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Estimate of a universal critical-amplitude ratio from its ε expansion up to ε^2

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The exponent α of the specific heat C vanishes at some value n_0 of the number n of components of the order parameter. n_0 is estimated (1.942 ± 0.026) from the available long series at $d=3$ in powers of the Φ^4 coupling. Knowing that the ratio $A^+/A^-=1$ ($A^{\pm}$ are the critical amplitudes of C above and below T_c) for $n=n_0$, the estimate at $d=3$ of this ratio for $n=2$ obtained from ε expansion up to ε^2 is improved. The cases $n=1$ and 3 are also considered.

Field theoretical techniques applied to critical phenomena[1] have provided precise estimates of many universal characteristics of critical behavior. Calculations may be done within two different frameworks based on Feynman graph expansion. First is the well-known Wilson-Fisher ε expansion[2] ($\varepsilon=4-d$, d is the space dimension) referred to in the following as I. Second is the usual perturbative expansion in powers of the Φ^4 coupling performed directly at integer values of d[3] (e.g., $d=3$), referred to as II. These two kinds of expansions lead to divergent[4,5] series whose known large-order behavior suggests resummation methods.[6,7]

It has recently been shown[8] that, provided that the series are long enough (five or six orders), the two frameworks lead to compatible estimates of critical exponents for various values of the number n of components of the order parameter. As for critical amplitudes, long enough series have been recently made available in scheme II for various n in the disordered phase[9] ($T > T_c$) and only for $n=1$ in the ordered phase[10] ($T < T_c$). In scheme I, the series are known in general for any n, but they are very short, at most up to ε^2.[11–13] Consequently, very little is accurately known for $n > 1$, on most universal critical amplitude combinations.

In an analysis of experimental data of a system near a second-order phase transition, not only critical exponents are important, leading and subleading critical amplitudes are too. Up to now the best system for a systematic study of universality of critical singularities is the superfluid transition of ^{4}He (Ref. 14), which corresponds to the case $n=2$. Usual experimental determination of α (the critical exponent of the specific heat C) in this system is still slightly in disagreement[15,16] with the theoretical estimate,[7] while the leading (subleading) amplitude ratio A^+/A^- (D^+/D^-) of C is, theoretically, essentially unknown owing to the shortness of the available series.

It is generally admitted that the two first terms of the ε expansion are sufficient to get an estimate of the quantity of interest. One considers either the simple sum of the series ([2,0] Padé) or the [1,1] Padé as giving the best values. For example, Table I displays the various Padé approximants of the series, up to ε^2, for the exponent γ at $n=1$, 2, and 3 (Ref. 1) compared to the standard values obtained by Le Guillou and Zinn-Justin[7] from scheme II. One observes a rather good agreement considering the shortness of the series. Unfortunately this is not, in general, the case for amplitude ratios, and this fact is very often ignored when one refers to "renormalization-group (RG) predictions." The main difference between exponents and amplitudes which could explain this situation is the following: ε expansion of critical exponents is essentially determined by the poles at $\varepsilon=0$ of the Feynman integrals for the symmetric theory ($T > T_c$), while for the critical amplitude ratios the complete integrals of the two phases contribute. One may thus expect a simpler structure for the ε expansion of exponents than for that of amplitude ratios.[17]

To illustrate this point let us consider the ratio A^+/A^-. Its expansion up to ε^2 is recalled in the Appendix, and the corresponding Padé approximants are presented in Table II. Only the value at $n=1$ is known[10] with accuracy from scheme II. Hence, for $n=2$ and 3 one is limited to the numbers presented in this Table. How far can one trust them?

First one sees that, for $n=1$, the values are not in a very good agreement with the recent estimate[10] at $d=3$. More important is the instability of the result with respect to the approximant considered. Such an instability is not ob-

TABLE I. Comparison between estimates for the exponent γ, for various values of n, obtained from Padé approximants (2nd column) of $O(\varepsilon^2)$ expansions given in the Appendix, and resummation of long series of scheme II (3rd column) considered as the standard estimates of RG theory. The three numbers given for each n in the 2nd column correspond, respectively, to the Padé (2,0), (1,1), (0,2). See text for a discussion of this table.

n	$O(\varepsilon^2)$	From Ref. 7
1	1.244 1.311 1.276	1.241 ± 0.002
2	1.30 1.40 1.35	1.316 ± 0.0025
3	1.35 1.48 1.42	1.386 ± 0.004

TABLE II. Comparison between estimates for the universal amplitude ratio A^+/A^-, for various values of n, obtained from Padé approximants (2nd column) of $O(\varepsilon^2)$ expansion given in the Appendix, and resummation of long series of scheme II (3rd column). Same presentation as for Table I. See text for a discussion of this table.

n	$O(\varepsilon^2)$	From Ref. 10
	0.394	
1	0.438	0.541 ± 0.014
	0.150	
	0.817	
2	0.880	...
	0.331	
	1.258	
3	1.326	...
	0.541	

served for critical exponents (see Table I).

The ratio A^+/A^- is particularly interesting since it must be exactly equal to unity when $\alpha=0$ (the critical singularity of C becomes logarithmic). Now, from the result of scheme II, the physical case of $n=2$ should correspond to a slightly negative value of α.[7] Hence $n=2$ is very close to n_0 at which value α vanishes. One may thus expect A^+/A^- to be very close to 1 for $n=2$. Moreover, assuming continuity at $\alpha=0$ and from indications given by the ε expansion, one should have $A^+/A^- > 1$ (for $n \geq 2$). Table II clearly shows that $O(\varepsilon^2)$ expansion does not reach this value.

The object of this paper is to propose estimates at A^+/A^- for $n=2$ which very likely are close to the values that one would obtain in frameworks I or II with long enough series. The primary ingredients will be the available series up to ε^2 and the large order behavior. The fundamental point is that $n=2$ is very close to n_0 [see Eq. (4) below].

The reason for the bad estimates of Table II is the shortness of the expansion. By summing term by term the series for A^+/A^- at $n=2$ and $\varepsilon=1$ (see Appendix), one obtains, respectively, at zeroth, first, and second order, 0.5, 1.035, and 0.817.

At large order k, the coefficients a_k of the ε expansion of A^+/A^- behave as[5]

$$a_k \simeq k!(-a)^k k^{b_0} c \quad \text{as } k \to \infty , \tag{1}$$

with

$$a = \frac{3}{n+8} , \tag{2}$$

and b_0 varies linearly with n as $n/2$ (c is a constant which strongly depends on the quantity considered). One may suppose that the change of sign observed, in A^+/A^- (see Appendix), between the first and second order is a direct consequence of Eq. (1) and that the unknown third order will have a positive sign, and so on. Hence the sum of the series will oscillate about unity without convergence [Eq. (1) shows that the series diverges]. Now, how far above unity is the value of A^+/A^- for $n=2$? Indeed, as already noted in experimental analysis,[16] the deviation from unity of A^+/A^- at small α depends essentially on the determination of α. Since α depends continuously on n, one may expect that the value of A^+/A^- at $n=2$ will be essentially determined by the knowledge of n_0 at which $\alpha(n_0)=0$. From ε expansion up to ε^2 (see Appendix) one gets[18]

$$n_0 = 4 - 4\varepsilon + O(\varepsilon^2) . \tag{3}$$

The information on n_0, so obtained, is not sufficient to get precise information on A^+/A^- at $n=2$ and $d=3$ ($\varepsilon=1$). Fortunately, longer series are available for α. Using the series of scheme II,[19] which are the longest, I obtain the following estimate:

$$n_0 = 1.942 \pm 0.026 \tag{4}$$

for $d=3$.

The error analysis of the procedure used[9,10] to get this result is not as complete as in the work of Le Guillou and Zinn-Justin,[7] although the method is essentially the same. For example, the corresponding estimate of α at $n=2$ is -0.0066 ± 0.0030 to be compared to the standard value,[7] -0.007 ± 0.006.

The knowledge of n_0 will now be used to estimate A^+/A^- as follows.

(1) Replace the ε expansion of A^+/A^- by a chosen function $f(\varepsilon,n,b)$ having the same three first terms and the same high-order behavior [with the parameter a given by Eq. (2) and $b_0=b$ considered as a free parameter] in ε.

(2) The free parameter b is chosen such that $f(1,n_0,b)=1$. This gives a value $b(n_0)$, with n_0 given by Eq. (4).

(3) Using the dependency on n of b_0, calculate, for $n=2$, $f(1,n,b(n_0)+(n-n_0)/2)$ to get the value of A^+/A^-.

I have made the following choice for $f(\varepsilon,n,b)$:

$$f(\varepsilon,n,b(s,r)) - \frac{n}{4} = Q\left[\int_0^\infty \frac{dx}{x}\frac{e^{-x}x^s}{(1+a\varepsilon x)^r} - \Gamma(s)\right], \tag{5}$$

in which $\Gamma(s)$ is the Euler function of s.

It is easy to verify that the high-order behavior of Eq. (5) has a form similar to Eq. (1) with $b_0=b(s,r)=r+s-2$. The two parameters Q and r are chosen, at fixed s (a free parameter), such that the two first terms of the expansion in powers of ε of Eq. (5) are equal to those of $(A^+/A^- - n/4)$. Hence, Q and r are functions of n and s.

Applying steps 2 and 3 described above, with n_0 given by Eq. (4), I get the following estimate for $n=2$:

$$\frac{A^+}{A^-} = 1.0294 \pm 0.0134 \quad (n=2) . \tag{6}$$

To get the error bar, the two cases of the direct and inverse series for A^+/A^- have been considered together with the error on n_0 of Eq. (4).

In order to appreciate the validity of this estimate, one can also perform step 3 for $n=1$, for which a result using a long series is available[10] (see Table II). It becomes

$$\frac{A^+}{A^-} = 0.524 \pm 0.010 \quad (n=1) . \tag{7}$$

The procedure used here which consists in estimating

A^+/A^-, starting from its exact value at $\alpha=0$, is not as good when $|\alpha|$ grows (i.e., when n goes away from n_0). Nevertheless, the value of Eq. (7) for $n=1$ is in better agreement with the result of Ref. 10 than the usual "best estimate" found from Padé approximants (see Table II). One may thus expect Eq. (6) for $n=2$ to be very close to the correct answer.

It is worth noticing that an analysis of experimental data on ^{4}He at various pressures yields,[16] for $\alpha=-0.007$, $A^+/A^-=1.029$, which is in excellent agreement with Eq. (6).

The same procedure may also be applied for $n=3$. I obtain

$$\frac{A^+}{A^-}=1.521\pm0.022 \quad (n=3) \ . \tag{8}$$

The quantity $(1-A^+/A^-)/\alpha$ is expected to be relatively insensitive to the value of α.[20] I have found it to be, respectively, for $n=1$, 2, and 3, equal to 4.307 ± 0.030, 4.455 ± 0.040, and 4.563 ± 0.089 to be compared to the value 0.46 found in Ref. 18.

This brief and rather simple study of the universal ratio A^+/A^- has been made possible because it is equal to unity at $\alpha=0$. The knowledge of long series for α has allowed estimating n_0 (at which $\alpha=0$) with accuracy. This has provided a strong constraint on the only parameter n (for $n\simeq n_0$) of the ε expansion of A^+/A^-, since this ratio takes on a known value at $n=n_0$. It is clear that estimates by Padé approximants from $O(\varepsilon^2)$ series for amplitude ratios do not always give correct values. One must be more cautious when referring to such estimates as being the RG prediction on the quantity considered.

APPENDIX

The ε expansions up to ε^2 of exponents γ and α are[21]

$$\gamma=1+\frac{(n+2)}{2(n+8)}\varepsilon+\frac{(n+2)(n^2+22n+52)}{4(n+8)^3}\varepsilon^2+O(\varepsilon^3) \ ,$$

$$\alpha=\frac{(4-n)}{2(n+8)}\varepsilon-\frac{(n+2)(n^2+30n+56)}{4(n+8)^3}\varepsilon^2+O(\varepsilon^3) \ .$$

The ε expansion of the ratio A^+/A^- up to ε^2 is[21]

$$\frac{A^+}{A^-}=\frac{n}{4}2^{\alpha}\left[1+\varepsilon+\varepsilon^2\left(\frac{3n^2+26n+100}{2(n+8)^2}+\frac{(4-n)(n-1)}{2(n+8)^2}\zeta(2)-\frac{3(5n+22)}{(n+8)^2}\zeta(3)-\frac{9(4-n)\lambda}{2(n+8)^2}\right)\right] \ ,$$

in which $\zeta(2)=1.645$, $\zeta(3)=1.2021$, and $\lambda=1.1719$.

To get the true ε expansion of A^+/A^- one must expand the factor 2^{α} by using the expression of α given just above.

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