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Linear and nonlinear electrical conduction in quasi-two-dimensional quantum wells

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The dc electrical transport parallel to the walls of a quasi-two-dimensional quantum well, with a magnetic field $\mathbf{B}=B\mathbf{z}$ applied normal to its barriers, is considered. The influence of scattering by optical phonons on the dc current is investigated at high temperatures and strong electric fields (nonlinear transport). Certain values of the electric field induce transitions of the carriers between neighboring Landau levels and the usual magnetophonon maxima, reported previously, convert into minima and vice versa. This behavior of the magnetophonon extrema has been recently observed in n^+-n^--n^+ structures. For scattering by impurities at very low temperatures and weak electric fields (linear transport) the dc conductivity σ_{xx} oscillates with period $(\varepsilon_F-\varepsilon_0 n^2)/\hbar\omega_0$, where ε_F is the Fermi level, n denotes energy-level quantization in the z direction, and ε_0 is the energy of the lowest level. For strong electric fields, transitions between neighboring Landau levels can occur, leading to an additional oscillatory structure. The Hall conductivity σ_{yx} is evaluated. The possibility of transitions between the levels n in wide wells is also investigated. The dependence of the conductivities (or currents), inverse scattering rates, and level widths on the magnetic field, the thickness of the well, and the temperature is shown explicitly.

I. INTRODUCTION

Recently, there has been considerable interest in understanding the nonlinear behavior of hot electrons in two-dimensional or quasi-two-dimensional systems.[1] Unusual effects, such as the breakdown of the integral quantum Hall effect[2] or a new type of conduction mechanism in n^+-n^--n^+ GaAs structures,[3] when the relevant currents or electric fields exceed certain values, have been reported. The results of Ref. 3 indicate that, for submicron devices, the conductivity σ_{xx} depends on the layer thickness.

In a previous paper,[4] following earlier investigations of the low-field (electric) dc transport in quantum wells[5–7] in the absence of a magnetic field, we studied magnetophonon oscillations in quantum wells for all types of phonons. The same approach has been used in Ref. 8 but with less explicit results. For optical phonons the well-known resonances, occuring when $\omega_L=P\omega_0$, where P is an integer and ω_L and ω_0 are the phonon and klystron frequencies, respectively, were readily obtained. Also, an approximate treatment of the elastic scattering by acoustical or piezoelectrical phonons at low temperatures led to expectation of resonances, in very pure samples, when $P\hbar\omega_0=\varepsilon_F-\varepsilon_0$, where ε_F and ε_0 are the Fermi level and lowest subband energy, respectively. The conductivity σ_{xx} and the inverse scattering rates were shown to depend on the thickness of the well.

The study, in this paper, of magnetophonon resonances in a quantum well, at strong electric fields is motivated by the unusual results of Ref. 3. After the presentation of the formalism in Sec. II, it will be shown in Sec. III that certain values of the electric field can induce transitions between neighboring Landau levels and this is connected with the conversion of the low-field magnetophonon maxima into minima and vice versa. Only optical and polar optical phonons are considered. Moreover, we are not aware of any treatment of the influence of impurity scattering in a quantum well, when a magnetic field is present. This is taken up in Sec. IV both for weak and strong electric fields, corresponding to linear transport and nonlinear transport, respectively. In the former case, it will be shown that the conductivity σ_{xx}, parallel to the walls of the well, oscillates, at very low temperatures, with period $(\varepsilon_F-\varepsilon_n)/\hbar\omega_0$, where ε_n is the highest occupied subband in the well ($\varepsilon_n=\varepsilon_0 n^2$).

The rest of the paper is organized as follows. In Sec. V, the Hall conductivity σ_{yx} is evaluated and is shown to depend on the thickness of the well. Conclusions follow in Sec. VI. Appendix A contains certain formulas necessary for the calculations. The inverse scattering rates (or level widths) are given in Appendix B. Finally, Appendix C contains an outline of the derivation of the formulas used for the conductivity or current density.

II. THE FORMALISM

A. Basic expressions

We consider a many-body system described by the Hamiltonian

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$$H=H^0+\lambda V-\mathbf{A}\cdot\mathbf{F}(t)\ . \tag{2.1}$$

H^0 is the largest part of H which can be diagonalized (analytically), λV is a binary-type interaction, assumed nondiagonal and small compared to H^0, and $-\mathbf{A}\cdot\mathbf{F}(t)$ is the external field Hamiltonian with $\mathbf{A}$ being an operator and $\mathbf{F}(t)$ a generalized force.

1. Linear transport

The average value of the current density operator J at time t, denoted by $\langle \cdots \rangle_t$, is given by

$$\langle J\rangle_t=\mathrm{T}[\rho(t)J]\ , \tag{2.2}$$

where $\rho(t)$ is the density operator associated with (2.1). For linear responses, i.e., small electric fields $\mathbf{E}(t)$, and within the Born approximation the current density associated with the diagonal part of $\rho(t)$ (in the representation of H^0) is given by[9]

$$\langle (J_\mu)_d\rangle_t=\frac{q}{\Omega}\sum_\zeta(-\mathscr{B}_\zeta\langle n_\zeta\rangle_t\alpha_{\mu\zeta}+\langle n_\zeta\rangle_t\dot{\alpha}_{\mu\zeta}),\quad \mu=x,y,z\ , \tag{2.3}$$

where Ω is the volume, q is the charge of the carriers (fermions), $\mathbf{F}(t)=q\mathbf{E}(t)$, and where $\mathbf{A}=\sum_i(\mathbf{r}_i-\langle\mathbf{r}_i\rangle_{\rm eq})=\sum_i\boldsymbol{\alpha}_i$, with $\langle\mathbf{r}_i\rangle_{\rm eq},\mathbf{r}_i$, being the positions of the ith carrier before and after the application of the electric field, respectively. Furthermore, $\alpha_{\mu\zeta}=(\zeta\,|\,\alpha_\mu\,|\,\zeta)$, $|\,\zeta)$ is the one-particle eigenstate of h^0 ($H^0=\sum h^0$) with eigenvalue ε_ζ and average occupancy $\langle n_\zeta\rangle_t$; $\mathscr{B}_\zeta\langle n_\zeta\rangle_t$ is the collision integral of the *quantum* Boltzmann equation[9] and d stands for diagonal. The second term of Eq. (2.3) is the usual ponderomotive current. The first term represents the many-body contribution of collisions to the current and has been termed "collisional current;" in a semiclassical treatment, this term is absent.

When we have only collisional current, the dc conductivity component $\sigma^d_{\mu\mu}(0)$ reads [cf. Ref. 9, Eq. (2.84)]

$$\sigma^d_{\mu\mu}(0)=\frac{\beta q^2}{2\Omega}\sum_{\zeta,\zeta',\mathrm{spin}}\langle n_\zeta\rangle_{\rm eq}(1-\langle n_{\zeta'}\rangle_{\rm eq})\times w_{\zeta\zeta'}(\alpha_{\mu\zeta}-\alpha_{\mu\zeta'})^2\ , \tag{2.4}$$

where $\beta=1/k_BT$ with k_B being Boltzmann's constant, and T the temperature; $w_{\zeta\zeta'}$ is the binary transition rate, given by the "golden rule," and $\langle n_\zeta\rangle_{\rm eq}$ is the Fermi-Dirac distribution function. Equation (2.4) is valid for both elastic and inelastic scattering. The component $\sigma^d_{\mu\nu}(0)$ is given by (2.4) with $\frac{1}{2}(\alpha_{\mu\zeta}-\alpha_{\mu\zeta'})^2$ replaced by $(\alpha_{\nu\zeta}-\alpha_{\nu\zeta'})\alpha_{\mu\zeta'}$.

The nondiagonal (nd) part of $\rho(t)$ leads to the following dc conductivity formula [cf. Ref. 9, Eq. (3.21)]

$$\sigma^{nd}_{\mu\nu}(0)=\Omega\hbar i\sum_{\zeta',\zeta''}{}'\langle n_{\zeta'}\rangle(1-\langle n_{\zeta''}\rangle_{\rm eq})(\zeta'\,|\,j_\nu\,|\,\zeta'')(\zeta''\,|\,j_\mu\,|\,\zeta')(1-e^{\beta(\varepsilon_{\zeta''}-\varepsilon_{\zeta'})})/(\varepsilon_{\zeta''}-\varepsilon_{\zeta'})^2\ , \tag{2.5}$$

where $j=q\dot{\alpha}/\Omega$ is the one-particle current operator. The prime on $\sum$ means $\zeta'\neq\zeta''$. The main feature of Eq. (2.5) is that it is independent of the nondiagonal part of the interaction, i.e., of λV; the diagonal part of the interaction, if any, will simply shift the energies ε_ζ appearing in Eq. (2.5). The total conductivity $\sigma_{\mu\nu}$ is given by $\sigma_{\mu\nu}=\sigma^d_{\mu\nu}+\sigma^{nd}_{\mu\nu}$.

The above formulas are very general and not tied to a $\mathbf{k}$ space description. So far they have been applied to situations as diverse as mobility in metal-oxide-semiconductor field-effect transistors (MOSFET's),[10] conduction through localized states in amorphous materials,[11] integral quantum Hall effect,[12] quantum wells,[4] etc. An outline of the derivation of these formulas, in Ref. 9, is given in Appendix C.

When electrons interact with phonons (assumed to remain at equilibrium) the transition rate $w_{\zeta\zeta'}$ is given by

$$w_{\zeta\zeta'}=\sum_{\mathbf{q}}[Q^+\langle n_{\mathbf{q}}\rangle_{\rm eq}+Q^-(1+\langle n_{\mathbf{q}}\rangle_{\rm eq}]\ , \tag{2.6}$$

where

$$Q^\pm=\frac{2\pi}{\hbar}\,|\,F(\mathbf{q})\,|^2\,|\,(\zeta'\,|\,e^{\pm i\mathbf{q}\cdot\mathbf{r}}\,|\,\zeta)\,|^2\delta(\varepsilon_\zeta-\varepsilon_{\zeta'}\pm E_{\mathbf{q}})\ . \tag{2.7}$$

Q^+ and Q^- correspond to absorption and emission of a phonon with wave vector $\mathbf{q}$, and energy $E_{\mathbf{q}}$, respectively. $\langle N_{\mathbf{q}}\rangle_{\rm eq}$ is the equilibrium number of phonons and $F(\mathbf{q})$ is the Fourier transform of the electron-phonon interaction (λV). We consider only longitudinal phonons in the deformation potential model.

When electrons interact with randomly distributed impurities (assumed to remain at equilibrium) the transition rate is given by

$$w_{\zeta\zeta'}=\frac{2\pi}{\hbar}\frac{N_I}{\Omega}\sum_{\zeta',\mathbf{q}}|\,U(\mathbf{q})\,|^2\,|\,(\zeta'\,|\,e^{i\mathbf{q}\cdot\mathbf{r}}\,|\,\zeta)\,|^2\delta(\varepsilon_\zeta-\varepsilon_{\zeta'})\ , \tag{2.8}$$

where N_I is the impurity concentration and $U(\mathbf{q})$ the Fourier transform of the impurity potential $U(\mathbf{r}-\mathbf{R})$; $\mathbf{r}$ and $\mathbf{R}$ are the positions of the electron and the impurity, respectively.

The application of the above formulas is a rather standard procedure provided that Born approximation and linear response theory are valid. Usually, the Born approximation is assumed to hold. The validity of linear response theory, however, i.e., the values of the electric field E_l for which the theory is valid, depends on the particular problem. In general, one assumes or proves under general conditions[13] that the potential energy eE_ll is much smaller than k_BT, where l is some characteristic length, e.g., the mean free path. For transport in strong magnetic fields and transitions between Landau levels, which is our case, the Born approximation applies[12] and l is the magnetic length $l=(\hbar/m^*\omega_0)^{1/2}$, ω_0 is the cyclotron frequency, and m^* is the effective mass. For T, m^*, and ω_0 given the condition $eE_ll\ll k_BT$ specifies the values E_l for which the linear response formulas (2.2)–(2.5) are valid.

Above E_l starts the hot carrier regime which is discussed below.

2. Nonlinear transport

For values of E above E_l the response of the system, described by (2.1), is no longer linear, i.e., it is qualitatively different: for $E>E_l$ the quantum states of the system are modified and the transport coefficients, in general, cannot be expressed as correlations of equilibrium quantities[9] leading to the simple expressions (2.3)–(2.5). As examples, we may mention phonon drag effects,[14] the breakdown of the integral quantum Hall effect,[2] etc. The problem of evaluating the conductivity tensor becomes much more complicated since the scattering system (phonons or impurities) may no longer be at equilibrium and the distribution functions are different from their equilibrium values. In general, it is assumed that the scattering system remains at equilibrium so that one does not have to solve coupled equations for, e.g., the distribution functions of electrons and of scatters.[14,15] In this case the evaluation of the current is still possible and, to all orders in the electric field, the following result obtains for the dc current density:[15–17]

$$\langle J_\mu \rangle = (q/2\Omega)\sum_{\zeta,\zeta'}(\alpha_{\mu\zeta'}-\alpha_{\mu\zeta})[f_\zeta(1-f_{\zeta'})w_{\zeta\zeta'} - f_{\zeta'}(1-f_\zeta)w_{\zeta'\zeta}]\ , \quad (2.9)$$

where the distribution function $\langle n_\zeta \rangle \equiv f_\zeta$ is determined by

$$\sum_{\zeta'} f_\zeta(1-f_{\zeta'})w_{\zeta\zeta'} - f_{\zeta'}(1-f_\zeta)w_{\zeta'\zeta} = 0\ . \quad (2.10)$$

For linear conduction in crossed electric and magnetic fields, Eq. (2.9) reduces[15,16] to $\langle J_\mu \rangle = \sigma_{\mu\mu}(0)E$, with $\sigma_{\mu\mu}(0)$ given by Eq. (2.4). This is also true for the present case [c.f. Eqs. (2.12)–(2.14)]. The main difference of Eqs. (2.4) and (2.9) is that in the latter the quantities $\alpha_{\mu\zeta}$, f_ζ, and $w_{\zeta\zeta'}$ depend on the electric field E, whereas in the former they do not. For more details concerning the derivation of Eq. (2.9) see Appendix C.

For nondegenerate statistics, i.e., for high temperatures, $1-f_\zeta \approx 1-f_{\zeta'} \approx 1$ and Eq. (2.9) takes the simpler form ($\langle J_\mu \rangle \equiv J_\mu$)

$$J_\mu = (q/2\Omega)\sum_{\zeta,\zeta'}(\alpha_{\mu\zeta'}-\alpha_{\mu\zeta})[f_\zeta w_{\zeta\zeta'} - f_{\zeta'} w_{\zeta'\zeta}]\ . \quad (2.11)$$

Strictly speaking, Eqs. (2.9) and (2.11) are valid when the scatterers are at equilibrium. As an approximation, however, they could be used for the case when the scatterers are not at equilibrium, i.e., for not too strong electric fields.

Equations (2.4)–(2.11) will be used to evaluate the conductivity (or the current density) in a quantum well. The necessary one-electron attributes are given below.

B. Quantum well in crossed electric and magnetic fields

We consider a quantum well with a magnetic field B applied perpendicular to its barriers (z direction). The distance between the barriers, assumed infinitely high, is L_z and an electric field is applied in the x direction. In the Landau gauge the one-electron Hamiltonian, states and eigenvalues read[18,4]

$$h^0 = (\mathbf{p}+q\mathbf{A})^2/2m^* + eEx, \quad \mathbf{A}=(0,Bx,0) \quad (2.12)$$

$$(\mathbf{r}\,|\,\zeta) = (2/L_yL_z)^{1/2}\phi_N(x-x_0) \times e^{ik_y y}\sin(n\pi z/L_z),\quad n=1,2,3,\ldots, \quad (2.13)$$

$$\varepsilon_\zeta \equiv \varepsilon_{N,n,k_y} = (N+\tfrac{1}{2})\hbar\omega_0 + n^2\varepsilon_0 - \hbar V_d k_y + m^* V_d^2/2,\quad N=0,1,2,\ldots, \quad (2.14)$$

where $V_d = E/B$ is the drift velocity, $\varepsilon_0 = \hbar^2\pi^2/2m^* L_z^2$), $-x_0 = l^2(k_y + eE/\hbar\omega_0)$, and where ϕ_N represents harmonic oscillator wave functions. N is the Landau level index and n denotes quantization of the energy spectrum in the z direction. We have assumed a spherical effective mass m^* but the results hold for $m_\perp^* \neq m_z^*$ as well. The dimensions of the sample are L_x, L_y, L_z; k_y is the wave vector in the y direction and $\mathbf{A}$ is the vector potential. The last two terms of Eq. (2.14) represent the potential and kinetic energy of the electrons in the electric field. In the absence of this field these terms are zero as well as the second terms of h^0 and x_0. Thus, the effect of including the electric field in h^0 is to lift the k_y degeneracy of the energy spectrum and to shift the center position of the orbits by $eEl^2/\hbar\omega_0$.

For the calculations of this paper we need the following matrix elements[18] in the representation (2.13):

$$(\zeta\,|\,x\,|\,\zeta') = -l^2(k_y - eE/\hbar\omega_0)\delta_{NN'}\delta_{k,k'} + (l/\sqrt{2})(\sqrt{N+1}\,\delta_{N',N+1} + \sqrt{N}\,\delta_{N',N-1})\delta_{k,k'}\ , \quad (2.15)$$

$$(\zeta\,|\,\dot{\alpha}_x\,|\,\zeta') = i(l\omega_0/\sqrt{2})(-\sqrt{N+1}\,\delta_{N',N+1} + \sqrt{N}\,\delta_{N',N-1})\delta_{k,k'}\ , \quad (2.16)$$

$$(\zeta\,|\,\dot{\alpha}_y\,|\,\zeta') = (l\omega_0/\sqrt{2})[(\sqrt{N+1}\,\delta_{N',N+1} + \sqrt{N}\,\delta_{N',N-1}) - (eE/m^*\omega_0)\delta_{NN'}]\delta_{kk'}\ , \quad (2.17)$$

where $\delta_{kk'} = \delta_{nn'}\delta_{k_y k'_y}$. Furthermore,

$$|(\zeta\,|\,e^{\pm i\mathbf{q}\cdot\mathbf{r}}\,|\,\zeta')|^2 = |F_{nn'}(\pm q_z)|^2 \times |J_{NN'}(u)|^2 \delta_{k_y,k'_y \pm q_y}\ , \quad (2.18)$$

$$F_{nn'}(\pm q_z) = (2/L_z)\int e^{\pm i q_z z}\sin(n\pi z/L_z) \times \sin(n'\pi z/L_z)dz\ , \quad (2.19)$$

$$|J_{NN'}(u)|^2 = (N'!/N!)e^{-u}u^{N'-N}[L_N^{N'-N}(u)]^2\ , \quad (2.20)$$

$$\int_{-\infty}^{\infty}|F_{nn'}(\pm q_z)|^2 dq_z = (\pi/L_z)(2+\delta_{nn'})\ , \quad (2.21)$$

where $u = l^2(q_x^2+q_y^2)/2$ and where $L_N^M(u)$ is a Laguerre polynomial. For details concerning the derivation of Eqs. (2.18)–(2.21) see Ref. 4, Sec. II B and references cited therein. Interestingly, the matrix element (2.18) is independent of the electric field. We also note that for the linear case E is not included in h^0 and the last term of Eq. (2.17) is zero.

We can now proceed to the evaluation of the conductivity components σ_{xx} and σ_{yy} for linear transport or of the current density for nonlinear transport. For linear transport, Eqs. (2.16) and (2.17) show that the ponderomotive current [second term of Eq. (2.3)] is zero but the "collisional" current is not. Thus as far as diagonal contributions to σ_{xx} are concerned we have to evaluate σ_{xx}^{d} using Eq. (2.4). The nondiagonal contribution σ_{xx}^{nd}, given by Eq. (2.5) for $\mu=\nu=x$, can be shown to vanish identically for the states (2.13) by a procedure identical with that of Ref. 12, Sec. II. Concerning the component σ_{yx}^{d} we remark that [cf. Eq. (2.4)] it vanishes because $\alpha_{\mu\zeta'}$ is zero[9] for $\mu=y$. Using Eqs. (2.16) and (2.17) we can also show that the ponderomotive contribution σ_{yx}^{d} vanishes identically [cf. Ref. 9, Eq. (2.55)]. We are thus left with σ_{yx}^{nd} as given by Eq. (2.5). This component, which in the first Born approximation does not depend explicitly on the interaction, will be evaluated in the last section. For nonlinear transport we will evaluate the current density using Eqs. (2.10) and (2.11).

III. SCATTERING BY OPTICAL PHONONS: NONLINEAR TRANSPORT

In the case of linear transport, the electron-LO phonon interaction in a quantum well has been investigated previously both in the absence[5-7] and in the presence of a magnetic field.[4,8] Following this work we assume that the vibrational spectrum of the quasi-two-dimensional system is identical with that in a bulk material, i.e., that the LO phonons, to a first approximation, are not affected by the presence of the quantum well. Deviations from this bulk behavior, such as interface[19] modes or slabmodes[20] are neglected. We also assume that the phonons are dispersionless, i.e., $E_g=\hbar\omega_L\approx$constant, where ω_L is the phonon frequency.

Since optical phonons are important at high temperatures, we use Eq. (2.11) in which we substitute Eqs. (2.6), (2.14), and (2.18)–(2.21). Since we are considering a uniform system, we take

$$f_\zeta=f_{N,n}=\exp[\beta_e(\varepsilon_F-\varepsilon_{N,n})] ,$$

which is a spatially uniform (k_y independent) solution of Eq. (2.10); $\varepsilon_{N,n}\equiv\varepsilon_{N,n,0}=(N+\frac{1}{2})\hbar\omega_0+n^2\varepsilon_0$ and $\beta_e=1/k_BT_e$ with T_e being the electron temperature.[21] The sum over k_y, performed with the use of periodic boundary conditions and the constraint $0\leq l^2(k_y+eE/\hbar\omega_0)\leq L_x$, gives a factor $A_0/2\pi l^2$. To get tractable integrals over q_z we neglect (only for polar optical phonons) the q_z dependence of the factor $|F(\mathbf{q})|^2$. Instead of the phonon equilibrium distribution function $N_0=[\exp(\beta\hbar\omega_L)-1]^{-1}$ we use the approximate nonequilibrium distribution

$$N_{\mathbf{q}}\approx N_0[1+\mathbf{q}\cdot\mathbf{V}d/(\hbar\omega_L-\mathbf{q}\cdot\mathbf{V}d)]\equiv\overline{N}_0$$

for nonlinear transport in strong magnetic fields.[14] It is expected that this distribution describes the scattering system (phonons) better than N_0. Further, we set $N'-N=-M$ in the emission term and $N'-N=+M$ in the absorption term. Then, the first term of Eq. (2.11), J_x^1, proportional to $w_{\zeta\zeta'}$, gives

$$J_x^1\approx(e/2L_z\hbar)\sum_{N,N',n,n'}e^{\beta_e(\varepsilon_F-\varepsilon_{N,n})}(2+\delta_{nn'})\times\sum_{q_\perp}|F(\mathbf{q})|^2q_y\,|J_{NN'}(u)|^2[\overline{N}_0\delta(-M\hbar\omega_0+\Delta_{nn'}+\hbar V_dq_y+\hbar\omega_L)-(1+\overline{N}_0)\delta(M\hbar\omega_0+\Delta_{nn'}-\hbar V_dq_y-\hbar\omega_L)] , \quad (3.1)$$

where $\Delta_{nn'}=(n^2-(n')^2)\varepsilon_0$. The integral over q_y can be done immediately, but the resulting integral over q_x must be done separately for each N and N' and is very difficult to evaluate analytically. To simplify the calculations, we replace q_y, in the argument of the δ function, in $\overline{N}_0$, and in front of $|J_{NN'}(u)|^2$, by $eB\,\Delta\bar{x}/\hbar$, where $\Delta\bar{x}$ is a constant, of the order of l, which will be specified later. This approximation is equivalent to assuming an effective phonon momentum: $eV_dq_y\approx eE\Delta\bar{x}$. The sum over $q_\perp$ is transformed into an integral using polar coordinates. Equation (3.1) takes the form

$$J_x^1\approx C\sum_{N,N',n}\Delta\bar{x}\left[e^{\beta_e(\varepsilon_F-\varepsilon_{N,n})}\int|J_{NN'}(u)|^2\,|F(\mathbf{q})|^2du\right]\{\overline{N}_0\delta(-M\hbar\omega_0+eE\,\Delta\bar{x}+\hbar\omega_L)-(1+\overline{N}_0)\delta(M\hbar\omega_0-eE\,\Delta\bar{x}-\hbar\omega_L)\}+2C\sum_{N,N',n}\Delta\bar{x}[\cdots]\sum_{n'}\{+\Delta_{nn'}\} , \quad (3.2)$$

where $[\cdots]$ is the same as in the first line and where the notation $\{+\Delta nn'\}$ indicates the same quantity as in $\{\cdots\}$ with $\Delta_{nn'}$ added to the argument of the δ functions. The constant C is equal to $Be^2A_0/2\pi l^2L_z\hbar$. The other term of Eq. (2.11), J_x^2, proportional to $w_{\zeta'\zeta}$, is given by Eq. (3.2) as is easily shown by interchanging ζ with ζ' in Eq. (2.11), since neither $(\alpha_{\mu\zeta'}-\alpha_{\mu\zeta})$ nor $|(\zeta|\exp(\pm i\mathbf{q}\cdot\mathbf{r})|\zeta')|^2$ depend on the electric field, cf. Eq. (2.18), and since f_ζ is an even function of the electric field[21] [$T_e=f(E^2)$].

A. Optical phonons

As usual, we take

$$|F(\mathbf{q})|^2=\hbar^2D^2/2\Omega\rho\hbar\omega_L)=D'/\Omega , \quad (3.3)$$

where ρ is the density of the material and where D is a constant.

1. *Narrow wells*

We assume that the thickness L_z of the well is so small that no transitions between the levels ε_n ($\varepsilon_n=\varepsilon_0 n^2$) can take place by, e.g., varying the magnetic or electric field. In this case $\Delta_{nn'}=0$ and the second term of Eq. (3.2) is twice the first term. The integrals over u are given by (A1) and (A2). If several levels ε_n are occupied, Eq. (3.2) leads to ($J_x=J_x^1+J_x^2$)

$$J_x \approx 3C'D' \sum_{N,M,n} \Delta\bar{x} e^{\beta_e(\varepsilon_F-\varepsilon_{N,n})}[\bar{N}_0-(1+\bar{N}_0)(N-M)!/N!]\delta(M-\bar{\omega}_L/\omega_0) , \tag{3.4}$$

where $C'=2C/\hbar\omega_0\Omega$ and $\bar{\omega}_L=\omega_L+eE\Delta\bar{x}/\hbar$. Comparing Eq. (3.4) with Eq. (3.10) of Ref. 4, valid for the linear case, we see that the approximation $\hbar V_d q_y \approx eE\Delta\bar{x}$ leads to a displacement of the resonance peaks from $M\omega_0=\omega_L$ to $M\omega_0=\omega_L+eE\Delta\bar{x}/\hbar$. Now the δ function is replaced by a Lorentzian of width Γ_N and shift zero and the sum over M is performed as in Appendix A. If we approximate the factor $(N-M)!/N!$ by an average value N^* we can perform the sum over N for both terms, if N is large, provided $\Delta\bar{x}$ is independent of N. We then obtain

$$J_x \approx 3C'D' \sum_n \Delta\bar{x} e^{\beta_e(\varepsilon_F-\varepsilon_0 n^2)}[\bar{N}_0-(1+\bar{N}_0)N^*] \left\{1+2\sum_{s=1}^{\infty} e^{-2\pi s(\Gamma_N/\hbar\omega_0)}\cos(2\pi s\bar{\omega}_L/\omega_0)\right\} \Big/ 2\sinh\alpha , \tag{3.5}$$

where $\alpha=\beta_e\hbar\omega_0/2$ and where Γ_N is given by (B1).

At resonance, $\bar{\omega}_L=p\omega_0$, p integer, and the quantity in the large parens is equal to $\coth(\pi\Gamma_N/\hbar\omega_0)$. For small electric fields, i.e., for $E\to 0$, Eq. (3.5) shows clearly the usual magnetophonon resonances, $\omega_L=p\omega_0$, for $\pi\Gamma_N \ll \hbar\omega_0$. For $\omega_L=p\omega_0$ the cosine factor, in Eq. (3.5), becomes $\cos(2\pi seE\,\Delta\bar{x}/\hbar)$. Hence, by varying the electric field a usual magnetophonon maximum ($\omega_L=p\omega_0$) can convert into a minimum and vice versa. This behavior, as well as the displacement of the resonant peaks with increasing electric field (current), has been recently observed in polar materials,[3] see Sec. III B in which the constant $\Delta\bar{x}$ is defined.

2. *Wide wells*

When the thickness L_z of the well increases the energy levels ε_n come closer to each other and transitions between them can take place by varying, e.g., the electric field. If we consider transitions only between neighboring ε_n levels, we have $n'=n$ or $n'=\pm 1$. We then obtain the results (3.4) and (3.5) plus two additional terms corresponding to an upward ($n'=n+1$) and a downward ($n'=n-1$) jump. These two terms are not the same because the energy spectrum ε_n is not equidistant. Corresponding to (3.5), we obtain

$$J_x \approx (3.5)+2C'D' \sum_n \Delta\bar{x}\, e^{\beta_e(\varepsilon_F-\varepsilon_0 n^2)} N^{\pm} \left\{1+2\sum_{s=1}^{\infty} e^{-2\pi s(\Gamma_N/\hbar\omega_0)}\cos(2\pi s\omega^{\pm}/\omega_0)\right\} \Big/ 2\sinh\alpha , \tag{3.6}$$

where $N^+=\bar{N}_0$, $N^-=-(1+\bar{N}_0)N^*$, and $\omega^{\pm}=\bar{\omega}_L-(1\pm 2n)\varepsilon_0/\hbar$. For simplicity we have assumed the same Γ_N in the additional terms.

The transition between the levels ε_n due to an increase in the electric field could be seen easier by studying the behavior of the magnetophonon extrema as a function of the electric field. In this case $p\omega_0=\omega_L$ and the factors $\cos(\cdots)$ in Eq. (3.6) become $\cos(2\pi seE\,\Delta\bar{x}/\hbar)$ and

$$\cos\{2\pi s[eE\,\Delta\bar{x}-(1\pm 2n)\varepsilon_0]/\hbar\omega_0\}$$

for the first and second term, respectively. Hence, starting from a usual magnetophonon extremum (corresponding to $E\approx 0$) and increasing the electric field, while keeping the magnetic field constant, we may see oscillations of the extremum amplitude in wide wells for which $\Delta_{n,n\pm 1} \ll \hbar\omega_0$.

B. Polar optical phonons

We proceed as in Sec. III A with

$$|F(\mathbf{q})|^2=(A/\Omega q^2)\approx A'/\Omega u , \tag{3.7}$$

where $A'=Al^2/2$ and where we assumed $q_\perp \gg q_z$ for transport in the (x,y) plane. This approximation allows us to do the integrals over q_z. A is the constant of the polar interaction. Again, we make the approximation $q_y \approx eB\,\Delta\bar{x}/\hbar$ in the argument of the δ function and the factor $(a_{x\zeta'}-a_{x\zeta})/u$. The integrals over u then are again given by (A1) and (A2). J_x is still given by Eqs. (3.4) or (3.5) with $D'\Delta\bar{x}$ replaced by $A'/\Delta\bar{x}$, provided $\Delta\bar{x}$ is independent of N. If $\Delta\bar{x}$ depends on N, then instead of Eq. (3.5) we obtain

$$J_x \approx 3C'A' \sum_{N,n} e^{\beta_e(\varepsilon_F-\varepsilon_{N,n})}[\bar{N}_0-(1+\bar{N}_0)N^*] \left\{1+2\sum_{s=1}^{\infty} e^{-2\pi s(\Gamma_N/\hbar\omega_0)}\cos(2\pi s\bar{\omega}_L/\omega_0)\right\} \Big/ \Delta\bar{x} . \tag{3.8}$$

The comments following Eq. (3.5) apply equally well to Eq. (3.8) but with Γ_N given by (B3). Magnetophonon maxima convert into minima by increasing the electric field and vice versa, as reported in Ref. 3. The values of the electric field for which the amplitude of the usual magnetophonon extremum, $\omega_L = p\omega_0$, becomes zero are given from the solution of

$$\sum_{s=1}^{\infty} e^{-2\pi s(\Gamma_N/\hbar\omega_0)} \cos(2\pi seE\,\Delta\bar{x}/\hbar\omega_0)=0 . \qquad (3.9)$$

Due to the exponential it is expected that the first term $(s=1)$ gives the largest contribution to the sum. Equation (3.9) is then approximately obeyed for

$$eE\,\Delta\bar{x} \approx (2m+1)\hbar\omega_0/4\ , \quad m=0,1,2,\dots . \qquad (3.10)$$

Now for transitions between neighboring Landau levels the wave-vector change $\Delta k_y = q_y$ is about $1/l$, cf. Eq. (2.15). But the spatial extension $\Delta\bar{x}_N$ of $\phi_N(x-x_0)$, as determined from the mean-square deviation,[22] is of the order $1/q_y$: $\Delta\bar{x}_N = (\sqrt{N+1/2})l$. Hence, we may take $\Delta\bar{x} = (\Delta\bar{x}_N + \Delta\bar{x}_{N+1})/2$. For $m=0$, the values of the electric field, for which Eq. (3.10) is obeyed, differ from the corresponding values, obtained from Eq. (3) of Ref. 3, by a multiplicative factor $\sqrt{2}/2$. Hence, we obtain almost the same relationship but for narrower samples, since the results of this section are valid for L_z of the order of a few hundred Å (for GaAs), whereas the narrowest sample of Ref. 3 had $L_z = 0.25\ \mu$m. The physical interpretation of Eq. (3.10), elaborated further in Ref. 3, is that the potential energy gained from the electric field is of the order of $\hbar\omega_0$ and that the wave functions of neighboring Landau levels develop a significant overlap at values of the electric field given by Eq. (3.10).

Finally, we notice that the oscillations in Eq. (3.8) are damped at strong electric fields since Γ_N, as given by (B4), is roughly proportional to E. Further damping may come from β_e, since, in general, $T_e = Tf(E^2)$, cf. Ref. 21.

Wide wells

Proceeding as in Sec. III A 2 we obtain

$$J_x \approx (3.8) + 2C'A' \sum_{N,n} e^{\beta_e(\varepsilon_F - \varepsilon_{N,n})} N^{\pm}/\Delta\bar{x} \times \left\{ 1 + 2\sum_{s=1}^{\infty} e^{-2\pi s(\Gamma_N/\hbar\omega_0)} \times \cos(2\pi s\omega^{\pm}/\omega_0) \right\} , \qquad (3.11)$$

where $N^{\pm}$ and $\omega^{\pm}$ are defined as previously, cf. Sec. III A 2.

IV. SCATTERING BY IMPURITIES

We assume that the electrons are scattered by randomly distributed impurities. We consider very short-range potentials, which correspond to perfect screening and give results in closed form, and long-range potentials with the screening treated as in inversion layers.[23]

A. Linear transport

1. Very short-range interaction

We take $U(\mathbf{r}-\mathbf{R}) = V_0\delta(\mathbf{r}-\mathbf{R})$. The Fourier transform of this potential is constant: $U(\mathbf{q}) = \bar{V}_0$. Using Eqs. (2.4), (2.8), and (2.13)–(2.15) with $E=0$, and Eqs. (2.18)–(2.21) we find after the summations over k_y and q_z the result

$$\sigma_{xx}^d = \beta C_I \sum_{N,N',n,n'} (2+\delta_{nn'}) f_{N,n}(1-f_{N',n'}) \times \int u\,|J_{NN'}(u)|^2 du \times \delta((N-N')\hbar\omega_0 + \Delta_{nn'})\ , \qquad (4.1)$$

where $C_I = (N_I/4\pi\hbar)(e\bar{V}_0/L_z l)^2$. For simplicity, we consider only narrow wells for which $n=n'$. The case $n \neq n'$ can be treated as in Sec. III A 2. The integral over u is given by (A3) and Eq. (4.1) takes the form

$$\sigma_{xx}^d = 3\beta\bar{C} \sum_{N,n} f_{N,n}(1-f_{N,n})(2N+1)\ , \qquad (4.2)$$

where $\bar{C} = C_I/2\pi\hbar\omega_0$.

Apart from the constant $\bar{C}$, Eq. (4.2) has the same form as Eq. (3.19) of Ref. 4.

For very low temperatures, $\beta f_{N,n}(1-f_{N,n}) \approx \delta(\varepsilon_{N,n} - \varepsilon_F)$. We can then perform the sum over N using the Poisson's sum formula; we obtain

$$\sigma_{xx}^d \approx 3\bar{C}(\hbar\omega_0)^{-1} \sum_n \bar{\varepsilon}_F \left[1 + 2\sum_{s=1}^{\infty} (-1)^s e^{-2\pi s(\Gamma_N/\hbar\omega_0)} \cos(2\pi s\bar{\varepsilon}_F) \right] , \qquad (4.3)$$

where $\bar{\varepsilon}_F = (\varepsilon_F - \varepsilon_0 n^2)/\hbar\omega_0$. Γ_N is given by (B7). Hence, for $\Gamma_N \ll \hbar\omega_0$, the conductivity oscillates with period $\bar{\varepsilon}_F$. This period, however, may change as the Fermi level moves through the subbands ε_n because the quantity $\varepsilon_F - \varepsilon_0 n^2$ may not be the same for all subbands. A change of the period, as the second subband gets occupied, has been observed in heterostructures.[24] It is interesting to take the zero temperature limit for $n=1$. In this case, $\varepsilon_F = (N+\frac{1}{2})\hbar\omega_0 + \varepsilon_0$, $\cos(2\pi s\bar{\varepsilon}_F) = (-1)^s$, the quantity in large parens is equal to $\coth(\pi\Gamma_N/\hbar\omega_0)$, and Eq. (4.3) becomes simpler:

$$\lim_{T\to 0} \sigma_{xx}^d = (3\bar{C}/\hbar\omega_0)\coth(\pi\Gamma_N/\hbar\omega_0)(N+\tfrac{1}{2}) \approx (3\bar{C}/\pi\Gamma_N)(N+\tfrac{1}{2}),\quad \pi\Gamma_N \ll \hbar\omega_0 . \qquad (4.4)$$

Thus, the zero temperature conductivity decreases with increasing magnetic field B and well thickness L_z, and it goes to zero for $B \rightarrow \infty$. The exact dependence, however, on these parameters is tied to whichever of the two approximations, $\coth x \approx 1/x$ or $\coth x \approx 1/x + x/3 - x^3/45$, $x = \pi\Gamma_N/\hbar\omega_0$ is used, cf. (B7) or (B2) with Δ replaced by

$$\Delta_I = (2\hbar^2 L_z/\pi N_I \bar{V}_0^2 3nm^* .$$

2. Long-range interaction

The Fourier transform of the potential due to a singly ionized impurity situated at z_i, $U(\mathbf{q},z_i)$, is given by[23]

$$U(\mathbf{q},z_i) = \frac{2\pi e^2}{qk}\frac{F(\mathbf{q},z_i)}{1+q_s F(\mathbf{q})/q} , \tag{4.5}$$

where $\mathbf{q}$ is the two-dimensional wave vector, k is the dielectric constant of the medium (well), q_s is the screening constant, and where

$$F(\mathbf{q},z_i) = \int_0^{L_z} \phi^*(z) e^{-q|z-z_i|}\phi(z)\,dz , \tag{4.6}$$

and

$$F(\mathbf{q}) = \int_0^{L_z}\int_0^{L_z} |\phi(z)|^2 e^{-q|z-z'|}|\phi(z')|^2\,dz\,dz' , \tag{4.7}$$

with

$$\phi(z) = \sqrt{2}\sin(n\pi z/L_z)/\sqrt{L_z} .$$

The last two expressions can be evaluated exactly [cf. Eqs. (A7)–(A9)], but the resulting integral over u [cf. Eq. (4.1)] becomes very cumbersome. In any case, the major contribution to this integral comes from very small values of $u = l^2q^2/2$. To linear order in q we have $F(\mathbf{q}) \approx 1 + 5qL_z/4\pi^2 n^2$ and $F(\mathbf{q},z_i) \approx 1$. Assuming then a random distribution of impurities, $N(z_i) \approx N_I = \text{const}$, we obtain

$$|U(\mathbf{q})|^2 \approx N_I(2\pi e^2/k)^2/q_\perp^2(b+q_s/q_\perp)^2 , \tag{4.8}$$

where $b = 1 + 5q_s L_z/4\pi^2 n^2$. Corresponding to (4.1) we now obtain

$$\sigma_{xx}^d \approx \beta D_I \sum_{N,N',n,n'} (2+\delta_{nn'}) f_{N,n}(1-f_{N',n'}) \int |J_{NN'}(u)|^2 (b + q_s^l/\sqrt{2u})^{-2} du\,\delta((N-N')\hbar\omega_0 + \Delta_{nn'}) , \tag{4.9}$$

where $D_I = -(N_I/2\pi\hbar)(ne^3/kL_z)^2$. We again consider only the case $n = n'$. Now the integral over u is very difficult to evaluate in closed form for $b \neq 0$ but its main contribution comes from very small values of u due to the exponential of the factor $|J_{NN'}(u)|^2$, cf. Eq. (2.20). Therefore, for values of L_z comparable to those of l and $q_s l \gg 1$ the factor b can be neglected. The integral then over u is easily done using (A3) and, corresponding to (4.2), we obtain

$$\sigma_{xx}^d \approx 3\beta D_I'(2/q_s^2 l^2) \sum_{N,n} f_{N,n}(1-f_{N,n})(2N+1) , \tag{4.10}$$

where $D_I' = D_I/2\pi\hbar\omega_0$. The similarity of this result, valid for $q_s l \gg 1$, to the one valid for perfect screening, i.e., Eq. (4.2), is evident. All the analysis following Eq. (4.2) applies equally well to Eq. (4.10). From either of these equations we can get another interesting result valid for very strong magnetic fields such that $e^{-\alpha} \ll 1$, $\alpha = \beta[\varepsilon_F - (N+\frac{1}{2})\hbar\omega_0 - \varepsilon_0]$ when only one ε_n level is occupied. Expanding the factors $f_{N,1}$ and $1 - f_{N,1}$ we obtain, to order $e^{-\alpha}$,

$$\sigma_{xx}^d \approx 3\beta D_I'(2/q_s^2 l^2)(2N+1)e^{-\alpha} , \tag{4.11}$$

since only the level N contributes appreciably. Thus, as a function of the temperature T, the conductivity σ_{xx} shows a simple activated behavior: $\sigma_{xx} \sim e^{-\varepsilon/T}/T$.

B. Nonlinear transport

1. Short-range interaction

The only difference from the Sec. IV. A 2 is that we start with Eq. (2.9) instead of Eq. (2.4) and that Eqs. (2.13)–(2.15) contain the electric field ($E \neq 0$). As in Sec. III, we approximate q_y, wherever it appears raised to power 1, by $eB\Delta\bar{x}/\hbar$. With $\Delta_{nn'} = 0$ the term of Eq. (2.9) proportional to $w_{\zeta\zeta'}$ (labeled J_x^1) becomes

$$J_x^1 \approx C' \sum_{N,N',n} \Delta\bar{x} f_{N',n}(1-f_{N',n}) \int |J_{NN'}(u)|^2 du \times \delta((N-N')\hbar\omega_0 + eE\,\Delta\bar{x}) , \tag{4.12}$$

where

$$C' = (3BN_I/4\pi\hbar)(e\bar{V}_0/L_z l)^2$$

and where $f_{N,n}$ is a spatially uniform distribution function given by the Fermi-Dirac function with T replaced by the electron temperature T_e. The integral over u is unity [cf. (A1)] and the sum over N' is performed approximately with the help of (A6). Equation (4.12) then gives

$$J_x^1 \approx \bar{C}' \sum_{N,n} \Delta\bar{x} f_{N,n}(1-f_{N+\delta,n}) \times \left[1 + 2\sum_{s=1}^{\infty} e^{-2\pi s(\Gamma_N/\hbar\omega_0)}\cos(2_0\pi s\delta)\right] , \tag{4.13}$$

where $\delta = eE\,\Delta\bar{x}/\hbar\omega_0$ and $\bar{C}' = C'/2\pi\hbar\omega_0$. Γ_N is given by (B7). The other term (J_x^2) of Eq. (2.9), proportional to $w_{\zeta'\zeta}$, is given by Eq. (4.13) with N and $N+\delta$ interchanged. The sum of the two terms can then be written as ($J_x = J_x^1 + J_x^2$)

$$J_x \approx \bar{C}' \sum_{N,n} \Delta\bar{x} f_{N,n}(1-f_{N,n})[f_{N+\delta,n}/f_{N,n} + (1-f_{N+\delta,n})/(1-f_{N,n})] \left[1 + 2\sum_{s=1}^{\infty} e^{-2\pi s(\Gamma_N/\hbar\omega_0)}\cos(2\pi s\delta)\right] . \tag{4.14}$$

At very low temperatures $\beta_e f_{N,n}(1-f_{N,n})\approx\delta(\varepsilon_{N,n}-\varepsilon_F)$, and, upon using (A6), Eq. (4.14) takes the simpler form

$$J_x \approx (2\bar{C}'/\beta_e\hbar\omega_0)\sum_n \Delta\bar{x}\left[1+2\sum_{s=1}^{\infty}(-1)^s e^{-2\pi s(\Gamma_N/\hbar\omega_0)}\cos(2\pi s\bar{\varepsilon}_F)\right]\left[1+2\sum_{s=1}^{\infty}e^{-2\pi s(\Gamma_N/\hbar\omega_0)}\cos(2\pi s\delta)\right], \tag{4.15}$$

where $\bar{\varepsilon}_F=(\varepsilon_F-\varepsilon_0 n^2)/\hbar\omega_0$. If $\Delta\bar{x}$ is a function of N then N must be replaced by $(\varepsilon_F-\varepsilon_0 n^2)/\hbar\omega_0-\frac{1}{2}$. Comparing (4.15) with (4.9) we see that (i) we have oscillations with period $\bar{\varepsilon}_F$, as in the linear case, and (ii) oscillations with period δ due to the electric field provided that $\bar{\varepsilon}_F$ does not vary much as a function of the electric field.

2. *Long-range interaction*

As in Sec. IV A 2., we limit ourselves again to the case $q_s l \gg 1$. The results then (4.12)–(4.15) can be taken over with $\bar{V}_0^2$ replaced by $(2\pi e^2/k)^2(2/q_s^2)$.

V. THE HALL CONDUCTIVITY σ_{yx} LINEAR TRANSPORT

If the nondiagonal matrix elements of $\rho(t)$ are neglected the Hall current density (and consequently the Hall conductivity) can be obtained directly from Eqs. (2.2) and (2.17) as shown in Refs. 16 and 18. As shown previously[11,12] this neglect can be avoided in the linear case. In what follows we give the result for σ_{yx} of a quantum well as described in Sec. II B.

As explained at the end of Sec. II B the only nonvanishing contribution to σ_{yx} is given by Eq. (2.5). The matrix elements of j_ν and j_μ ($\nu=x$, $\mu=y$) are given by (2.16) and (2.17) with $E=0$, respectively. Due to the Kronecker deltas in these equations we find from (2.14) that

$$\varepsilon_{\zeta''}-\varepsilon_{\zeta'}=(N''-N')\hbar\omega_0=\pm\hbar\omega_0 \ . \tag{5.1}$$

Using periodic boundary conditions in the y direction and the constraint $0\le -x_0=l^2k_y\le L_x$ we find

$$\sum_{ky}\rightarrow(Ly/2\pi)\int dk_y=A_0/2\pi l^2 \ , \tag{5.2}$$

where spin was not taken into account; if spin is included (5.2) is multiplied by 2. We take again a spatially uniform distribution $\langle n_\zeta\rangle=f_{N,n}$. Using Eqs. (2.5), (2.16), (2.17), (5.1), (5.2), and proceeding exactly as in Ref. 12, Sec. II A, we obtain $[\sigma_{yx}(0)\equiv\sigma_{yx}]$.

$$\sigma_{yx}=(e^2/hL_z)\sum_{n,N}(N+1)f_{N,n}(1-f_{N+1,n})\times(1-e^{-\beta\hbar\omega_0}) \ . \tag{5.3}$$

We now remark[25] that

$$f_{N,n}(1-f_{N+1,n})\exp(-\beta\hbar\omega_0)=(1-f_{N,n})f_{N+1,n} \ ,$$

and with that we can rewrite (5.3) as

$$\sigma_{yx}=(e^2/hL_z)\sum_{n,N}(N+1)(f_{N,n}-f_{N+1,n}) \ . \tag{5.4}$$

The change $N+1\rightarrow N$ is made in the second term and this leads to the final result

$$\sigma_{yx}=(e^2/hL_z)\sum_{n,N}f_{N,n} \ . \tag{5.5}$$

Hence, σ_{yx} depends on the thickness L_z through the factors $1/L_z$ and $f_{N,n}$, where $f_{N,n}=\{1+\exp[\beta(\varepsilon_{N,n}-\varepsilon_F)]\}^{-1}$. For free electrons, the equilibrium density n_0, as determined from

$$n_0=\sum_{N,n,k_y}\int\delta(\varepsilon-\varepsilon_{N,n})f(\varepsilon)d\varepsilon/\Omega \ ,$$

is equal to $\sum_{N,n,k_y}f_{N,n}/\Omega$ and (5.5) takes the well-known form $\sigma_{yx}=en_0/B$. At zero temperature $f_{N,n}=1$ and (5.5) becomes

$$\sigma_{yx}=(e^2/hL_z)(N+1)n \ . \tag{5.6}$$

For $n=1$, very low temperatures and strong magnetic fields such that $e^{-\alpha}\ll 1$, $\alpha=\beta(\varepsilon_F-\varepsilon_{N,n})$, Eq. (5.5) becomes

$$\sigma_{yx}\approx(e^2/hL_z)(N+1)(1-e^{-\alpha}) \ , \tag{5.7}$$

the exponential factor representing the deviation from the zero temperature value. From (5.7) and (4.11) we obtain $[\Delta\sigma_{yx}(T)=\sigma_{yx}(0)-\sigma_{yx}(T)]$

$$\Delta\sigma_{yx}(T)/\sigma_{xx}(T)=(e^2q_s^2l^2/6\beta D_I' hL_z)\times(N+1)/(2N+1) \ . \tag{5.8}$$

This kind of relationship has been observed in the studies of the integral quantum Hall effect.[26] If βN_I is constant ($D_I'\propto N_I$), i.e., if thermal activation of the carriers occurs, the ratio $\Delta\sigma_{yx}(T)/\sigma_{xx}(T)$ is independent of the temperature.

At high temperatures,

$$f_{N,n}\approx\exp[\beta(\varepsilon_F-\varepsilon_{N,n})] \ ,$$

$$\varepsilon_{N,n}=(N+\tfrac{1}{2})\hbar\omega_0+\varepsilon_0 n^2 \ ,$$

the sum over N, in Eq. (5.5), is easily performed for high N, and Eq. (5.5) takes the form

$$\sigma_{yx}\approx(e^2/hL_z)[2\sinh(\beta\hbar\omega_0/2)]^{-1}\sum_n e^{\beta(\varepsilon_F-\varepsilon_0 n^2)} \ . \tag{5.9}$$

Comparing Eqs. (5.7) and (5.9) we see that the temperature dependence of σ_{yx} is more complicated at high temperatures due to the factor $\sinh(\beta\hbar\omega_0/2)$.

With regard to the thickness dependence of σ_{yx} we remark that only the result (5.6), at zero temperature, shows a simple dependence on L_z. Since the energy spectrum depends on L_z so does ε_F and this makes the L_z dependence of the results (5.5), (5.7), and (5.9) more complicated.

VI. CONCLUDING REMARKS

In this paper we have evaluated dc conductivities parallel to the walls of a quantum well, in the presence of a magnetic field normal to its walls, when linear response theory is valid, i.e., for weak electric fields, and dc currents for nonlinear responses.

In the linear case, we have shown that scattering by short-range or long-range ($q_s l \gg 1$) impurity potentials leads to oscillations of the conductivity σ_{xx} with period $(\varepsilon_F - \varepsilon_0 n^2)/\hbar\omega_0$. To our knowledge, this result is new, and the only pertinent approximation made in obtaining it was the replacement, at very low temperatures, of $\beta f_{N,n}(1-f_{N,n})$ by $\delta(\varepsilon_{N,n}-\varepsilon_F)$. Essentially, the same result (with $n=1$) was reported previously for elastic scattering by acoustical phonons but it was obtained with an additional approximation.[4] Since usually, impurity scattering dominates at very low temperatures, it is quite possible that the (acoustical) phonon contribution to σ_{xx} is masked by the impurity contribution. It any case, it would be desirable to perform experiments, similar to those of Ref. 3, in much shorter n^+-n^--n^+ structures, of the order of a few hundred Å, to check the validity of the result as well as the thickness dependence of the conductivity.

The Hall conductivity σ_{yx} has been evaluated for the linear case and is shown to depend on the thickness of the well. To our knowledge, this result is new. We have not investigated the possibility of the integral quantum Hall effect in a quantum well but in analogy with superlattices, where the effect has been observed[27] and treated theoretically,[28] we expect it to be the case if only the lowest subband ($n=1$) is occupied and the Fermi level is pinned in the gaps of the energy spectrum since the sample becomes quasi-two-dimensional. If that is the case, the results (5.5)–(5.8) could be used, and it would be interesting to see whether the finite thickness of the quasi-two-dimensional layer has any measurable effect on σ_{yx} as it does, for example, in the activation energy for σ_{xx}, in the fractional quantum Hall effect.[29]

We have also investigated the possibility of transitions between the well levels ε_n by, e.g., varying the electric field. This is likely to be realized in wide wells ($L_z \leq 1000$ Å) at very strong magnetic fields such that $\Delta_{nn'} \ll \hbar\omega_0$. So far we are not aware of any relevant experimental work.

We further notice that the activation energies for σ_{xx}, Eq. (4.11), and $\Delta\sigma_{yx}$, Eq. (5.7), are the same and that they depend on the thickness of the well. The prefactors, however, in all the expressions for σ_{xx} have different dependence on the well thickness than the corresponding expressions for σ_{yx}. This is due to the fact that σ_{xx} depends on the scattering (impurity density, etc.), whereas σ_{yx} does not, at least in the first Born approximation.

All the results for the nonlinear case are tied to the approximation $eV_d q_y \approx eE\,\Delta\bar{x}$ ($\Delta\bar{x} \sim l$). This led to an additional oscillatory structure, for scattering by impurities or phonons, which depends on the electric field strength. For certain values of the electric field [cf. Eq. (3.10)] the wave functions of neighboring Landau levels overlap and transitions between them are possible: maxima convert into minima and vice versa. Although the results of Ref. 3 make this clear we cannot make a more quantitative comparison of our theory with the experiment, and thus check the error made by $eV_d q_y \approx eE\,\Delta\bar{x}$, since the present theory applies to much narrower wells and since we are not aware of any pertinent experimental work.

Finally, we notice that, with regard to the results for σ_{xx}, the usual divergences of the Born approximation have been avoided by replacing, as usual, the δ functions by Lorentzians (or Gaussians). The relevant level widths have been estimated from the inverse scattering rates but they have a simple form (cf. Appendix B) only at the resonance (for off-resonance Γ_N values, for electron-phonon interaction, see Ref. 4). These results could be used in estimating the mobility of the samples through $\mu = e\tau/m^*$. Interestingly, the results for Γ_N or $1/\tau$ depend explicitly only on the number of the occupied subbands n, but not on the Landau level index N.

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APPENDIX A

The first two integrals below are found in tables.[30] For the explicit evaluation of the next three ones, see Refs. 4 and 11.

$$\int_0^\infty e^{-x} x^M [L_N^M(x)]^2 dx = (N+M)!/N!,\quad M>0\,, \tag{A1}$$

$$\int_0^\infty e^{-x} x^M [L_{N-M}^M(x)]^2 dx = 1,\quad M>0\,, \tag{A2}$$

$$\int_0^\infty e^{-x} x^{M+1} [L_N^M(x)]^2 dx = (2N+M+1)(N+M)!/N!\,, \tag{A3}$$

$$\int_0^\infty e^{-x} x^{M-1} [L_N^M(x)]^2 dx = (N+M)!/N!M,\quad M\neq 0\,, \tag{A4}$$

$$\int_{-\infty}^\infty \cos(2\pi s x)[(x+a)(x^2+b^2)]^{-1} dx = a e^{-2\pi s b}/(a^2+b^2),\quad s\geq 0\,. \tag{A5}$$

(A4) and (A5) are used for the derivation of (B3) and (B4).

The following relation is derived in Ref. 4 using Poisson's summation formula:

$$\sum_{M=0}^\infty \delta(M-\omega_L/\omega_0) \approx 1 + 2\sum_{s=1}^\infty e^{-2\pi s(\Gamma_N/\hbar\omega_0)} \times \cos(2\pi s\omega_L/\omega_0)\,, \tag{A6}$$

where Γ_N is the width of the Lorentzian (of shift zero) by which the δ function is approximated. In a similar manner one can carry out the sum $\sum_M \phi(M)\delta(g(M))$.

$F(\mathbf{q})$, as given by Eq. (4.7), has been evaluated in Ref. 31. The result is

$$F(\mathbf{q}) = 2(x^2+8\pi^2 n^2)(x^2+4\pi^2 n^2)^{-2}(1-e^{-x}) + x/(x^2+4\pi^2 n^2) + 2[1-(1-e^{-x})/x]/x,\quad x = qL_z\,. \tag{A7}$$

To linear order in x, $F(\mathbf{q}) \approx 1 + 5x/4\pi^2 n^2$. From Eq. (4.6) we obtain

$$F(\mathbf{q},z_i) = 4\pi^2 n^2 e^{-qz_i}(e^x - 1) \times (x^2+4\pi^2 n^2)^{-1}/x,\quad z<z_i\,, \tag{A8}$$

$$F(\mathbf{q},z_i)=-4\pi^2n^2e^{qz_i}(e^{-x}-1)\times(x^2+4\pi^2n^2)^{-1}/x,\ z>z_i\ . \quad \text{(A9)}$$

To linear order in x, $F(\mathbf{q},z_i)\approx 1$ for both $z<z_i$ or $z>z_i$.

APPENDIX B

The damping factors Γ_N, appearing in the text, are estimated from the inverse scattering rates $1/\tau$ according to $\Gamma_N\approx\hbar/\tau$. The procedure is identical with that of Ref. 4 so we only give the results for $n=n'$.

A. Optical phonons

For $\bar{\omega}_L=p\omega_0$ the Γ_N are determined from the graphical solution of

$$\Gamma_N\approx 3\hbar KD'(1+2N_0)n\coth(\pi\Gamma_N/\hbar\omega_0)\ . \quad \text{(B1)}$$

For $\pi\Gamma_N\ll\hbar\omega_0$ and $\coth x\approx 1/x+x/3-x^3/45$ we obtain

$$\Gamma_N\approx(15\{1-3\Delta+[(1-3\Delta)^2+36]^{1/2}\}/2\pi^2)^{1/2}\hbar\omega_0\ , \quad \text{(B2)}$$

where

$$\Delta=2\hbar^2L_z/3\pi m^*D'(1+2N_0)n$$

and

$$K=\tfrac{1}{2}\hbar L_z l^2\ .$$

B. Polar optical phonons

Corresponding to (B1) we obtain

$$x\approx\hbar\Lambda\delta(\pi/\hbar\omega_0)^2\coth x/(\delta^2+x^2),\quad x=\pi\Gamma_N/\hbar\omega_0\ , \quad \text{(B3)}$$

where $\Lambda=3KA'(1+2N_0)nl^2/2$ and $\delta=\pi\bar{\omega}_L/\omega_0$. For $\coth x\approx 1/x$, Eq. (B3) gives

$$\Gamma_N\approx\{[\delta(\delta^2+4\Lambda')^{1/2}-\delta]/2\pi^2\}^{1/2}\hbar\omega_0\ ,\quad \Lambda'=\pi\Lambda/\bar{\omega}_L\omega_0\ . \quad \text{(B4)}$$

C. Impurity scattering.

1. Short-range interaction

Using Eq. (2.8), with $U(\mathbf{q})=\bar{V}_0$, and Eqs. (2.16)–(2.19) we obtain ($\hbar V_d q_y\approx eE\Delta\bar{x}$)

$$\frac{1}{\tau}=(N_I\bar{V}_0^2 3nm^*/2\hbar^3L_z)\sum_M\int|J_{N,N+M}(u)|^2\times du\,\delta(M-eE\,\Delta\bar{x}/\hbar\omega_0) \quad \text{(B5)}$$

The integral over u is 1, cf. (A1), and the sum over M is given by (A6) with $\omega_L\to eE\Delta\bar{x}$. For $eE\Delta\bar{x}=p\hbar\omega_0$, p integer, we get

$$\Gamma_N\approx(N_I\bar{V}_0^2 3nm^*/2\hbar^2L_z)\coth(\pi\Gamma_N/\hbar\omega_0)\ , \quad \text{(B6)}$$

so that, Γ_N is given, for $\pi\Gamma_N\ll h\omega_0$, by an expression similar to (B2). If, however, only one term is kept in the expansion of $\coth x$, i.e., if $\coth x\approx 1/x$, then

$$\Gamma_N\approx[(N_I\bar{V}_0^2 3nm^*/2\hbar^2\pi L_z)\hbar\omega_0]^{1/2}\ . \quad \text{(B7)}$$

2. Long-range interaction

The only difference with Appendix B C 2 is that $U(q)$ is given by the approximate result (4.8). For $q_s l\gg 1$, however Γ_N is again given by (B6) or (B7) with $\bar{V}_0$ replaced by $2\pi e^2/kq_s$.

APPENDIX C

Below we outline the derivation of formulas (2.3)–(2.5). The details are to be found in Refs. 9 and 32. Moreover, we indicate briefly how formula (2.9) is derived in Ref. 17.

In Ref. 32, the Hamiltonian (2.1) is inserted into the von Neumann equation for the density operator $\rho(t)$, and the latter is split into a diagonal part (in the representation of H^0) $\rho_d(t)$, and a nondiagonal part $\rho_{\text{nd}}(t)$. Then, by application of projection operators to the von Neumann equation two coupled equations are obtained, one for $\rho_d(t)$ and one for $\rho_{\text{nd}}(t)$. These equations are decoupled by application of the Van Hove limit for weakly interacting systems.

$$\lambda\to 0,\quad t/\tau_t\to\infty,\quad \lambda^2 t\ \text{finite}\ , \quad \text{(C1)}$$

where $\tau_t\approx\hbar/\Delta t$ is the time for a transition between two eigenstates of H^0 to take place. This limit is equivalent to the first Born approximation. The assumptions are then made that (i) V is a two-body interaction potential and as such translationally invariant, and (ii) A commutes with λV and $\dot{A}$ is translationally invariant. This leads, for linear responses of the system from its equilibrium state (ρ_{eq}), i.e., for weak electric fields, to two inhomogeneous master equations, one for $\rho_d(t)$ and one for $\rho_{\text{nd}}(t)$. The solution of the first equation leads to the conductivity formula

$$\sigma^d_{\mu\nu}(i\omega)=\beta\Omega\int_0^\infty dt\,e^{-i\omega t}\mathrm{T_r}[\rho_{\text{eq}}J^R_{d\nu}J^R_{d\mu}(t)]\ , \quad \text{(C2)}$$

where the reduced current J^R_d (R denotes the Van Hove limit) is given by

$$J^R_d=\frac{q}{\Omega}\left|-\Lambda_d\sum_i(\mathbf{r}_i-\mathbf{r}_i^{\text{eq}})_d+\sum_i V_{id}\right|\ . \quad \text{(C3)}$$

Here Λ_d is the master superoperator in Liouville space defined

$$\Lambda_d K=\sum_{\gamma,\gamma'}|\gamma\rangle\langle\gamma|\,[W_{\gamma'\gamma}\langle\gamma'|K|\gamma'\rangle-W_{\gamma\gamma'}\langle\gamma|K|\gamma\rangle] \quad \text{(C4)}$$

where $|\gamma\rangle$ are the eigenstates of H^0, $W_{\gamma\gamma'}$ is given by the golden rule, and where $\mathbf{V}_i$ is the velocity operator. Equations (C2) to (C4) are valid at the many-body level. Their reduction to the one-body level is made in Ref. 9 under the assumption that the scattering system remains at equilibrium. Equation (C3) reduces to Eqs. (2.3) and (C2), for $\mu=\nu$, $\omega=0$ reduces to Eq. (2.4) when the ponderomotive current [second term of Eq. (C3)] is zero.

The solution of the equation for $\rho_{\rm nd}(t)$ leads to a many-body formula for the conductivity $\sigma_{\mu\nu}^{nd}(i\omega)$ similar to (C2). The dc version of its one-body analog is given by Eq. (2.5).

As to Eq. (2.9), it has been derived from the master equation for the Landau states in Refs. 15 and 16. Its derivation for a general state is given in Ref. 17. The starting point is the Liouville equation. Then a density matrix is introduced and the equation for its elements is solved by iteration. The lowest order in the interaction, the current J_μ is proportional to λ^2. The assumption that the scattering system remains at equilibrium leads directly to Eq. (2.9) for all values of the electric field.[33]

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