is the current due to diffusion and $\bar{\mathbf{v}}$ is some average velocity. In general, there is more than one way of choosing $\bar{\mathbf{v}}$. One is to impose the condition that the diffusive flux of particles, irrespective of their species, vanish. This gives the equation

$$\sum_i(\mathbf{J}_i)_{\text{diff}}=\mathbf{J}_1+\mathbf{J}_2-n\bar{\mathbf{v}}=0\ . \tag{55}$$

The currents of interest, the diffusion currents $(\mathbf{J}_i)_{\text{diff}}$, are then given by

$$(\mathbf{J}_i)_{\text{diff}}=\mathbf{J}_i-n_i\bar{\mathbf{v}} \tag{56}$$

with $\bar{\mathbf{v}}$ being determined from Eq. (55). If one is interested only in the difference $(\mathbf{J}_1)_{\text{diff}}-(\mathbf{J}_2)_{\text{diff}}$, as one is in the case of spin diffusion, one finds

$$\mathbf{J}_{\text{diff}}=(\mathbf{J}_1)_{\text{diff}}-(\mathbf{J}_2)_{\text{diff}}=\sum_i\mathbf{v}_i\left[\sigma_i-\frac{m}{n}\right]\delta\bar{n}_{\mathbf{p}i}\ , \tag{57}$$

where $\delta\bar{n}_{\mathbf{p}i}=\mathbf{n}_{\mathbf{p}i}-n_i^0(\epsilon_{\mathbf{p}i})$. In our notation for $J$ in Eq. (39) this amounts to having $\mathbf{O}_{\mathbf{p}i}=v_i(\sigma_i-m/n)\hat{\mathbf{p}}$, and hence, $G_i=v_i(\sigma_i-m/n)$, $X_0=1$, $D_0(\hat{\mathbf{p}})=D_Q(\hat{\mathbf{p}})=\hat{\mathbf{p}}\cdot\hat{\mathbf{n}}$. The integral (43) in this case is the same as in the case of viscosity and is given by Eq. (48). One then finds

$$\mathbf{J}_{\text{diff}}=-\tfrac{1}{6}\sum_{i,j,k}\sqrt{N_i(0)\tau_i(0)N_k(0)\tau_k(0)}v_iv_k\left[\sigma_i-\frac{m}{n}\right]\left[\sigma_k-\frac{m}{n}\right]S_{ij}R^+(\lambda_j)\tilde{S}_{jk}\nabla\delta\mu\ , \tag{58}$$

where $R^+(\lambda)$ is given by Eq. (50), and the sum over $i,j,k$, runs over the two components.

The diffusion coefficient relates currents to gradients of concentrations, and therefore the final step is to express $\nabla\delta\mu$ in terms of $\nabla(n_1-n_2)$. Under the conditions we are studying, the pressure gradient vanishes, and we may therefore write

$$\nabla(n_1-n_2)=\chi_P\nabla\delta\mu\ , \tag{59}$$

where $\chi_P=2[\partial(n_1-n_2)/\partial(\mu_1-\mu_2)]_P$ is a generalized susceptibility of the two-component system at constant pressure. An expression for $\chi_P$ in terms of Landau parameters is derived in Appendix B. The diffusion current is therefore given by

$$\mathbf{J}_{\text{diff}}=-D\nabla m\ , \tag{60}$$

with

$$D=\tfrac{1}{3}(\chi_P)^{-1}\sum_{i,j,k}\sqrt{N_i(0)\tau_i(0)N_k(0)\tau_k(0)}v_iv_k\left[\sigma_i-\frac{m}{n}\right]\left[\sigma_k-\frac{m}{n}\right]S_{ij}R^+(\lambda_j)\tilde{S}_{jk}\ . \tag{61}$$

In the one-component case, Eq. (61) reduces to the familiar result

$$D=\tfrac{1}{3}N(0)v_F^2\chi^{-1}\tau_D \tag{62}$$

where

$$\tau_D=R^+(\lambda)\tau(0)\ .$$

The results above were derived for the case when $\bar{\mathbf{v}}$ is chosen such that the net flux of particles in a frame moving with velocity $\bar{\mathbf{v}}$ vanish. Another choice, which is in some respects a more natural one, is to take $\bar{\mathbf{v}}$ to be the center of mass velocity of the particles. In other words, in a frame moving with velocity $\bar{\mathbf{v}}$, the total momentum density vanishes. $\bar{\mathbf{v}}$ is thus given by

$$\bar{\mathbf{v}}=\frac{\mathbf{g}}{\rho}\ , \tag{63}$$

where $\mathbf{g}$ is the total momentum density in the original frame and $\rho=n_1m_1+n_2m_2$ is the total mass density. Here $m_1$ and $m_2$ are the bare masses of the particles. For a system which is Galilean invariant, the momentum density is given by

$$\mathbf{g}=\sum_i\sum_{\mathbf{p}}\mathbf{p}\delta n_{\mathbf{p}i}\ , \tag{64}$$

where $\delta n_{\mathbf{p}i}=n_{\mathbf{p}i}-n_i^0(\epsilon_{\mathbf{p}i}^0)$. Note that it is the energy $\epsilon_{\mathbf{p}i}^0$, which contains no nonequilibrium contributions, that occurs here. $\delta n_{\mathbf{p}i}$ may be expressed in terms of the quantity $\delta\bar{n}_{\mathbf{p}i}=n_{\mathbf{p}i}-n_i^0(\epsilon_{\mathbf{p}i})$ which occurs in the collision term [cf. Eq. (4)], since

$$\begin{aligned}\delta\bar{n}_{\mathbf{p}i}&=\delta n_{\mathbf{p}i}-\frac{\partial n_{\mathbf{p}i}^0}{\partial\epsilon_{\mathbf{p}i}}(\epsilon_{\mathbf{p}i}-\epsilon_{\mathbf{p}i}^0)\\&=\delta n_{\mathbf{p}i}-\frac{\partial n_{\mathbf{p}i}^0}{\partial\epsilon_{\mathbf{p}i}}\sum_j\sum_{\mathbf{p}'}f_{\mathbf{p}\mathbf{p}'}^{ij}\delta n_{\mathbf{p}'j}\ ,\end{aligned} \tag{65}$$

where $f^{ij}_{\mathbf{pp}'}$ is the Landau interaction function. From Galilean invariance it follows that[11]

$$p_i = m_i v_i + \frac{f^{ij}_1}{3} N_j (m_j v_j) \ , \tag{66}$$

where

$$\frac{f^{ij}_1}{3} = \int d\mu \, \mu f^{ij}_{\mathbf{pp}'} \ ,$$

with $\mu = \hat{\mathbf{p}} \cdot \hat{\mathbf{p}}'$, and $p$ and $p'$ are taken to be on the respective Fermi surfaces. From this condition one then finds

$$\mathbf{g} = (m_1 n_1 + m_2 n_2) \bar{\mathbf{v}} = \sum_i m_i \sum_{\mathbf{p}} \mathbf{v}_{\mathbf{p}i} \delta \bar{n}_{\mathbf{p}i} \tag{67}$$

$$= \sum_i m_i \mathbf{J}_i \ . \tag{68}$$

Thus the diffusive current $\mathbf{J}_{\text{diff}}$ in Eq. (58) is given by

$$\mathbf{J}_{\text{diff}} = \sum_i \sum_{\mathbf{p}} \mathbf{v}_{\mathbf{p}i} \left[ \sigma_i - \frac{m_i m}{\rho} \right] \delta \bar{n}_{\mathbf{p}i} \ , \tag{69}$$

or, in other words, the quantity $G_i$ is $v_i(\sigma i - m_i m/\rho)$ for this case, and the diffusion coefficient is given by Eq. (61), but with $(\sigma_i - m/n)$ replaced by $(\sigma_i - m_i m/\rho)$. When the bare mass of the two components are identical, the two choices of $\bar{\mathbf{v}}$ are identical, and thus the two diffusion coefficients are also identical. Note that this is true even if the effective masses for the two components are different.

In the case of liquid metallic hydrogen considered in Ref. 6, the two components, electrons and protons, are distinguishable. Since the exchange interaction between the two different components does not exist (although such interactions are present within each component system), the diffusion of spin of this two-component system reduces in form to that of the single-component case. The case we have considered is more general, and hence the expression for the diffusion coefficient is more involved.

We note that in the case of diffusion in a multicomponent system, there must exist an eigenvalue, $\lambda = 1$. This corresponds to the bulk motion discussed earlier. The contribution to the diffusion coefficient from this term is formally indeterminate, since both the numerator and one of the denominators in the series vanish. The matrix elements in the numerator vanish, since by definition, diffusion currents contain no contributions from bulk motion, and, as a result, the corresponding contribution to the diffusion coefficient vanishes. Accordingly, in summing over the eigenvalues, $j$ in Eqs. (55) and (61), one must exclude the $\lambda_j = 1$ term. To demonstrate that there is an eigenvalue, $\lambda = 1$, we note that for $l = 1$ symmetry and parity, $P = +1$, momentum conservation and Eqs. (28)–(30) give

$$\xi^i_{ii} = 1 \ , \tag{70}$$

$$p_i \xi^i_{ij} + p_j \xi^j_{ij} = p_i \quad (i \neq j) \ . \tag{71}$$

Using these conditions in Eqs. (24) and (26), we obtain

$$1 - \lambda_{ii} = \left[ \frac{m^*_j}{m^*_i} \right]^2 \left[ \frac{p_j}{p_i} \right]^{3/2} \left[ \frac{\overline{W}_j}{\overline{W}_i} \right]^{1/2} \lambda_{ij} \ , \tag{72}$$

which leads to the following result:

$$(1 - \lambda_{ii})(1 - \lambda_{jj}) = \lambda^2_{ij} \ . \tag{73}$$

This is the eigenvalue condition, as given by Eq. (A2) in Appendix A, for an eigenvalue of $\lambda = 1$. The other eigenvalue is $\lambda_{11} + \lambda_{22} - 1 = \lambda_{11}\lambda_{22} - \lambda^2_{12}$. This condition provides a useful consistency check on numerical calculations of $\lambda_{ij}$. By similar argument, one can show that for more than two components, there is likewise an eigenvalue, $\lambda = 1$. There will not generally be an eigenvalue of $\lambda = 1$ in the case of thermal conductivity for which the parity is $P = -1$, or in the case of viscosity, which corresponds to $l = 2$ angular symmetry.

## V. DISCUSSION

We have solved the Boltzmann equation for a multicomponent Fermi liquid to obtain the transport coefficients at low temperatures, thereby generalizing a number of earlier calculations. A particularly noteworthy feature of our results is that they are compact, and involve the same functions which occur in the transport coefficients for single-component Fermi liquids, rather than some new functions given as infinite series. This form makes it rather convenient for obtaining transport coefficients for any number of components. As examples, we have derived explicit expressions for thermal conductivity and shear viscosity for any number of components coupled to one another, and for diffusion in the case of two coupled components. The extension of the calculation for diffusion to more than two components follows easily from our results. For the two-component case, we have provided some explicit details. The eigenvalues $\lambda_i$ and the orthogonal transformation matrix $S$ are given in Appendix A, while the susceptibility at constant pressure $\chi_P$ needed in the computation of diffusion is derived in Appendix B. Again, the extension to a larger number of components follows in a straightforward manner from the discussion in the Appendixes.

Throughout our discussion above we have confined our attention to scattering processes which do not change the numbers of particles of each component, but our results may be extended straightforwardly to more general processes, such as scattering by the dipole-dipole interaction in liquid $^3$He and $\beta$ processes in mixtures of neutrons, protons, electrons, and electron neutrinos.

In our derivation of the transport properties, we have kept the scattering probabilities $W_{ij}(\theta,\phi)$ completely general. So, our calculation can have two sorts of use. One use is, given the expressions for $W_{ij}(\theta,\phi)$ for the physical system under consideration, to predict the transport coefficients. Another use, which has been pioneered in the case of liquid $^3$He and in dilute solutions of $^3$He in $^4$He, is to deduce from experiments the properties of $W_{ij}(\theta,\phi)$.

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## APPENDIX A: ORTHOGONAL TRANSFORMATION

We wish to find the orthogonal matrix $S$ which diagonalizes the symmetric matrix $\lambda_{ij}$, thereby giving the eigenvalues $\lambda_i$ [Eq. (31)]. For this we simply have to solve an eigenvalue problem, namely

$$\lambda_{ij}\alpha_j = \lambda\alpha_j \ , \tag{A1}$$

where $\alpha_j$ is a vector, and the $\lambda$'s are the eigenvalues, both to be determined. The eigenvalue condition is

$$\det |\, \lambda_{ij} - \lambda\delta_{ij} \,| = 0 \ , \tag{A2}$$

and the eigenvector corresponding to the $t$th eigenvalue $\lambda_t$ is obtained from

$$\sum_j (\lambda_{ij} - \lambda_t\delta_{ij})\alpha_j^t = 0 \ . \tag{A3}$$

The eigenvectors determine the matrix $S$.

For the two-component case, $\lambda_{ij}$ is a $2\times 2$ matrix and $\alpha$ a two-component vector. Then the eigenvalues $\lambda_t$ are

$$\lambda_\pm = \lambda_s \pm (\lambda_a^2 + \lambda_{12}^2)^{1/2} \ , \tag{A4}$$

where

$$\lambda_{s\,(a)} = \frac{\lambda_{11} \pm \lambda_{22}}{2} \ . \tag{A5}$$

The quantities $\lambda_{11}$, $\lambda_{22}$, and $\lambda_{12}$ are given [Eqs. (23)–(30)]; they are related to scattering amplitudes and to the Fermi momenta and the effective masses of the two components. The eigenvectors $\alpha^+, \alpha^-$ corresponding to the eigenvalues $\lambda_+$ and $\lambda_-$, respectively, give the transformation matrix $S$

$$S = \begin{bmatrix} \cos\zeta/2 & -\sin\zeta/2 \\ \sin\zeta/2 & \cos\zeta/2 \end{bmatrix} , \tag{A6}$$

where

$$\begin{aligned} A\cos\zeta &= \lambda_a \ , \\ A\sin\zeta &= \lambda_{12} \ , \end{aligned} \tag{A7}$$

with $A$ being a normalization constant. $\alpha^+$ and $\alpha^-$ are the first and second columns of $S$, respectively. $\tilde{S}$ is the transpose of $S$.

In Eq. (A6), $\zeta = 0$ for $\lambda_{12} = \lambda_{21} = 0$; $S$ is then a unit matrix. In other words, in the absence of any coupling between the components, the bases are left unrotated in Eq. (42), and the transport coefficients of the multicomponent system are those of a set of uncoupled Fermi liquids.

## APPENDIX B: SUSCEPTIBILITY FOR A TWO-COMPONENT SYSTEM

We use thermodynamic arguments and Fermi liquid theory to derive the susceptibility at constant pressure $\chi_P$ for a two-component Fermi liquid, which enters into the expression for the diffusion coefficient, Eq. (61), in the text. Our calculation follows the work of Bedell[12] who also obtained other responses in spin-polarized $^3$He.

Consider a two-component Fermi system, with components having densities $n_1$ and $n_2$. We wish to determine $2[\partial(n_1 - n_2)/\partial(\mu_1 - \mu_2)]_P$. To do this we evaluate the change in the energy density, which is given quite generally by

$$\delta E = \sum_{\mathbf{p},i} \epsilon^0_{\mathbf{p}i}\delta n_{\mathbf{p}i} + \tfrac{1}{2} \sum_{\mathbf{p}i,\mathbf{p}'j} f^{ij}_{\mathbf{pp}'}\delta n_{\mathbf{p}i}\delta n_{\mathbf{p}'j} \ . \tag{B1}$$

Here we are interested only in changes of the size of the Fermi surface, and not of its shape, so we may express the energy in terms of changes in the total density. Only the angular averages of $f^{ij}_{\mathbf{pp}'}$ appear, and we denote these by $f^{ij}_0$. One finds

$$\delta E = \mu_i^0\delta n_i + \sum_i \frac{\delta n_i^2}{2N_i(0)} + \tfrac{1}{2}\sum_{i,j} f_0^{ij}\delta n_i\delta n_j \ , \tag{B2}$$

where $\mu_i^0$ is the chemical potential of species $i$ in the absence of the density changes. The chemical potentials are given by $\mu_i = \partial E/\partial n_i$, and therefore

$$\delta\mu_i = \frac{\delta n_i}{N_i(0)} + \sum_j f_{ij}\delta n_j \ . \tag{B3}$$

We wish to evaluate the susceptibility keeping the pressure fixed. The condition for this is

$$\delta P = \sum_i n_i\delta n_i = 0 \ , \tag{B4}$$

or

$$\sum_i n_i \frac{\delta n_i}{N_i(0)} + \sum_{i,j} f_{ij} n_i \delta n_j = 0 \ . \tag{B5}$$

Thus one finds

$$\delta n_1 = \frac{A_2}{A}\delta(n_1 - n_2)$$

and (B6)

$$\delta n_2 = -\frac{A_1}{A}\delta(n_1 - n_2) \ ,$$

where

$$A_i = \frac{n_i}{N_i(0)} + f_{ij}n_j \ , \tag{B7}$$

and $A = A_1 + A_2$. From Eqs. (B3), (B5), and (B6) we find

$$\delta(\mu_1 - \mu_2) = \left[ A_2 \left[ \frac{1}{N_1(0)} + f_{11} - f_{12} \right] + A_1 \left[ \frac{1}{N_2(0)} + f_{22} - f_{12} \right] \right] \frac{\delta(n_1 - n_2)}{A} \ . \tag{B8}$$

This then gives for the susceptibility,

$$(\chi_P)^{-1}=\frac{n}{2}\left[\frac{[1+N_1(0)f_0^{11}][1+N_2(0)f_0^{22}]-N_1(0)N_2(0)f_0^{12^2}}{n_1N_2(0)[1+N_1(0)f_0^{11}]+n_2N_1(0)[1+N_2(0)f_0^{22}]+nN_1(0)N_2(0)f_0^{12}}\right]. \tag{B9}$$

For the case of spin diffusion in an unpolarized Fermi liquid, $n_1=n_2=n/2$, $N_1(0)=N_2(0)=\frac{1}{2}N(0)$, $f_0^{11}=f_0^{22}$, and Eq. (B9) reduces to the familiar result

$$\chi^{-1}=\frac{1+F_0^a}{N(0)}, \tag{B10}$$

where

$$F_0^a=N(0)\left[\frac{f_0^{11}-f_0^{12}}{2}\right]. \tag{B11}$$

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