Meissner effect in anisotropic superconductors

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The Meissner effect in non-s-wave superconductors is studied. In contrast to the isotropic s-wave case, it is shown that even in transverse gauge one must, in general, include contributions from collective modes of the order parameter, in particular the phase mode, in the expression for the screening currents induced in a superconductor by a magnetic field. Expressions for the screening currents applicable to the case of heavy-fermion superconductors are given. Anisotropy of London penetration depth is shown to occur in general for anisotropic superconductors.

The discovery of unconventional superconductivity in the heavy-fermion compounds[1] has led to renewed interest in the properties of non-s-wave superconductors. Theoretical studies, in various approximations, of such properties as ultrasonic attenuation, upper critical field, and proximity and Josephson effects have already appeared. For a critical discussion of this literature see Ref. 2. Here, we add to this literature a microscopic calculation of the linear response for a superconductor in a static magnetic field. We show that the calculation of the Meissner effect in anisotropic superconductors differs in an important and apparently previously neglected way from the standard calculation used for isotropic superconductors. We also obtain formulas for the screening currents in terms of the structure of the gap matrix and the Fermi surface. We show that for non-s-wave superconductors there is anisotropy in the London penetration depth. Its precise magnitude depends on details. What can be said in general is that in a cubic crystal, the anisotropy will be largest near T_c and will go to zero as the temperature $T\rightarrow 0$.

The Meissner effect has been studied previously,[3] but the expression given for the screening current in Ref. 3 is incorrect because it omits the contributions of collective modes of the order parameter. After this work was submitted, another work appeared[4] in which the Meissner effect was analyzed using a phenomenological hydrodynamic formalism. The calculation presented here amounts to a derivation of the formalism of Ref. 4 from a microscopic theory.

It is shown in this work that to ensure current conservation, one must, in general, include all collective modes in the linear-response kernel, but that in the heavy-fermion systems, pinning of the order parameter by lattice and spin-orbit effects combined with the large effective mass means that the only important collective mode is the phase mode (except possibly near $T=T_c$, as discussed below). The effects of this mode can be easily calculated, and shown to guarantee current conservation.

One may understand the necessity of a collective mode contribution to the current from the continuity equation, which in the static limit requires that the current be purely transverse. Now in isotropic s-wave superconductors one studies the Meissner effect by calculating the screening currents $\mathbf{j}$ induced by a vector potential $\mathbf{A}$. One may use any gauge. In a transverse gauge, isotropy ensures that $\mathbf{j}$ is also transverse. The screening currents may then be calculated by the naive linear response in which collective mode contributions are neglected. In a gauge with longitudinal components of $\mathbf{A}$, however, the naive method leads to a longitudinal term in $\mathbf{j}$. As Anderson[5] first showed, in this case the phase mode gives rise to a purely longitudinal current which cancels the longitudinal term in $\mathbf{j}$ given by the naive terms. In an anisotropic situation, even a transverse vector potential will give rise to a longitudinal term in the current (in the absence of collective mode terms). A "backflow" collective mode term must then be excited to cancel this unphysical longitudinal term; but in an anisotropic situation this collective mode term will not be purely longitudinal. Its transverse part will then contribute to the physical current.

We now proceed to derive equations for the linear response of an anisotropic superconductor coupled to a static magnetic field, which is specified by a vector potential $\mathbf{A}(\mathbf{r})$ which couples to the current density $\mathbf{j}(\mathbf{r})$ in the usual way via a perturbing Hamiltonian

$$H_p=\int d^3r\,\mathbf{A}(\mathbf{r})\cdot\mathbf{j}(\mathbf{r})\ . \tag{1}$$

We make the standard assumptions of weak coupling superconductivity, and also assume that the superconducting order parameter is uniform throughout the material. Whether the superconductor is spin-singlet or spin-triplet paired is irrelevant here; the treatment that follows applies to either case. Within these assumptions we can write an unperturbed Hamiltonian

$$H_0=\sum_{\mathbf{k},\alpha}\varepsilon(\mathbf{k})C^{\dagger}_{\mathbf{k}\alpha}C_{\mathbf{k}\alpha}+\sum_{\mathbf{k},\alpha,\beta}[\Delta^{\dagger}_{\alpha\beta}(\mathbf{k})C_{\mathbf{k}\beta}C_{-\mathbf{k}\alpha}+\text{H.c.}]\ . \tag{2}$$

Here α,β are pseudo-spin-indices (used instead of spin indices in the presence of spin-orbit scattering in a crystal[6]) and Δ is the gap matrix defined by

$$\Delta_{\alpha\beta}(\hat{\mathbf{k}},\mathbf{q})=\tfrac{1}{2}\sum_{\mathbf{p}}V(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})\langle C_{\mathbf{p}-1/2\mathbf{q},\alpha}C_{-\mathbf{p}-1/2\mathbf{q},\beta}\rangle\ . \tag{3}$$

In (3) $\hat{\mathbf{k}}$ is the relative coordinate for paired electrons ($\mathbf{k}$ is confined to the Fermi surface and so only its direction is relevant) and $\mathbf{q}$ is the center of mass coordinate.

Note that $\Delta_{\alpha\beta}(\hat{\mathbf{k}},\mathbf{q})=-\Delta_{\beta\alpha}(-\hat{\mathbf{k}},\mathbf{q})$. In equilibrium $\Delta_{\alpha\beta}(\hat{\mathbf{k}},\mathbf{q})=0$ unless $\mathbf{q}=0$; hence we write the unperturbed gap matrix as $\Delta_{\alpha\beta}(\hat{\mathbf{k}})$. $V(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})$ is the usual weak-coupling pairing potential, taken to be spin-independent to save writing. To study "pure" L-wave superconductivity in an isotropic Fermi liquid, one would take $V(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})=V_0P_L(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})$, with $P_L(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})=\sum_m [Y_L^{*m}(\hat{\mathbf{k}})]Y_L^m(\hat{\mathbf{p}})$. Here P_L is the order L Legendre polynomial and the Y_L^m are the spherical harmonics, which form a basis for the angular momentum L representation of the rotation group O(3). In the crystals studied here, the symmetry group is some crystal-symmetry group, which will have various representations, which we denote by L. In particular, we will refer to the case in which the pairing potential and the gap function are invariant under the full crystal-symmetry group as s-wave superconductivity. For the various representations we can define analogs of P_L and Y_L^m. We will not need the explicit forms of these functions; we will assume that we have the crystalline analog of pure L-wave superconductivity. The effects of mixing different Ls have been studied in the case of an isotropic Fermi liquid and found to be negligible.[7,8] In the crystalline analog of the pure L-wave case we have

$$\int dS_k \Delta_{\alpha\beta}(\hat{\mathbf{k}};\mathbf{q})V(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})=V_0\Delta_{\alpha\beta}(\hat{\mathbf{k}},\mathbf{q}) . \tag{4}$$

Here dS_k is the measure on the Fermi surface (i.e., up to a constant, the angle dependent density of states), and V_0 is the strength of the pairing potential.

We assume, as is common in studies of anisotropic superconductivity, that the unperturbed gap matrix is unitary in the sense that $\Delta^\dagger_{\alpha\beta}(\hat{\mathbf{k}})\Delta_{\beta\gamma}(\hat{\mathbf{k}})=\Delta^2(\hat{\mathbf{k}})\delta_{\alpha\gamma}$.[7] $\Delta^2(\hat{\mathbf{k}})$ is a number, the magnitude of the energy gap in direction $\hat{\mathbf{p}}$, and we denote the average of the gap over the Fermi surface Δ^2. Unitarity is not a trivial assumption: nonunitary states can have persistent currents in the ground state,[8] and hence may have a more complicated response to a magnetic field than the unitary states we consider here.

It is also convenient to define the Gorkov G and F functions via

$$G^{\alpha\beta}(\mathbf{k},\omega)=\frac{\delta_{\alpha\beta}(i\omega-\varepsilon(\mathbf{k}))}{\omega^2+\varepsilon(\mathbf{k})^2+\Delta^2(\hat{\mathbf{k}})} , \tag{5a}$$

$$F^{\alpha\beta}(\mathbf{k},\omega)=\frac{\Delta_{\alpha\beta}(\hat{\mathbf{k}})}{\omega^2+\varepsilon(\mathbf{k})^2+\Delta^2(\hat{\mathbf{k}})} . \tag{5b}$$

We now study the system via "self-consistent linear response," in which it is assumed that the perturbation (1) causes also a perturbation in the gap function, $\Delta_{\alpha\beta}(k)\rightarrow\Delta_{\alpha\beta}(k)+\Delta^{(1)}_{\alpha\beta}(k,q)$, and hence an additional term in the perturbing Hamiltonian,

$$H'_p=\sum_{k,q,\alpha,\beta}[\Delta^{+(1)}_{\alpha\beta}(\hat{\mathbf{k}},\mathbf{q})C_{k-1/2q,\beta}C_{-k-1/2q,\alpha}+\text{H.c.}] \tag{6}$$

Then one studies the response of the system to the combined perturbations (1) and (6), determines the variation in the gap function self-consistently, and finally computes the current. This is equivalent to solving diagrammatic equations for vertex functions in the ladder approximation.[6]

Straightforward calculations yield

$$\begin{aligned}\Delta^{(1)}_{\alpha\beta}(\hat{\mathbf{k}},\mathbf{q})=-\sum_{\substack{p,\omega\\ \gamma,\delta}} V(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})[\,&F^{\alpha\gamma}(\mathbf{p}-1/2\mathbf{q},-\omega)\Delta^{\dagger(1)}_{\gamma\delta}(\hat{\mathbf{p}},\mathbf{q})F^{\delta\beta}(\mathbf{p}-1/2\mathbf{q},\omega)\\ &+G^{\beta\gamma}(-\mathbf{p}-1/2\mathbf{q},-\omega)\Delta^{(1)}_{\gamma\delta}(\hat{\mathbf{p}},\mathbf{q})G^{\delta\alpha}(\mathbf{p}-1/2\mathbf{q},\omega)]\\ &+i\sum_{\substack{p,\omega\\ \gamma}} V(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})\mathbf{p}\cdot A(\mathbf{q})G^{\alpha\gamma}(\mathbf{p}-1/2\mathbf{q},-\omega)F^{\gamma\beta}(\mathbf{p}+1/2\mathbf{q},\omega) ,\end{aligned} \tag{7}$$

$$\begin{aligned}J(q)=&-i\sum_{\substack{p,\omega\\ \gamma,\delta}} V(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})\mathbf{p}[F^\dagger_{\alpha\gamma}(\mathbf{p}-1/2\mathbf{q},-\omega)\Delta^{(1)}_{\gamma\delta}(\hat{\mathbf{p}},\mathbf{q})G^{\delta\beta}(+\mathbf{p}-1/2\mathbf{q},\omega)-G^{\beta\gamma}(\mathbf{p}+1/2\mathbf{q},-\omega)\Delta^{\dagger(1)}_{\gamma\delta}(\hat{\mathbf{p}},\mathbf{q})F^{\delta\alpha}(\mathbf{p}-1/2\mathbf{q},\omega)]\\ &-\sum_p \mathbf{p}[\mathbf{p}\cdot A(\mathbf{q})][G^{\alpha\beta}(\mathbf{p}+1/2\mathbf{q},-\omega)G^{\beta\alpha}(\mathbf{p}-1/2\mathbf{q},\omega)+F^{\dagger\alpha\beta}(\mathbf{p}+1/2\mathbf{q},\omega)F^{\beta\gamma}(\mathbf{p}-1/2\mathbf{q},\omega)] .\end{aligned} \tag{8}$$

Here ω is a Matsubara frequency and the equation for $\Delta^\dagger$ is the complex conjugate of that for Δ.

Consider first the self-consistent equations for $\Delta^\dagger$ and Δ. They are complicated; their full solution requires specification of all collective modes of the system, an impossible task without further assumptions. We shall assume that the order parameter is rigidly pinned by crystal-field and spin-orbit effects, so that the only relevant mode is the phase mode, for which

$$\Delta^{(1)}_{\alpha\beta}(k,r)=(e^{i\phi(r)}-1)\Delta_{\alpha\beta}(k) . \tag{9}$$

This is reasonable because gapless modes arise from the breaking of a continuous symmetry; in a crystal there is no continuous rotation symmetry for the spatial degrees of freedom and spin-orbit effects lock the spin degrees of freedom to the spatial degrees of freedom; thus the phase mode is the only gapless mode left. Note that by construction, $\mathbf{q}\cdot\mathbf{j}=0$; also, we compute only the reactive part of the response kernel. Thus plasmon effects[5] do not enter; the phase mode is gapless for our purposes. In the non-s-wave case (in contrast to the s-wave case), nonphase modes may contribute to the current, however, these other

modes will have finite energy gaps E_g, where $E_g \propto \Delta$, and so will contribute to the current via terms of order qv_F/E_g, where q is the wave vector of the perturbation and v_F the Fermi velocity. These terms will be negligible in the limit $qv_F < \Delta$. This limit, however, may be the relevant limit for heavy fermions, because the standard result for penetration depth, λ, may be written[7,9] as $\lambda = c(m^*/m)^{1/2}\Delta^{-1}$, where c is a constant and m^*/m is the ratio of effective mass to free-electron mass. Thus, as $q \propto \lambda^{-1}$ and $v_F \propto (1/m^*)$, the parameter qv_F/Δ is $(m^*/m)^{3/2}$ times smaller than in a free-electron system with the same number of electrons. For $m^*/m = 100$ this ensures that $qv_F/\Delta \ll 1$.

This approximation may break down as $T \to T_c$ if the pinning energies go to zero faster than Δ does, because q is proportional to the inverse penetration depth, which is of order Δ. This occurs for those triplet phases which are listed by Blount[5] as transforming according to a two-dimensional representation of the crystal group. In these cases, the discussion given below does not apply as $T \to T_c$. This point is discussed further in the Appendix.

Applying the rigid pinning approximation to the above equations, and using the gap equation and unitarity, one obtains [here $\phi(q)$ is the Fourier transform of the spatially varying part of the overall phase of the superconductor introduced in Eq. (9)]

$$\phi(q) = \frac{-m}{q} \frac{\int dS_p R(\hat{\mathbf{p}})(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})\hat{\mathbf{p}}\cdot\mathbf{A}(\mathbf{q})}{\int dS_p R(\hat{\mathbf{p}})(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})^2} , \tag{10}$$

where

$$R(\hat{\mathbf{p}}) = \sum_{\omega} \frac{\Delta^2(\hat{\mathbf{p}})}{[\omega^2+\Delta^2(\hat{\mathbf{p}})]^{1/2}[\omega^2+\Delta^2(\hat{\mathbf{p}})+1/4v_F(\hat{\mathbf{p}}\cdot\mathbf{q})^2]} . \tag{11}$$

We may now set $q=0$ in (10) and (11) because even at $T=0$ the fact that Δ may vanish along a line on the Fermi surface leads only to a very weak singularity of the form $x^2\log(x)$ $(x = v_F q/\Delta \ll 1)$, and thus to corrections which are negligible. Note that Eq. (10) is valid in any gauge. The textbook procedure for finding the response to a magnetic field involves finding a gauge in which $\phi(q)$ vanishes. As can be seen from (10), this gauge will, in general, not be easy to specify, nor will the Maxwell equation relating current and vector potential in this gauge be simple. If instead one works in transverse gauge, $\mathbf{q}\cdot\mathbf{A}(\mathbf{q})=0$, one finds that $\phi(q)=0$ for a general direction of $\mathbf{A}$ only if the crystal is cubic, and one is either at $T=0$ or one has s-wave superconductivity.

Substituting (10) and the rigid pinning condition into (8) yields

$$\mathbf{J}(\mathbf{q}) = \int dS_p (R(\hat{\mathbf{p}})\hat{\mathbf{p}}[\hat{\mathbf{p}}\cdot\mathbf{A}(\mathbf{q}) + \hat{\mathbf{p}}\cdot\hat{\mathbf{q}}(q/m)\phi(q)] . \tag{12}$$

From (11) and (12) one sees that $\mathbf{q}\cdot\mathbf{j}(\mathbf{q})=0$, as is required by current conservation. Without the contributions from the phase mode, this would not in general be the case.

More interestingly, it follows from (12) that $\mathbf{j}$ is not necessarily parallel to $\mathbf{A}$: the penetration depth, and indeed the full magnetic response kernel, are anisotropic. In a noncubic crystal this is to be expected; in a material of cubic symmetry, such as UBe_{13}, anisotropy of the penetration depth would imply non-s-wave superconductivity. In a cubic crystal the anisotropy of penetration depth is driven by the anisotropy in $R(p)$. This is maximal at $T \simeq T_c$ and goes to zero as $T \to 0$.

The magnetostatic problem posed by Eq. (12) and the relevant Maxwell equation $\nabla\cdot\mathbf{B}=0$ is very difficult to solve in an arbitrary geometry. For example, the fields produced by the screening currents in a sphere of anisotropic superconductor in a uniform field would not have the perfect dipole form obtained for an s-wave superconductor. In the conventional plane-parallel Meissner effect geometry, in which the superconductor occupies the half-space $z<0$ and one takes a field in the x-y plane, which may vary in the z direction (and which approaches a constant as $z \to \infty$), one discovers that for $z>0$ the magnetic field is constant in space, but for $z<0$ we may choose x and y axes so that Eq. (12) may be written as

$$j_i(z) = \lambda_i^2 A_i(z) \quad (i = x,y) , \tag{13}$$

where λ_x and λ_y may easily be computed. For a non-s-wave superconductor λ_x will not, in general, equal λ_y. The field will then change direction as it attenuates in the crystal. In Fig. 1 we have plotted the ratio, λ_x/λ_y as a function of temperature, for a polar[8] superconductor oriented so that $\Delta^2(\hat{\mathbf{p}}) = \Delta^2\hat{\mathbf{p}}_x^2$. In this case one discovers $\phi(q)=0$, so we need only the first term in (12). For simplicity we have assumed that $\Delta = T_c/(1-T/T_c)^{1/2}$, and have used a spherical Fermi surface in performing the angle integrals.

Note, however, that although anisotropy in the penetration depth is in general expected for non-s-wave superconductors, there exist many special orientations of the order parameter with respect to material surfaces for which $\mathbf{j}$ is parallel to $\mathbf{A}$ and for which the penetration depth is isotropic over the relevant range of orientations of $\mathbf{A}$. One possibly important example is a superconductor in an axial state[8] with axis oriented parallel to $\mathbf{q}$. Then one has ro-

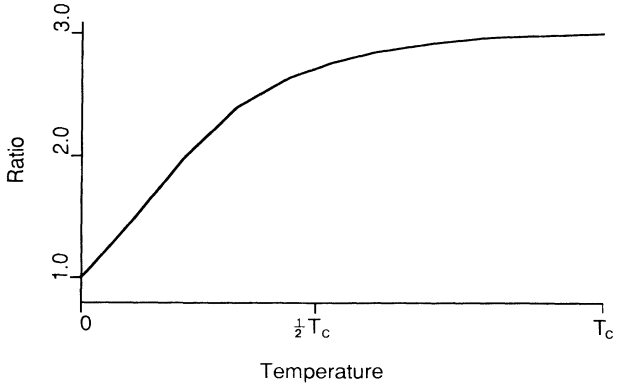

FIG. 1. Plot of the ratio of maximum to minimum London penetration depth as a function of temperature for a superconductor in a polar state in the geometry discussed in the text.

tational invariance in the plane perpendicular to q. But it has been argued[10] that an axial superconductor will orient itself so that its axis is normal to an interface, while in a Meissner effect experiment the field varies in the direction normal to the interface and is directed parallel to the interface. Thus in this situation $\lambda_x = \lambda_y$.

In conclusion, this paper presents formulas for the current induced in a superconductor by a magnetic field. The formalism includes collective mode contributions which are necessary to ensure current conservation. The penetration depth may be easily calculated from these formulas.

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APPENDIX

We consider solving Eqs. (7) and (8) near $T = T_c$. In this limit the equations reduce to the equations one would obtain by solving the Landau-Ginzburg equations in a weak magnetic field by assuming a solution of the form $\Delta = \Delta^{(0)} + \Delta^{(1)}$ (where $\Delta^{(0)}$ is the solution to the Landau-Ginzburg equations in zero field) and working to linear order in $\Delta^{(1)}$ and $\mathbf{A}$. The equations are notationally complicated: one should expand $\Delta^{(1)}$ in terms of the Y_m^L functions discussed in the text, with the coefficients functions of q; one then obtains a matrix equation for the vector whose components are these coefficients. One may write the resulting equations schematically as

$$q^2 F \Delta^{(1)} + G \Delta^{(1)} = qH(\mathbf{A}) . \tag{A1}$$

Here H is a vector (in the space of Y_m^L depending linearly on A and arising from the derivative terms in the Landau-Ginzburg equation. F and G are matrices acting on the vector $\Delta^{(1)}$. F comes from the derivative terms and G from the nonderivative terms. G contains the pinning forces which force the gap to have a particular angular form and a particular orientation with respect to the crystalline axes. It is in general of order $(T - T_c)$. One then obtains $\Delta^{(1)} = (q^2 F + G)^{-1} q H(\mathbf{A})$. The discussion given in the text shows that this is small for heavy fermions. However, if $\Delta^{(1)}$ is merely a phase change, then $G\Delta^{(1)}$ must vanish, because the free energy is invariant under phase changes. In this case, the one treated in the text, $\Delta^{(1)} \propto (1/q) F^{-1} H(\mathbf{A})$, which gives a nonnegligible contribution to the current.

In general, the phase mode is the only soft mode in a crystal with spin-orbit scattering. However, for certain of the triplet phases enumerated by Blount[5] the continuous rotation symmetry is only broken in the sixth-order term of the free-energy functional. Then as $T \to T_c$ we would have $G \propto (T - T_c)^{3/2}$ while $q^2 \propto (T - T_c)$.[1] Then the estimate given in the text does not apply, and the nearly soft modes may contribute to the response kernel.

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