PHYSICAL REVIEW B VOLUME 34, NUMBER 11 1 DECEMBER 1986

Universal Poisson's ratio in a two-dimensional random network of rigid and nonrigid bonds

David J. Bergman and Edgardo Duering
School of Physics and Astronomy, The Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, 69978 Tel Aviv, Israel

(Received 25 July 1986)

Two different elastic moduli near the percolation threshold of a two-dimensional random honeycomb network of rigid and nonrigid bonds were calculated as a function of the correlation length ξ and the width L of the network whose length N is very large ($N \gg L$, $N \gg \xi$). For L and ξ large enough, the ratio μ/C_{11} is found to depend only on the ratio ξ/L. For $\xi/L < 1$, the ratio tends to a value 0.46 ± 0.02, which corresponds to a rather low (though positive) value of Poisson's ratio σ, namely, $\sigma = 0.08 \pm 0.04$.

Recently, the critical behavior of random elastic networks has been the subject of a number of investigations, some of which have indicated that the ratio of different moduli tends to a limiting value that is independent of the values of the microscopic elastic parameters.[1-3] This was found both in the case of a diluted network,[1,3] where the elastic moduli tend to zero as $p \to p_c^+$ (here p is the fraction of occupied bonds while p_c is the percolation threshold), and in the case of a normal-rigid network,[2] where the elastic moduli diverge as $p \to p_c^-$ (here p is the fraction of totally rigid bonds). In both cases the microscopic elastic properties included a bond-stretching force constant k as well as a bond-bending force constant m between neighboring bonds, so that the percolation threshold p_c is identical with the threshold for solid, elastic, or rigid behavior. Another common characteristic of these numerical investigations was that each random network was constructed in the form of a long $N \times L$ two-dimensional strip (with $N \gg L$) at $p = p_c$, and the macroscopic elastic properties of the strip were then determined by a transfer-matrix method. This was done for different values of the width L, and then the finite-size scaling idea was used to determine the critical exponent of, e.g., the elastic modulus C_{11} at the rigidity threshold of a normal-rigid network according to $C_{11} \sim L^{S/\nu}$ (see Ref. 2). Inherent in this approach is the fact that the correlation length ξ is always much greater than L. Indeed, for an infinite random network at $p = p_c$, ξ would be infinite. By contrast, real random systems for which the random network might be a reasonable model will usually be at some $p \neq p_c$ such that ξ is much less than the macroscopic linear dimensions. In that case the critical behavior takes the form $C_{11} \sim \xi^{S/\nu} \sim (p_c - p)^{-S}$ and is independent of the size of the system.

The finite-size scaling idea describes these two regimes, as well as the transitions between them, by the following *Ansätze:*

$$C_{11} = \xi^{S/\nu} F\left(\frac{\xi}{L}\right) , \tag{1a}$$

where

$$F(x) \to \begin{cases} A, & \text{for } x \to 0, \\ A' x^{-S/\nu}, & \text{for } x \to \infty \end{cases} , \tag{1b}$$

and A, A' are positive constants. A similar *Ansatz* can be written for another elastic coefficient, the shear modulus μ,

$$\mu = \xi^{S/\nu} G\left(\frac{\xi}{L}\right) , \tag{2a}$$

where

$$G(x) \to \begin{cases} B, & \text{for } x \to 0, \\ B' x^{-S/\nu}, & \text{for } x \to \infty \end{cases} , \tag{2b}$$

and B, B' are again positive constants. The fact that the same exponent $S \cong 1.30$ appears in both cases was one of the conclusions of Ref. 2. Another result of that paper was that the ratio μ/C_{11} tends to a value of about $\frac{2}{3}$ for increasing L,[1,4] independent of the microscopic force constants k, m. From the *Ansätze* (1) and (2), we can determine two asymptotic values for the ratio μ/C_{11},

$$\frac{\mu}{C_{11}} \cong \begin{cases} A/B, & \text{for } L \gg \xi, \\ A'/B', & \text{for } L \ll \xi . \end{cases} \tag{3}$$

Clearly, the simulations of Ref. 2 determined that $A'/B' \cong \frac{2}{3}$. Because the Poisson ratio σ is simply related to the ratio μ/C_{11}, that result would mean that

$$\sigma = 1 - \frac{2\mu}{C_{11}} \cong -\frac{1}{3} , \tag{4}$$

which was somewhat unexpected:[2] Although negative values of σ are not forbidden by stability considerations, nevertheless, in all naturally occurring homogeneous and isotropic solids the Poisson ratio is found to be positive.[5]

In this Rapid Communication, we report on some simulations where the ratio μ/C_{11} in a rigid-nonrigid network was determined for nonzero values of L/ξ, with the ultimate aim of learning about systems at the other extreme condition $L \gg \xi$, when $\mu/C_{11} = A/B$. Clearly, this could be different from the value $\frac{2}{3}$ found in Ref. 2. Following Eqs. (1) and (2), we would expect the ratio μ/C_{11} to depend on L and ξ as follows:

$$\frac{\mu}{C_{11}} = \frac{G(\xi/L)}{F(\xi/L)} \equiv H(\xi/L) . \tag{5}$$

This would hold, however, only when both $\xi \gg 1$ and $L \gg 1$. Otherwise we might expect some residual dependence on L, as well as on the microscopic force constants k,m.

The simulations reported here were done using the same transfer-matrix method that was used in Ref. 2, where its important elements were described. Here we will only mention that the network was a two-dimensional honeycomb network in the shape of a long ($N \gg L$) strip with periodic boundary conditions in the short directions. The bonds were chosen randomly and independently to be either rigid (with probability $p < p_c$) or normal (with probability $1-p$). A rigid bond has $k=\infty$, while an angle between adjacent bonds has $m=\infty$ only if both bonds are rigid. Simulations were done at a number of different values of L and ξ/L, where ξ was assumed to be given by

$$\xi = \left(\frac{p-p_c}{p_c} \right)^{-\nu}, \tag{6}$$

with $\nu=\frac{4}{3}$.[6] The strip length N should ideally be large enough so that a unique result is obtained for the elastic moduli at fixed values of the other parameters. In practice, this sometimes requires extremely long strips, as found already in Refs. 2 and 3 and in other transfer-matrix calculations. In Fig. 1 we show in one case how the results for μ/C_{11} fluctuate when N is increased in steps of 50. Evidently there are small and rapid fluctuations as well as large and slow fluctuations, and this makes it difficult to estimate the accuracy of the supposedly unique result. In this Rapid Communication we will not attempt to gauge the accuracy of each calculation. It will become apparent below that the accuracy is sufficient for the conclusions that we reach.

In Fig. 2 we show a limited collection of results for μ/C_{11} vs L for various fixed values of ξ/L and for two different values of k/m, which would correspond to

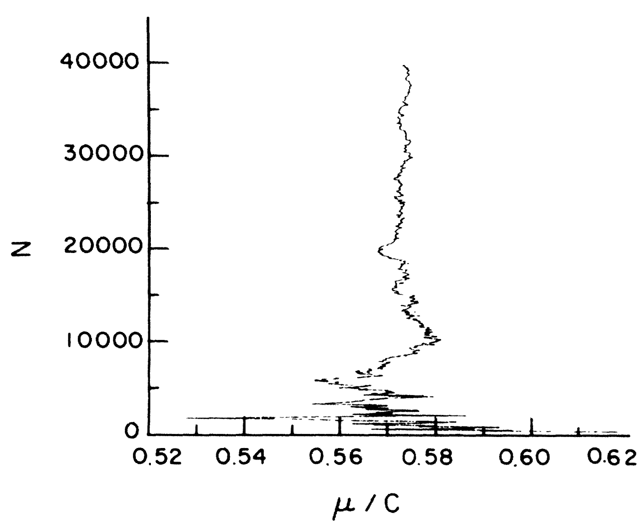

FIG. 1. Results for μ/C_{11} when $k/m=1.5$, $L=20$, $N_{max}=40000$ and $p_c-p=0.0138$ (i.e., $\xi=8{,}55$) plotted vs N in steps of 50. Note that although the fluctuations settle down rather quickly, there remain fluctuations on many scales even for large N. This becomes even more serious when $p \to p_c$.

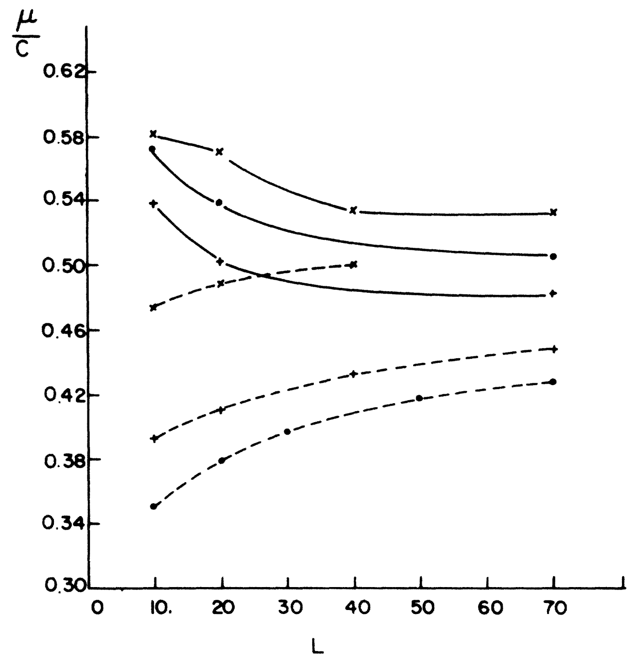

FIG. 2. Values of μ/C_{11} plotted vs the strip width L for several values of ξ/L and for two values of k/m, the points are the results of the numerical simulations for $\xi/L=0.323(\bullet)$, $0.940(+)$, and $8.55(\times)$. The lines were drawn through the points in order to guide the eye and to differentiate between the results for $k/m=1.5$ (full line) and 15.82 (dashed line). Note that at the lowest value of ξ/L, even for the largest L (i.e., 70) we are at $p_c-p\cong 0.063$, i.e., about 10% below the percolation threshold $p_c\cong 0.6527$, so that it is questionable whether the critical region has been reached.

$\sigma=-0.23$ and $\sigma=0.29$, respectively, if the network had no rigid bonds. It is evident that, at each value of ξ/L, there is a tendency for the results to converge to a value independent of k/m as L increases, and that those asymptotic values will depend on ξ/L. In particular, it seems that the asymptotic value of μ/C_{11} will decrease with decreasing ξ/L. The precise asymptotic values of μ/C_{11} are not derivable from Fig. 2 since there is still a considerable dependence on k/m and on L.

In Fig. 3 we show all of the results we have obtained for μ/C_{11} with the two values of k/m mentioned above, plotted versus ξ/L for various values of L. The following points are noteworthy.

(a) For a given value of L, as $\xi/L \to 0$ the ratio μ/C_{11} eventually tends to the value it should have for a homogeneous network with the appropriate value of k/m. This usually requires a qualitative departure from the asymptotic behavior, which is expected to be independent of k/m when both ξ and L are large.

(b) As L increases, the two sets of lines (corresponding to the two values of k/m) tend to approach each other.

(c) Consequently, it is not difficult to draw a semiquantitative plot of μ/C_{11} vs ξ/L for $L\to\infty$, and this is also shown in Fig. 3. This line, which must reach $\mu/C_{11}\cong\frac{2}{3}$ for $\xi/L\to\infty$ (this is not exhibited in Fig. 3), is seen to reach the approximate value 0.46 as $\xi/L\to 0$. Conse-

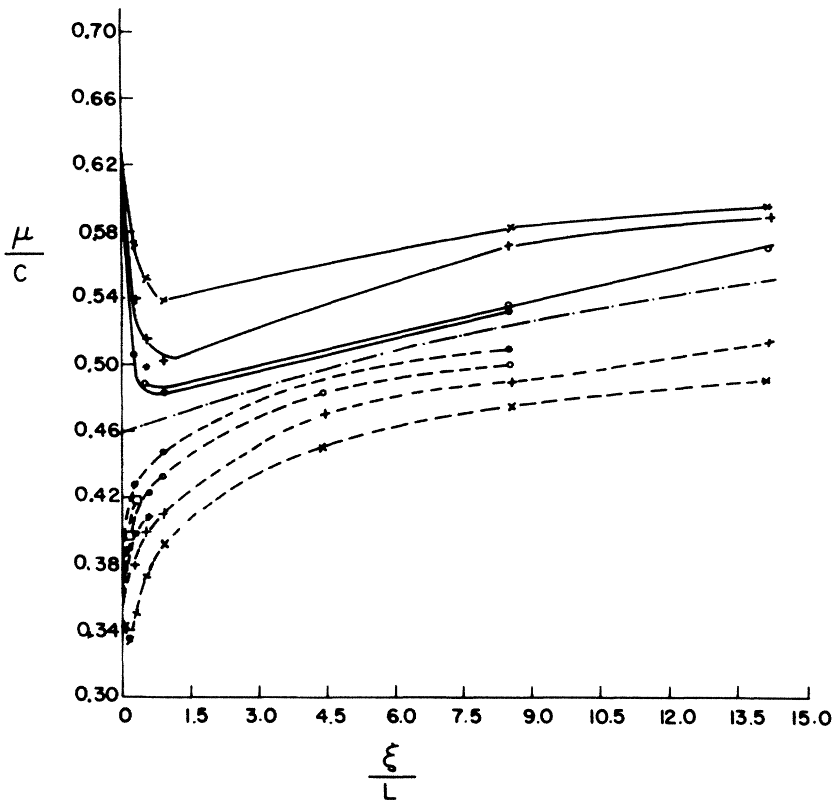

FIG. 3. Values of μ/C_{11} plotted vs ξ/L for different values of L and for two values of k/m. The points are the numerical simulation results for $L=10(\times)$, $20(+)$, $30(*)$, $40(\circ)$, $50(\square)$, and $70(\bullet)$. Two sets of lines were drawn through the points in order to guide the eye and to differentiate between the results for $k/m=1.5$ (full line) and 15.82 (dashed line). Based on these lines, another line (dot-dash) was drawn to indicate semiquantitatively the asymptotic dependence of μ/C_{11} on ξ/L for $L\rightarrow\infty$. Note that for sufficiently small values of ξ, i.e., when p is far enough below p_c, the dependence of μ/C_{11} on ξ changes qualitatively so as to reach the correct value for the homogeneous network when $\xi\rightarrow 0$. Thus, when $k/m=1.5$, μ/C_{11} changes from a decreasing function to an increasing function of ξ at $\xi=L$.

quently, we conclude that

$$\frac{A}{B}=\frac{\mu}{C_{11}}\bigg|_{\xi\gg L}=0.46\pm0.02\ . \tag{7}$$

This means that the asymptotic value of Poisson's ratio becomes

$$\sigma=0.08\pm0.04\ . \tag{8}$$

While this value is not negative, it is much closer to zero than in most natural homogeneous and isotropic solids, which is rather remarkable.

In order to get more precise results for the ratio μ/C_{11}, longer and wider trips should be simulated. This would have to be done on a supercomputer, since the present results already required about 150 h of CPU time on a CDC-Cyber 885. (Most of this time was used to evaluate the $L=70$ strips out to a length of $N=7000$.) It is known that such calculations can be speeded up enormously by using a vector computer,[3] so that is clearly how these calculations should be continued.

It is a pleasure to acknowledge useful discussions with Y. Kantor. This research was supported in part by a grant from the Israel Academy of Sciences.

---

[1]D. J. Bergman, Phys. Rev. B **33**, 2013 (1986).

[2]D. J. Bergman, Phys. Rev. B **31**, 1696 (1985). See also Ref. 4.

[3]J. G. Zabolitzky, D. J. Bergman, and D. Stauffer, J. Stat. Phys. (to be published). References to earlier and other works in this field can be found in Refs. 1–3.

[4]Because of an uncorrected misprint, it was erroneously stated in Ref. 2 that for $m/k=0.063\,22$ the asymptotic value obtained for μ/C_{11} is 0.76. The actual value obtained was 0.66. This result, together with the asymptotic value $\mu/C_{11}\cong 0.63$ found for $m/k=\frac{2}{3}$, formed the basis for the conclusion regarding universality of the ratio μ/C_{11}.

[5]L. D. Landau and E. M. Lifshitz, *Theory of Elasticity* (Pergamon, London, 1970), p. 14.

[6]B. Nienhuis, J. Phys. A **15**, 199 (1982).