Helium in Vycor, constrained randomness, and the Harris criterion

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Standard arguments for the relevance or otherwise of random disorder at bulk criticality are reconsidered in relation to the crossover seen in the superfluid transition at low coverages of helium adsorbed in Vycor, a porous glass. It is argued that Vycor should be modeled by *constrained randomness* characterized by pair correlations vanishing with wave vector, $\mathbf{q}$, as $|\mathbf{q}|^{\theta}$, with, physically, $\theta=2$. Simple scaling and Harris arguments yield a randomness crossover exponent $\phi_R=\alpha-\theta\nu$ (α and ν being the specific-heat and correlation-length exponents) but must be *corrected,* in the light of previous renormalization-group arguments, by taking proper account of nonlinear scaling fields or quadratic terms in the variation of transition temperature with disorder. *In fact,* constrained randomness is still relevant for α positive (as at a Gaussian fixed point describing ideal Bose behavior in dimensions $d<4$) but the quantitative effects are significantly reduced with, in particular, $\phi_R=\frac{1}{2}\alpha$. The interpretation of the experiments as indicative of crossover to ideal Bose criticality is discussed in the light of the results.

I. INTRODUCTION AND SUMMARY

The effect of quenched impurities or quenched randomness on the critical properties of a thermodynamic system has been a subject of considerable experimental and theoretical interest in the past decade. Various models such as random-bond Ising models,[1,2] random uniaxial anisotropy in Heisenberg models,[3] and various forms of quadratic random couplings in continuous spin s^4 models,[4–6] have been studied in detail. An important insight was obtained by Harris[1] who developed a type of Ginzburg criterion. Harris's self-consistency argument indicates that, at least for sufficiently weak disorder, the critical behavior of the pure or uniform system will survive if the specific-heat exponent α is negative, i.e.,

$$\alpha<0 \Longrightarrow \text{randomness irrelevant} . \tag{1.1}$$

This conclusion is born out by scaling arguments and renormalization-group calculations[4,5] in which it is found that short-ranged bond randomness is either relevant or irrelevant according to whether the appropriate crossover exponent ϕ_R, which is found to equal α, is, as usual, either positive or negative.

The present analysis is a continuation of previous work[7] in which current critical-point theory was applied to analyze the experimental results of Crooker, Hebral, Smith, Takano, and Reppy[8,9] on the critical behavior of the superfluid density of ^{4}He adsorbed into Vycor in the limit of very low coverages. Vycor is a porous, spongelike glass with a characteristic microscopic "pore" size of 50 to 100 Å in the experiments of Crooker *et al.*, however, it is irregular and amorphous in microscopic structure. Thus helium in Vycor may reasonably be regarded, in the regime in question, as a three-dimensional weakly interacting Bose gas in a random external potential.[7–9] By modeling the randomness of Vycor glass through the introduction of *effective* parameters for a *uniform* system, one indeed obtains remarkably good agreement between theory and experiment.[7] The interacting Bose gas in three dimensions probably satisfies the Harris criterion for irrelevance of the randomness since experimental data for bulk ^{4}He (which are consistent with the best theoretical estimates) yield $\alpha\simeq-0.02\pm0.02$. On the other hand, in the weakly interacting, low-density limit, the analysis[7] exhibits crossover to ideal-Bose-gas criticality. However, for the ideal Bose gas itself, one actually has $\alpha=\frac{1}{2}\epsilon$ where $\epsilon=4-d$; thus, the appropriate specific-heat exponent, as regards the *crossover* behavior, is *positive* in the dimensionality range of interest, namely, $2<d<4$. If randomness does indeed represent a relevant perturbation[10] for criticality in an ideal Bose fluid, which is described by a *Gaussian fixed point,* what fixed point will govern the low-coverage critical behavior of helium in Vycor? The experiments[8,9] are not inconsistent with the assumption that it remains Gaussian in nature.[7] Indeed, we show in a general context, in Sec. III, that even if the randomness *is* technically relevant, the crossover in criticality could, in principle, still be controlled by the ideal Gaussian fixed point. This occurs despite the rather pathological nature of the *ideal* Bose gas in a random potential, whose properties are reviewed briefly at the end of this section. The possibility of such "dominated crossover" arises because the effective magnitudes of the interactions, which determine the departures from ideality, *and* of the randomness are *both* controlled by the transition temperature T_c, which goes to zero as the density is reduced. We will argue, in Sec. IV, that the criterion for dominated crossover is *not* manifestly valid so that it remains unclear if or why a Gaussian fixed point validly describes the observed phenomena. Nevertheless, even though a definitive theoretical answer to the problem has not been found, we believe that some insight can be gained by looking more closely into the physical nature of Vycor and exploring some of its implications.

One of our purposes here, therefore, is to discuss a class of models, specified generally in Sec. II, in which the randomness is *constrained* in a certain, natural way that serves to reduce the magnitude and effects of the fluctuations characterizing the disorder. As we explain shortly below, there is evidence that Vycor can be described validly by this category of models.

Now, as is demonstrated in Secs. IV A and IV B, it turns out that the standard Harris and scaling arguments applied to such constrained random systems produce a new, *weaker* criterion for relevance expressed by

$$\alpha - 2\nu < 0 \Longrightarrow \text{randomness irrelevant} , \qquad (1.2)$$

where ν is the correlation-length exponent. At first sight, this is very encouraging since 2ν exceeds α in all known cases! However, Weinrib and Halperin,[6] though mainly concerned with the effects of *long-range* correlations in the randomness, have derived general renormalization-group recursion relations which turn out to cover the constrained case as well. Their results show that (1.2) is actually *incorrect,* and that, in fact, the original Harris criterion (1.1) still holds; however, as we show in Sec. IV C, the crossover exponent of the randomness changes to

$$\phi_R = \tfrac{1}{2}\alpha , \qquad (1.3)$$

which is smaller than before, giving $\phi_R = \frac{1}{4}$ in place of $\phi_R = \frac{1}{2}$ for $d = 3$, so that the effects of the randomness are *quantitatively reduced.* The misleading answers produced by the initial Harris and scaling arguments are disturbing theoretically; we show in Secs. IV D–IV F, however, that when properly extended and more carefully analyzed, both arguments can be corrected, the faulty steps being identified and the new result (1.3) being recaptured. Rather more detailed arguments, presented in Sec. IV G, are needed to establish, within the framework of the constrained models as set up, that the dominated crossover mechanism of Sec. III does *not* operate. The current theoretical situation in the light of these results and of other theoretical approaches to the effects of randomness on superfluid (or *XY*-like) ordering is discussed in Sec. V: the *superfluid onset* problem *at* $T = 0$ (see below) is a profound one and probably cannot be described simply in ideal-Bose-gas terms; however, *criticality* at $T_c > 0$ might still be correctly described by crossover from a Gaussian fixed point as T_c increases from zero.

A. Constrained randomness

In order to motivate in more detail the class of constrained models considered, and to explain why they are of physical interest in their own right even though they turn out to lie in the same universality class as the standard models of randomness, consider the following idealized picture of helium in Vycor. Following Kohn and Luttinger,[11] who considered impurity scattering in metals, and Kac and Luttinger,[12] who analyzed an ideal Bose gas in a random system of hard spheres, let us picture Vycor as a set of randomly placed scattering centers, each with short-range interactions with the helium atoms. The constrained random nature of the overall configuration may then be prescribed as follows: first, place the centers on a regular lattice with spacing

$$c_0 = \rho_0^{-1/d}, \quad \rho_0 = N_0 / V , \qquad (1.4)$$

where N_0 is the number of scattering centers and V is the volume of the system, then suppose that each center is independently displaced through a *finite* distance. One might imagine the displacements caused by some isotropic process of radiation damage. Alternatively, each center might be subjected to diffusion away from its lattice site to some random location (without interacting with other centers). More precisely, we may describe the final configuration by a probability density $p(\mathbf{x})$ in which $\mathbf{x}$ is the displacement from the initial lattice site. For example, if each center undergoes independent Brownian motion for the time t_0, we would have

$$p(\mathbf{x}) = (2\pi D t_0)^{-d/2} \exp(-|\mathbf{x}|^2 / 2Dt_0) , \qquad (1.5)$$

where D is the diffusion constant. For D finite this $p(\mathbf{x})$ represents a kind of anticlustering constraint; essentially, at most $(Dt_0/c_0^2)^{d/2}$ scattering centers can overlap one another. Similarly, empty regions of diameter larger than $c_0 + c_1 (Dt_0)^{1/2}$ are highly unlikely.

B. Vycor glass

To see why this model should describe Vycor, at least in a qualitative way, one should understand how Vycor is made, namely, via a process of *spinodal decomposition.*[13] A melt consisting primarily of boron oxide and silica above the consolute or phase-separation temperature T_c is quenched to within the liquid spinodal regime below T_c. Phase separation is allowed to proceed, but before it goes to completion, the melt is further quenched to below the glass transition temperature. Subsequently, the boron-rich component is leached out chemically, leaving an intricate, spongelike SiO_2 structure which photomicrographs suggest consists of a series of interconnecting "pores." The resulting glass sponge may be characterized by its structure factor $S(\mathbf{q})$, which can, in essence, be regarded as a frozen version of $S_t(\mathbf{q})$, the time-dependent structure factor which reflects the instantaneous configuration of the spinodally decomposing liquid glass mixture at time t after the initial quench below T_c. Because the order parameter for the spinodal decomposition (in this case, say, the density of one of the component glasses) is a *conserved* quantity, the decomposition process is diffusion limited. General considerations[14] then lead to the conclusion that $S_t(\mathbf{q}=\mathbf{0})$ is a constant of the motion. Thus if $S_{t=0}(\mathbf{0}) = 0$, as for a uniform, unseparated binary glass melt, it follows that $S_t(\mathbf{0})$ must vanish for all t- although for long times, $S_t(\mathbf{q})$ becomes more and more strongly peaked, close to $\mathbf{q} = 0$, signaling the incipience of long-range order, $S_t(\mathbf{q}) \simeq \delta(\mathbf{q})$, as $t \to \infty$. This behavior is seen explicitly in experiments on other binary glass mixtures.[15,16] Preliminary measurements of $S(\mathbf{q})$ for Vycor itself by x-ray scattering do indeed indicate that it vanishes as $\mathbf{q} \to \mathbf{0}$.[17]

Now the structure factor $S(\mathbf{q})$ is directly related to the pair correlation function for the randomness. We shall see in Sec. II that the vanishing of this correlation function when $\mathbf{q} \to \mathbf{0}$ indeed characterizes constrained randomness. We therefore conclude that Vycor is, in a real sense, less random than one might, *a priori,* have expected. Unfortunately, it is still sufficiently random as to destabilize the standard ideal-Bose-gas, or Gaussian, critical fixed point!

C. Random ideal Bose gas

The behavior of an ideal Bose gas in a random system of scattering centers has been considered by Kac and Luttinger.[12] They studied the unconstrained model with, essentially, $t_0=\infty$, so that, in fact, the centers are placed with a *uniform* distribution over the entire system and the reference lattice becomes quite irrelevant. In their first paper,[12(a)] Kac and Luttinger showed that, for purely repulsive scattering centers, Bose-Einstein condensation still occurs. More specifically, for low temperatures, $T<T_c$, replacing the sum over discrete energy states in a finite system by an integral for the infinite system no longer accounts properly for the total density of bosons; i.e., if $g(\varepsilon)$ is the density of states per unit volume in the thermodynamic limit, then with $\beta=1/k_BT$,

$$\int g(\varepsilon)(e^{\beta\varepsilon}-1)^{-1}d\varepsilon<\rho \quad \text{for} \quad \beta_c<\beta<\infty \ . \tag{1.6}$$

In their second paper,[12(b)] they considered specifically the case of hard-sphere centers and conjectured that the single-particle partition function, which is essentially the Laplace transform of $g(\varepsilon)$, behaves as

$$Q(\beta)\equiv\sum_i e^{-\beta\varepsilon_i}\sim e^{-A_d\beta^{d/(d+2)}} \quad \text{as } \beta\to\infty \ . \tag{1.7}$$

This result was subsequently proven rigorously by Donsker and Varadhan:[18] it corresponds to

$$g(\varepsilon)\sim\exp(-B_d\varepsilon^{-d/2}) \quad \text{as } \varepsilon\to 0 \ , \tag{1.8}$$

so that there is an exponentially small low-energy "tail" to the density of states.

These low-energy states in the tail are, in fact, localized to regions of linear size $l\sim(2m\varepsilon/\hbar^2)^{-1/2}$ in which, by a random fluctuation, there are *no* spheres. The very low density of states represents the small chance of finding such a region. On the other hand, in our constrained model there is a still smaller probability of finding such an open region: If $p(\mathbf{x})$ remains nonzero over an unbounded region, the tail will still extend to zero energy, but the density of states will be even more strongly suppressed; however, if $p(\mathbf{x})$ has compact support, only finite open regions are permitted and the tail will not extend to zero energy. One expects, instead, a similar tail terminating at some energy $\varepsilon_0>0$, corresponding to the lowest-energy extended state consistent with some more-or-less regular pattern of spheres. Thus the concept of an "open region" translates into a large region closely approximating a regular pattern: such regions will play a similar role when we reanalyze and extend the Harris argument in Secs. IV A, IV D, and IV E.

Now one may compare (1.8) to the result $g(\varepsilon)\propto\varepsilon^{(d-2)/2}$ for a uniform ideal Bose system. For large d this also vanishes faster than linearly when $\varepsilon\to 0$. One might, thus, expect the random ideal gas described by (1.8) to behave like an ideal Bose gas of high dimensionality. Moreover, since the upper critical dimensionality for the ideal Bose transition is $d=4$, one should expect a mean-field-like transition. However, in order to discuss the nature of the random ideal Bose critical behavior more fully, one should specify the order parameter and its conjugate field. In the uniform case these are, respectively, the "condensate wave function"

$$\Psi_0=\left\langle V^{-1}\int d^dr\,\psi(\mathbf{r})\right\rangle , \tag{1.9}$$

where $\psi(\mathbf{r})$ is the Bose field operator, and the conjugate off-diagonal field ν_0.[19] More generally, if ideal Bose condensation takes place into a ground state with a normalized wave function $\varphi(\mathbf{r})$, the order parameter will be

$$\Psi_\varphi=\left\langle V^{-1/2}\int d^dr\,\psi(\mathbf{r})\varphi^*(\mathbf{r})\right\rangle , \tag{1.10}$$

with a corresponding conjugate field ν_φ. This prescription is not actually very effective for the infinite Kac-Luttinger model since one cannot properly specify $\varphi(\mathbf{r})$: the lowest energy $\varepsilon=0$ is never actually achieved in the model, and the limit $\varepsilon\to 0$ is not really continuous since the corresponding states will not be close to one another in any useful sense. However, for any finite realization of the model there will be a ground state (with positive energy) and, to a good approximation, ideal Bose condensation will take place into this state. Then the solution technique is identical to that for the uniform case[19] and one indeed finds a mean-field-like transition. However, at low transition temperatures $T_c(\rho)$ [defined by *equality* in (1.6)], one finds, instead of the usual power-law relation $T_c(\rho)\sim\rho^{2/d}$, the logarithmic dependence

$$T_c(\rho)\sim[\ln(\rho^{-1})]^{-(d+2)/d} \tag{1.11}$$

and a corresponding exponential decrease in the uncondensed fraction at low temperatures, namely,

$$1-n_\varphi(T)\equiv 1-|\Psi_\varphi(T)|^2\sim\exp(-A_dT^{-d/(d+2)}) \quad (T<T_c) \ . \tag{1.12}$$

There is clearly no long-range order associated with this transition since the order parameter profile $\langle\Psi(\mathbf{r})\rangle=\sqrt{V}\Psi_\varphi\varphi^*(\mathbf{r})$ is localized if $\varphi(\mathbf{r})$ is localized. Indeed, there will be an *infinite* density of particles in the corresponding localized region when $V\to\infty$ for $T<T_c$. Moreover, the model cannot exhibit superfluidity since a localized state is insensitive to the boundary conditions while ρ_s is really an elastic "helicity" modulus found by varying the phase of $\psi(\mathbf{r})$ via boundary conditions.[20]

D. Helium in Vycor

From these considerations it is apparent that the ideal Bose-Einstein transition in a random medium is quite pathological: it does not correspond at all to what is observed in Vycor! Of course, this is hardly surprising: helium atoms have strongly repulsive cores that set limits on the maximum realizable density. This immediately precludes a transition of the Kac-Luttinger type. The actual behavior at low overall helium densities may be seen by appealing to experiment.[8,9] Adsorbing small quantities of ^{4}He into Vycor glass does *not* produce superfluidity at low temperatures. Rather, there is an overall *onset density* ρ_0, below which all the helium is essentially immobile or "frozen" on the Vycor surface but above which true superfluidity appears at low enough temperature. The simplest interpretation of this onset phenomenon[8,9] is that the excess helium, beyond the onset coverage, forms an independent, mobile fluid, "insulated" from the glass substrate by the frozen or localized layers. The localized layers are thus presumed to play no further role. If this were really the case there would be essentially no difference between the true system and an ideal model with no localized layers. Contrary to this presumption, we will summarize, in the final section, some of what has been said in the literature which is relevant to localization in an *interacting Bose system.* The few theoretical results available give strong hints that the localized layers *do* indeed play a role but might, perhaps, still lead back to ideal Bose *critical behavior* of the normal type, as consistent with the observations in Vycor. Nevertheless, it seems quite likely that the full *onset transition* at $T=0$ is more complex and not merely determined by some effective ideal Bose or Gaussian fixed point. At present, however, this is an open issue so that a complete theory of helium in Vycor remains to be established.

II. MODELS FOR CONSTRAINED RANDOMNESS

It was shown in Ref. 7 that the investigation of the critical region of superfluid helium in the limit of low densities can be reduced explicitly, by an appropriate mapping, to the study of continuous-spin s^4 models. All of what follows can be carried through without making this reduction but it then entails complications which serve mainly to obscure the relevant physics. We therefore discuss the reduced spin Hamiltonian

$$\mathscr{H}/k_BT \equiv \overline{\mathscr{H}} = \overline{\mathscr{H}}_0 + \overline{\mathscr{W}} + \overline{\mathscr{U}} \ , \tag{2.1}$$

in which we take, for d spatial dimensions,

$$\overline{\mathscr{H}}_0 = \tfrac{1}{2} \int d^dx \left(|\nabla \vec{\sigma}(\mathbf{x})|^2 + r\, |\vec{\sigma}(\mathbf{x})|^2 \right) , \tag{2.2}$$

$$\overline{\mathscr{W}} = \tfrac{1}{2} \int d^dx\, w(\mathbf{x})\, |\vec{\sigma}(\mathbf{x})|^2 , \tag{2.3}$$

$$\overline{\mathscr{U}} = \int d^dx \int d^dx'\, |\vec{\sigma}(\mathbf{x})|^2 u(\mathbf{x}-\mathbf{x}')\, |\vec{\sigma}(\mathbf{x}')|^2 , \tag{2.4}$$

where $\vec{\sigma}(\mathbf{x})$ denotes a classical, n-component continuous-spin variable with n components, $\sigma^{\mu}(\mathbf{x})$, satisfying $-\infty < \sigma^{\mu} < \infty$ ($\mu = 1,2,\ldots,n$). The parameter r corresponds as usual, to the primary temperaturelike variable, while $u(\mathbf{x})$ corresponds to a pairwise "particle-particle" interaction potential. Finally, $w(\mathbf{x})$ is a *random,* "single-particle" external potential: The standard s^4 model[10] is recovered by setting $w=0$ [and $u(\mathbf{x}) = u_{(0)}\delta(\mathbf{x})$]. It should be emphasized that, in spin language, $w(\mathbf{x})$ represents "bond disorder," *not* a "random field" which would couple directly to $\vec{\sigma}(\mathbf{x})$ rather than to $|\vec{\sigma}(\mathbf{x})|^2$ (or other even terms). The strict absence of random ordering-field terms in spin models describing superfluid helium follows from the gauge invariance [under $\psi(\mathbf{r}) \to e^{i\phi}\psi(\mathbf{r})$] of real quantal systems, i.e., our inability to realize off-diagonal fields physically. Further aspects of the mapping from helium will be mentioned below.

Mean thermodynamic and correlation properties follow, as usual, by computing statistical expectations, etc., for a given realization $\mathscr{W}$ of the random fields and then averaging over the, as yet unspecified, ensemble of random potentials; this latter operation we denote by $\langle\!\langle \cdot \rangle\!\rangle$. The well-known replica device allows one to interchange the order of random ensemble and thermodynamic statistical averaging (see, e.g., Ref. 4). Integrating over the randomness in this way yields a nonrandom replicated Hamiltonian $\mathscr{H}^{(p)}$, depending only on p sets of spin variables. At the end of the calculation the formal limit $p \to 0$ is taken. There are no strong reasons to doubt the validity of this procedure in regimes where no long-range order or broken symmetries arise. Following Aharony,[4] one obtains

$$\overline{\mathscr{H}}^{(p)} = \sum_{i=1}^{p} (\overline{\mathscr{H}}_{0,i} + \overline{\mathscr{U}}_i) + \sum_{i,j=1}^{p} \overline{\mathscr{W}}_{i,j} + O(\sigma^6) \ , \tag{2.5}$$

where the indices i and j label the p equivalent replicas. The terms $\overline{\mathscr{H}}_{0,i}$ and $\overline{\mathscr{U}}_i$ have forms identical to (2.2) and (2.3) except for the replacement of the original spin variables by the replicated variables $\vec{\sigma}_j(\mathbf{x})$. The last term is given by

$$\overline{\mathscr{W}}_{i,j} = -\tfrac{1}{8} \int d^dx \int d^dx'\, |\vec{\sigma}_i(\mathbf{x})|^2 G_2(\mathbf{x},\mathbf{x}')\, |\vec{\sigma}_j(\mathbf{x}')|^2 , \tag{2.6}$$

where the kernel G_2 derives from the randomness via the second-order cumulant

$$G_2(\mathbf{x},\mathbf{x}') = \langle\!\langle w(\mathbf{x}) w(\mathbf{x}') \rangle\!\rangle - \langle\!\langle w(\mathbf{x}) \rangle\!\rangle \langle\!\langle w(\mathbf{x}') \rangle\!\rangle \ . \tag{2.7}$$

The coupling between the replicas engendered by $\overline{\mathscr{W}}_{i,j}$ may lead to new critical behavior. The terms of order σ^6 indicated in (2.5) involve higher cumulants of w and should represent only irrelevant perturbations relative to $\overline{\mathscr{W}}_{i,j}$.

Consider, for simplicity, the fully translationally invariant situation

$$G_2(\mathbf{x},\mathbf{x}') = G(\mathbf{x}-\mathbf{x}') \equiv \int_{\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{x}} \hat{G}_{\mathbf{q}} \ , \tag{2.8}$$

where, for brevity we write, here and below,

$$\int_{\mathbf{q}} \Longleftrightarrow \int \frac{d^dq}{(2\pi)^d} \ , \tag{2.9}$$

the wave vector running over an appropriate Brillouin zone, or more generally, carrying an upper cutoff $q_{\Lambda} \simeq O(1)$ (where it is convenient to regard $\mathbf{q}$ as dimensionless[7]). Then, in terms of the Fourier-transformed replica spin variables $\hat{s}_{i,\mathbf{q}}$ and of the Fourier transform $\hat{u}_{\mathbf{q}}$ of the pair potential $u(\mathbf{x})$, we find, in the usual way,[4,10]

$$\overline{\mathscr{H}}^{(p)} \simeq \tfrac{1}{2} \sum_{i=1}^{p} \int_{\mathbf{q}} (r+q^2)\, |\hat{\sigma}_{i,\mathbf{q}}|^2 + \sum_{i,j=1}^{p} \int_{\mathbf{k}} \int_{\mathbf{k}'} \int_{\mathbf{q}} (\hat{u}_{\mathbf{q}}\delta_{ij} - \tfrac{1}{8}\hat{G}_{\mathbf{q}})\, \hat{\sigma}_{i,\mathbf{k}+\mathbf{q}} \cdot \hat{\sigma}_{i,-\mathbf{k}} \hat{\sigma}_{j,\mathbf{k}'-\mathbf{q}} \cdot \hat{\sigma}_{j,-\mathbf{k}'} \ . \tag{2.10}$$

This provides a convenient basis for subsequent considerations.

Randomness correlator

The function $\hat{G}_{\mathbf{q}}$ describes the statistical properties of the random potential $w(\mathbf{x})$: what form should it take? A standard choice is to suppose $\langle\langle w(\mathbf{x})\rangle\rangle=0$ and to take

$$\langle\langle w(\mathbf{x})w(\mathbf{x}')\rangle\rangle=\tilde{w}^2\delta(\mathbf{x}-\mathbf{x}') , \tag{2.11}$$

which represents "white noise." Rather more generally one may specify the correlations in the randomness through a translationally invariant probability weight functional

$$\mathscr{P}[\mathscr{W}]=\mathscr{N}^{-1}\exp\left[-\int d^dx\int d^dx' w(\mathbf{x})K(\mathbf{x}-\mathbf{x}')w(\mathbf{x}')\right] \tag{2.12}$$

of Gaussian character in which $\mathscr{N}$ is a normalization factor. In terms of the kernel $K(\mathbf{x})$ one then finds

$$1/\hat{G}_{\mathbf{q}}=\hat{K}_{\mathbf{q}}\equiv\int d^dx\, e^{i\mathbf{q}\cdot\mathbf{x}}K(\mathbf{x}) ,\qquad G(\mathbf{x})=\int_{\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{x}}/\hat{K}_{\mathbf{q}} . \tag{2.13}$$

The choice (2.12), in fact, eliminates the higher-order terms in (2.5) since the third- and higher-order cumulants of a Gaussian distribution vanish identically. Non-Gaussian distributions will, however, lead to higher-order terms in (2.12) and, hence, higher-order terms will appear in (2.5) and (2.10).

To obtain a more concrete representation of *constrained randomness* we examine more closely the scattering center model sketched in the Introduction. If $\varphi(\mathbf{r})$ represents the potential due to one center, which we will assume is of short range (in a sense to be made more precise when necessary), the total random potential is taken as

$$w(\mathbf{x})=\sum_{i=1}^{N_0}\varphi(\mathbf{x}-\mathbf{R}_i) , \tag{2.14}$$

where $\mathbf{R}_i$, with $i=1,2,\ldots,N_0$, denotes the position of the ith center. If the centers are distributed with probability density $P(\{\mathbf{R}_i\})$, averages of a function $A(\{\mathbf{R}_i\})$ are defined by

$$\langle\langle A\rangle\rangle=\prod_{i=1}^{N_0}\int d^dR_i P(\{\mathbf{R}_i\})A(\{\mathbf{R}_i\}) . \tag{2.15}$$

Kohn and Luttinger[11] took $P=V^{-N_0}$, where V is the system volume, so that the centers were placed in a uniform, uncorrelated manner. On the other hand, for the lattice-based model envisaged in Sec. I, we can write

$$P(\{\mathbf{R}_i\})=\prod_{j=1}^{N_0}p(\mathbf{R}_j-\mathbf{R}_j^0) , \tag{2.16}$$

where $\mathbf{R}_j^0$ is the fixed reference position of the jth center and we will suppose that $\mathscr{L}\equiv\{\mathbf{R}_j^0\}$ is a regular space lattice. The single-center distribution $p(\mathbf{x})$ may be supposed to have mean $\mathbf{0}$ and will be characterized by some finite-range parameter b: see, e.g., the diffusion form (1.5) for $p(\mathbf{x})$ for which $b=(Dt_0)^{1/2}$.

If we define the characteristic function

$$\hat{p}_{\mathbf{q}}=\int d^dx\, e^{i\mathbf{q}\cdot\mathbf{x}}p(\mathbf{x})\equiv\langle\langle \exp[i\mathbf{q}\cdot(\mathbf{R}_1-\mathbf{R}_1^0)]\rangle\rangle , \tag{2.17}$$

all the required cumulants of $w(\mathbf{x})$ can be expressed in terms of $\hat{p}_{\mathbf{q}}$ and $\hat{\varphi}_{\mathbf{q}}$, the Fourier transform of $\varphi(\mathbf{x})$. Thus we find

$$\langle\langle \hat{w}_{\mathbf{q}}\rangle\rangle\equiv\int d^dx\langle\langle w(\mathbf{x})\rangle\rangle e^{i\mathbf{q}\cdot\mathbf{x}} =\rho_0\sum_{\mathbf{Q}\in\mathscr{Q}}\delta(\mathbf{q}-\mathbf{Q})\hat{\varphi}_{\mathbf{Q}}\hat{p}_{\mathbf{Q}} , \tag{2.18}$$

where $\mathscr{Q}$ is the reciprocal lattice of $\mathscr{L}\equiv\{\mathbf{R}_j^0\}$ and, as before, $\rho_0=N_0/V$. Similarly, we obtain

$$\hat{G}_2(\mathbf{q},\mathbf{q}')\equiv\int d^dx\int d^dx' e^{i(\mathbf{q}\cdot\mathbf{x}+\mathbf{q}'\cdot\mathbf{x}')}G_2(\mathbf{x},\mathbf{x}') =\rho_0\sum_{\mathbf{Q}\in\mathscr{Q}}\delta(\mathbf{q}+\mathbf{q}'-\mathbf{Q})\hat{\varphi}_{\mathbf{q}}\hat{\varphi}_{\mathbf{Q}-\mathbf{q}}(\hat{p}_{\mathbf{Q}}-\hat{p}_{\mathbf{q}}\hat{p}_{\mathbf{Q}-\mathbf{q}}) . \tag{2.19}$$

The last result differs from the fully translationally invariant situation, for which $\hat{G}_2(\mathbf{q},\mathbf{q}')=\delta(\mathbf{q}+\mathbf{q}')\hat{G}_{\mathbf{q}}$, by the appearance of nonzero reciprocal-lattice vectors $\mathbf{Q}$. However, in the same spirit that such umklapp terms were tacitly neglected in writing the form (2.10) for the replicated Hamiltonian, we anticipate that only the $\mathbf{Q}=0$ terms in (2.19) will be directly relevant.

Finally, therefore, if we neglect the $\mathbf{Q}\neq\mathbf{0}$ terms in (2.19) and note the normalization relation $\hat{p}_0=1$, we obtain the randomness correlator

$$\hat{G}_{\mathbf{q}}=\rho_0|\hat{\varphi}_{\mathbf{q}}|^2(1-|\hat{p}_{\mathbf{q}}|^2) . \tag{2.20}$$

Note that if $p(\mathbf{x})$ approaches a δ function, $\delta(\mathbf{x})$, one has $\hat{p}_{\mathbf{q}}\to1$ (all $\mathbf{q}$) and then $\hat{G}_{\mathbf{q}}$ vanishes identically as it should for a periodic array of fixed "scatterers" or "impurities." More generally, provided $\varphi(\mathbf{x})$ is integrable, as we will assume, and that $p(\mathbf{x})$ has a second moment, as in the example (1.5), we see directly that $\hat{G}_{\mathbf{q}}$ *vanishes when* $\mathbf{q}\to0$ as $\rho_0|\hat{\varphi}_0|^2b^2q^2$. This, clearly, represents the *constrained character* of the randomness in the model.

More generally, then, we will suppose

$$\hat{G}_{\mathbf{q}}=\overline{w}^2b^d\gamma(b\mathbf{q}) , \tag{2.21}$$

where $\overline{w}^2\equiv G_2(\mathbf{x},\mathbf{x})=G(\mathbf{0})$, while as $\mathbf{y}\to0$,

$$\gamma(\mathbf{y})\approx c|\mathbf{y}|^\theta \text{ with } \theta\geq0 . \tag{2.22}$$

A constrained system in which all the correlations and potentials are of short range will, evidently, be characterized by $\theta=2$. We believe such a model is appropriate for describing Vycor.

III. APPLICATION TO SUPERFLUID HELIUM

In Ref. 7 the crossover in critical behavior from an interacting Bose fluid, representing helium, to an ideal Bose fluid was studied. If g is a coupling parameter measuring the strength of the interparticle interactions, ideal behavior is approached when $g\to0$ and, on general grounds, the crossover should obey a scaling form; for example, the dominant ordering susceptibility, namely, the off-diagonal susceptibility,[7] should be described by

$$\chi(T,g)\approx t^{-\gamma_0}X(g/t^{\phi}) , \tag{3.1}$$

where

$$t=(T-T_c^0)/T_c^0 , \tag{3.2}$$

in which T_c^0 is the ideal critical temperature, while γ_0 is the ideal susceptibility exponent and ϕ is the crossover exponent for the interactions at the ideal critical point. For observations at constant chemical potential (which is the simplest situation conceptually), one has[7]

$$\gamma_0=1, \quad \phi=\tfrac{1}{2}(4-d) \quad \text{for } d\le 4 , \tag{3.3}$$

where we have supposed that g acts like u in (2.4).

Now the full form for the coupling constant g can be found by general considerations. If the (effective) particle-particle interaction potential (for helium within Vycor[7]) is $v(\mathbf{r})$, only the integral $v_0=\int d^dr\, v(\mathbf{r})$ should matter near the ideal limit; alternatively, if a^* is the (effective) scattering length, one has $v_0\propto(\hbar^2/2m^*)(a^*)^{d-2}$ where m^* is the effective mass. At nonzero temperature the interactions can enter only as v_0/k_BT. Finally, the only length that can be defined in the ideal Bose limit is the thermal de Broglie wavelength

$$\Lambda_T=h/(2\pi m^* k_BT)^{1/2} . \tag{3.4}$$

The dimensionless coupling constant must thus be

$$g\propto v_0/k_BT\Lambda_T^d\propto(a^*/\Lambda_T)^{d-2} , \tag{3.5}$$

as is confirmed by detailed calculations.[7] If, as seems reasonable, the temperature dependence of a^* and m^* is not anomalous,[7] one evidently has

$$g\approx Bv_0T_c^{\psi} \quad \text{with } \psi=\tfrac{1}{2}(d-2) , \tag{3.6}$$

for $d\le 4$, where B can be regarded as a constant. Thus when T_c is driven to zero by reducing the filling fraction of helium in Vycor,[7-9] the coupling constant g also vanishes (for $d>2$) and ideal behavior is approached.

Now if g_R characterizes the strength, supposed not too large, of the *randomness,* we may extend (3.1) to the scaling form

$$\chi(T,g,g_R)\approx t^{-\gamma_0}X(g/t^{\phi};g_R/t^{\phi_R}) , \tag{3.7}$$

in which ϕ_R is the crossover exponent for randomness *about* the *ideal* critical point. If ϕ_R is *negative* the randomness is *irrelevant* since the scaling combination $y_R=g_R/t^{\phi_R}\equiv g_Rt^{|\phi_R|}$ vanishes as $t\to 0$ and (3.1) is asymptotically recaptured. *Positive* ϕ_R, however, implies that the randomness is *relevant* and it can then be neglected *only if* g_r/t^{ϕ_R} remains sufficiently small in the accessible critical region: small g_R *and* small ϕ_R may, in fact, lead to such a situation as we now demonstrate.

One must clearly enquire as to the explicit form of g_R. Following (3.6) we may anticipate a dependence

$$g_R\approx B_R\overline{w}_0^2T_c^{\psi_R} , \tag{3.8}$$

where ψ_R could, *a priori,* be positive, negative, or zero. Next, notice that in place of the pair of scaled variables $y=g/t^{\phi}$ and $y_R=g_R/t^{\phi_R}$ in (3.7) one could use y and the new variable

$$z=y_R/y^{\phi_R/\phi}=g_R/g^{\phi_R/\phi} , \tag{3.9}$$

which is independent of t and hence serves to measure the importance of the randomness *at* criticality. On combining (3.6) and (3.8) we get

$$z=B_z\overline{w}_0^2T_c^{\omega} \quad \text{with } \omega=\psi_R-\psi\phi_R/\phi . \tag{3.10}$$

From this we see that it is possible for the randomness to remain effectively irrelevant in the situation that $T_c\to 0$ *even if* ϕ_R is positive so that the randomness is actually relevant (when regarded as an *independent* parameter). The required condition for *effective irrelevance* is $\omega>0$ or

$$\psi_R/\psi>\phi_R/\phi , \tag{3.11}$$

where we have supposed $\psi>0$ as, in fact, applies in (3.6) for $d>2$; evidently $\psi_R>0$ is a necessary but not sufficient condition. [To complete the argument one should recall that $\phi_R^{(c)}$, the randomness exponent for interacting ($g\neq 0$) criticality, is negative, although small, so that crossover from ideality still takes place to normal superfluidity as described by a pure XY-like, $n=2$ fixed point.[7]]

To see if such a scenario is pertinent, suppose that the effective random potential acting on helium in Vycor is $w_0(\mathbf{r})$. As before, this can enter at criticality only as $w_0(\mathbf{r})/k_BT$. However, as seen, the effects of the randomness will be controlled by the corresponding correlator $\hat{G}_{\mathbf{k}}^0$ which, in analogy to (2.21) and (2.22), varies as $(\overline{w}_0/k_BT)^2b_0^d\,|b_0\mathbf{k}|^{\theta}$ when $\mathbf{k}\to\mathbf{0}$, where b_0 is the range of correlation of the randomness and one has $\theta>0$ for constrained randomness. Again, near the ideal Bose limit lengths must be scaled by Λ_T. Hence, we conclude that the dimensionless coupling constant describing the randomness can be taken as

$$g_R=\frac{\overline{w}_0^2}{(k_BT)^2}\left[\frac{b_0}{\Lambda_T}\right]^{d+\theta}\propto\overline{w}_0^2\left[\frac{b_0^2m^*}{\hbar^2}\right]^2\left[\frac{b_0}{\Lambda_T}\right]^{d+\theta-4} . \tag{3.12}$$

This is also confirmed by detailed calculations (extending Ref. 7). From this follows $\psi_R=\frac{1}{2}(d+\theta-4)$ so that, in the simplest constrained case, one has $\psi_R=\frac{1}{2}(d-2)$ which is positive (for $d>2$). Thus the scenario of effective irrelevance is not, *a priori,* excluded. Note, incidentally, that ordinary randomness, described by $\theta=0$, entails $\psi_R=-\frac{1}{2}(4-d)$ which is negative (for $d<4$) so that the effects of randomness are actually *enhanced* as $T_c\to 0$.

To complete the picture we must, evidently, obtain ϕ_R: in doing so, however, we will learn that the scaling formulation presented here is oversimplified in a manner that is normally harmless; specifically, we have neglected the *nonlinear* aspects of the full scaling fields which, among other features, can entail the *mixing* of different linear scaling fields. However, the detailed analysis will show that nonlinearity plays a crucial role when the randomness is constrained.

IV. RELEVANCE OF CONSTRAINED RANDOMNESS

In discussing the relevance of constrained randomness, with helium in Vycor in mind, it is instructive to consider first a natural extension of the original self-consistency argument given by Harris.[1] Then we will indicate how the simplest scaling arguments yield the same criterion. Finally, however, we will see that these arguments are too naive: renormalization-group ϵ-expansion calculations and properly extended Harris and scaling arguments yield the correct answers.

A. Harris argument reformulated

Harris argues[1] that inhomogeneous randomness $\mathscr{W}$ leads to a local shift $\delta T_c(\mathbf{r};\mathscr{W})$ in the critical temperature which depends on temperature through an averaging of the random potential over distances proportional to the correlation length $\xi(T)$ which diverges as $t^{-\nu}$ in the "pure" or nonrandom system ($\nu\equiv\nu_0$). If, as $t\to 0$, the root-mean-square deviation in reduced critical temperature, say Δt_c, defined via

$$(\Delta t_c)^2=\langle\langle[\delta T_c(\mathbf{r})/T_c]^2\rangle\rangle-\langle\langle\delta T_c(\mathbf{r})/T_c\rangle\rangle^2 \tag{4.1}$$

is of magnitude smaller than t, then, it is concluded, the randomness is irrelevant; conversely, if $\Delta t_c/t\to\infty$ as $t\to 0$, the randomness will be relevant.

To formalize this argument, we postulate that the local shift in critical temperature can be expressed in terms of the random potential $w_0(r)$ via

$$\Delta T_c(\mathbf{r};\mathscr{W})/T_c=\int\frac{d^dr'}{\xi^d(T)}\Theta\left[\frac{\mathbf{r}-\mathbf{r}'}{\xi(T)}\right]\frac{w_0(\mathbf{r}')}{k_BT_c}, \tag{4.2}$$

in which $\Theta(\mathbf{x})$ is a short-range smoothly varying kernel with a Fourier transform $\hat{\Theta}(\mathbf{q})$. [Note that a sharp cutoff or nondifferentiability in $\Theta(x)$ would be unphysical.] Then, following the notation of (2.21), we find

$$\begin{aligned}(\Delta t_c)^2&=\int_{\mathbf{k}}|\hat{\Theta}(\mathbf{k}\xi)|^2\hat{G}^0_{\mathbf{k}}/(k_BT_c)^2,\\&=(\bar{w}_0/k_BT_c)^2(b_0/\xi)^d\Gamma_0(b_0/\xi),\end{aligned} \tag{4.3}$$

where, as before, b_0 is the range of the randomness and the form of $\hat{G}^0_{\mathbf{k}}$, the correlator for the randomness, enters through

$$\Gamma_0(x)=\int_{\mathbf{q}}|\hat{\Theta}(\mathbf{q})|^2\gamma_0(x\mathbf{q}). \tag{4.4}$$

For constrained randomness we have, following (2.22), $\gamma_0(\mathbf{y})\approx c_0|\mathbf{y}|^\theta$. Since $\Theta(\mathbf{x})$ is smooth, the moments

$$\Theta_l\equiv\int_{\mathbf{q}}|\mathbf{q}|^l|\hat{\Theta}(\mathbf{q})|^2 \tag{4.5}$$

exist so that we finally obtain

$$(\Delta t_c)^2\approx c_0\Theta_\theta(\bar{w}_0/k_BT_c)^2[b_0/\xi(T)]^{d+\theta}, \tag{4.6}$$

as $\xi\to\infty$.

When $\theta=0$ this result yields $\Delta t_c\sim 1/(V_\xi)^{1/2}$ where V_ξ is the volume of a sphere of radius equal to the correlation length. If one regards the randomness as arising from an uncorrelated distribution of scattering centers of density ρ_0, as discussed in Sec. II, then this simply represents the expected statistical fluctuation in the number N_ξ of centers within V_ξ. A constraint enforces a more uniform distribution of centers and hence reduces the mean fluctuations in N_ξ by a factor proportional to $1/\xi^{\theta/2}$.

The Harris criterion for the *irrelevance* of randomness, namely, $\Delta t_c/t\to 0$, now yields

$$(d+\theta)\nu>2 \quad\text{or}\quad \alpha-\theta\nu<0\,. \tag{4.7}$$

The second inequality here follows from the hyperscaling relation $d\nu=2-\alpha$ in which $\alpha\equiv\alpha_0$ is the specific-heat exponent in the pure system; hyperscaling should be valid for d less than the upper borderline dimensionality $d_>$ ($=4$, for simple criticality). For simple constrained randomness, i.e., $\theta=2$, the modified criterion (1.2) is reproduced: Since one always has $\alpha<1$ and since $\nu>\frac{1}{2}$ holds when $d<d_>$, one is led to believe that simple constrained randomness would *always* be irrelevant! To see how far such a conclusion might be trusted we consider a scaling argument using the replica Hamiltonian approach of Sec. II.

B. Scaling analysis

We may follow a discussion presented by Aharony.[4] If $f_s(T,g_R)$ denotes the singular part of the free energy of a system in which g_R measures the strength of the randomness we anticipate, as in Sec. III, the scaling behavior

$$f_s\approx t^{2-\alpha}W(g_R/t^{\phi_R})\,. \tag{4.8}$$

If we examine the first derivative $Q(T,g_R)$ of the free energy with respect to the randomness and evaluate it in the limit of the pure system $g_R=0$, we must thus expect a singularity of the form

$$Q_s(T)=(\partial f_s/\partial g_R)_0\approx W_1t^{2-\alpha-\phi_R}, \tag{4.9}$$

where the subscript zero denotes $g_R=0$, while $W_1=(dW/dy)_{y=0}$.

On the other hand, if we use the replicated Hamiltonian (2.5) and (2.6) and note, as explained, that g_R measures the amplitude of the randomness correlator $\hat{G}_{\mathbf{q}}$, we obtain, on computing the free energy,

$$\begin{aligned}g_RQ(T)&=\lim_{p\to 0}\frac{1}{8p}\sum_{i,j=1}^{p}\int d^dx\,G(\mathbf{x})\\&\qquad\times\langle|\vec{\sigma}_i(\mathbf{x})|^2|\vec{\sigma}_j(\mathbf{0})|^2\rangle_0,\\&=\lim_{p\to 0}\tfrac{1}{8}\int d^dx\,G(\mathbf{x})[\langle|\vec{\sigma}(\mathbf{x})|^2|\vec{\sigma}(\mathbf{0})|^2\rangle_0\\&\qquad+(p-1)\langle|\vec{\sigma}(\mathbf{0})|^2\rangle_0^2],\end{aligned} \tag{4.10}$$

in which we have first used translational invariance and then the equivalence and independence of the different replicas when $g_R\to 0$ (which is certainly valid for $t>0$). In the limit $p\to 0$ the correlations in square brackets reduce simply to the energy-energy correlation function in the pure $g_R=0$ system, say, $C_{\mathscr{E}}(\mathbf{x})$. The spatial integral of $C_{\mathscr{E}}(\mathbf{x})$ yields the specific heat and hence the Fourier transform obeys the scaling relation[21]

$$\hat{C}_{\mathscr{E}}(t,\mathbf{q})\approx t^{-\alpha}\hat{Z}(q\xi)\,, \tag{4.11}$$

as $t\to 0$. On using this, (4.10) yields

$$g_R Q(T) \approx \tfrac{1}{8} t^{-\alpha} \int_{\mathbf{q}} \hat{G}_{\mathbf{q}} \hat{Z}(q\xi) = \tfrac{1}{8} t^{-\alpha} \bar{w}^{2} b^{d} \int_{\mathbf{q}} \gamma(b\mathbf{q}) \hat{Z}(q\xi) , \quad (4.12)$$

in which we have invoked (2.21). For constrained randomness with $\gamma(\mathbf{y}) \approx c \mid \mathbf{y} \mid^{\theta}$ as $\mathbf{y} \to 0$, this finally leads to

$$Q_s(T) = t^{-\alpha} \xi^{-d-\theta} \int_{\mathbf{z}} \mid \mathbf{z} \mid^{\theta} \Delta \hat{Z}(z) , \quad (4.13)$$

in which $\Delta\hat{Z}(z)$ is equal to $\hat{Z}(z)$ except for the subtraction[22] of the long-range parts, which decay slowly as

$$z^{-\alpha/\nu}(Z_0 + Z_{1-\alpha}/z^{(1-\alpha)/\nu} + \cdots + Z_k/z^{k/\nu} + \cdots) ,$$

with $k = 1,2, \ldots, k_{\max}$ and $k_{\max} < (d+\theta)\nu - \alpha$.

Now comparison of (4.13) with (4.9) yields the crossover exponent

$$\phi_R = 2 - (d+\theta)\nu = \alpha - \theta\nu , \quad (4.14)$$

the second part following, as before, from hyperscaling. The criterion for irrelevance is $\phi_R < 0$ which thus reproduces (4.7).

C. Renormalization-group recursion relations

Weinrib and Halperin[6] have recently analyzed the Hamiltonian (2.10) using renormalization-group theory. They were concerned primarily with the effects of randomness characterized by *long-range power-law correlations.* This translates into a correlator $\hat{G}_{\mathbf{q}}$ which *diverges* as $\mathbf{q} \to 0$ and hence corresponds to $\theta < 0$ in (2.22). For this situation they find, within an $[\varepsilon = (4-d)]$-expansion approach, that (4.14) is *valid.* For $\theta > 0$, however, one must reexamine the differential recursion relations which they calculated explicitly to first order in ε (see the Appendix of Ref. 6) but which are valid for general θ. If l is the renormalization-group flow parameter, the recursion relation for the full randomness correlator, $\hat{G}_{\mathbf{q}}(l)$, can be written[6]

$$\frac{d\hat{G}_q}{dl} = \delta(q)\hat{G}_q - \frac{8(n+2)}{(1+r)^2}\hat{u}_0\hat{G}_q + \frac{16}{(1+r)^2}\hat{G}_q\hat{G}^{\Lambda} + \frac{16}{(1+r)^2}(\hat{G}^{\Lambda})^2 , \quad (4.15)$$

where spherical symmetry has been assumed, while $\hat{G}^{\Lambda}$ denotes $\hat{G}_q$ *evaluated at the cutoff* $q = q_{\Lambda} = O(1)$ and

$$\delta(q) = \epsilon - \partial \ln \hat{G}_q / \partial \ln q . \quad (4.16)$$

Now suppose the initial, $l = 0$ Hamiltonian has constrained randomness so that $\hat{G}_{q=0}(l=0)$ vanishes identically. The crucial deduction from (4.15) is for the initial flow rate which is given by

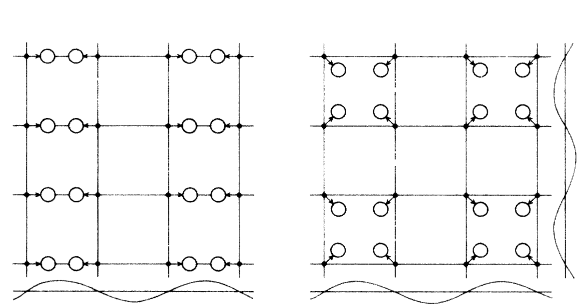
$$\frac{d\hat{G}_{q=0}}{dl} = \frac{16}{(1+r)^2}(\hat{G}^{\Lambda})^2 \quad (l=0) . \quad (4.17)$$

Thus, $\hat{G}_{q=0}$ is renormalized away from zero by fluctuations in the randomness with wave numbers *at* the cutoff. In other words, $\hat{G}_{q=0}(l)$ is *not* a constant of the renormalization-group flows; on the contrary, under renormalization, constrained randomness becomes *un*constrained and the critical behavior will hence be determined by the same fixed point as for normal short-range randomness. Note, however, that the departure from the pure situation is only *quadratic* in the overall strength of the randomness: this observation justifies the conclusion (1.3), namely, $\phi_R = \frac{1}{2}\alpha$ (for $\theta \geq 0$). Evidently the Harris and scaling arguments are inadequate as given: the true criterion for irrelevance for general θ ($\gtrless 0$) should read

$$\max\{\alpha - \theta\nu, \alpha\} < 0 . \quad (4.18)$$

D. Intuitive picture

As discovered here, the effective destruction of the constraint in the randomness by short wavelength fluctuations of the potential $w_0(\mathbf{r})$ is a purely formal renormalization-group result. However, it can be given an intuitive interpretation. The issue is why the root-mean-square fluctuations in $\delta T_c(\mathbf{r})$ are still proportional to the averaging volume, even though, in the point-impurity or scattering-center model, say, the random displacements are small so that deviations from local uniformity in the random potential are restricted [e.g., by small Dt_0 in (1.5)]. To answer this question, consider Fig. 1 in which the solid dots represent the regular lattice array, at density ρ_0, of the mean positions of scattering centers constrained to have only small displacements, say, less than $\frac{1}{3}c_0$, with $c_0 = \rho_0^{1/d}$ the lattice spacing. Figs. 1(a) and 1(b) illustrate two coherent patterns of local displacements of the scattering centers (open circles) which, if extended uniformly to the whole lattice, would result in critical temperatures $T_c^{(a)}$ and $T_c^{(b)}$, almost certainly differing somewhat from the critical point $T_c^{(0)} \equiv T_c(\rho_0)$ of an array with all scatterers *on* the reference lattice site. As indicated by the marginal sine waves, both patterns correspond to fluctuations of the random potential $w_0(\mathbf{r})$, with wave vectors on the edge of the Brillouin zone (which, at some stage of renormalization, will become the cutoff). Evidently, the independent, random-but-constrained placement of the centers will generate patterns which approximate quite closely those in Fig. 1 over finite regions of space. Furth-



(a) (b)

FIG. 1. Two regular configurations of the array of "scattering centers" (open circles) which satisfy the requirements of constrained randomness with respect to the underlying reference lattice (solid dots). The marginal sine waves indicate that such configurations represent short wavelength fluctuations in the randomness. If extended to an infinite system the corresponding critical temperatures $T_c^{(a)}$ and $T_c^{(b)}$ will differ, if only slightly, from that of the regular lattice array.

ermore, the typical sizes of such $T_c^{(a)}$ or $T_c^{(b)}$ domains will clearly scale as the square root of the volume examined. Thus the fluctuations in $\delta T_c(\mathbf{r})$ will once again follow unconstrained or Poisson statistics. Nevertheless, the *scale* of δT_c, which should be of order $(T_c^{(a)}-T_c^{(0)})$ or $(T_c^{(b)}-T_c^{(0)})$, must be much smaller than that generated by the unconstrained placement of the centers, which might be estimated as $[T_c(\frac{1}{2}\rho_0)-T_c^{(0)}]$ or even $[T_c(0)-T_c^{(0)}]$.

E. Harris argument revisited

It is interesting, even after the fact, to see how our extension of the Harris argument, if pushed a step further, can yield the correct results. The first point is that the expression (4.2) for the local temperature shift $\delta T_c(\mathbf{r};\mathscr{W})$ is too naive in that it supposes that the shift can be adequately represented by a *linear* functional of the random potential $w_0(\mathbf{r})$. Let us, instead, regard (4.2) as merely the first-order functional expression for $\delta T_c(\mathbf{r};\mathscr{W})$ and add to the right-hand side a second-order correction of the form

$$\delta T_c^{(2)}(\mathbf{r};\mathscr{W})/T_c=\int\int \frac{d^dr\,d^dr'}{[a\xi(T)]^d}\Theta_2\left[\frac{\mathbf{r}-\mathbf{R}}{\xi};\frac{\mathbf{r}'-\mathbf{r}''}{a}\right]\times\frac{w_0(\mathbf{r}')}{k_BT_c}\frac{w_0(\mathbf{r}'')}{k_BT_c}\,, \tag{4.19}$$

where $\mathbf{R}=\frac{1}{2}(\mathbf{r}+\mathbf{r}')$. The kernel $\Theta_2(\bar{\mathbf{x}};\mathbf{x})$ entails, as in (4.2), the correlation length ξ, and thus embodies Harris's conception that the local critical point is determined by a spatial average on the scale of $\xi\propto a/t^{\nu}\rightarrow\infty$. The second argument, however, in which a denotes a *lattice spacing* or *reciprocal cutoff* for the spin degrees of freedom, embodies the idea that the effective randomness can be modulated via short-range correlations mediated by the coupling of "neighboring" spins separated only on the scale a. This picture really entails some temperature dependence in the kernel $\Theta_2(\mathbf{x};\mathbf{x}')$ [and, perhaps, also in $\Theta(\mathbf{x})$] but we will suppose this is not crucial in the critical neighborhood.

Now assuming, for simplicity, that the odd cumulants of $w_0(\mathbf{r})$ vanish, the first-order expression (4.3) for the variance $(\Delta t_c)^2$ must be supplemented by a second-order term

$$(\Delta t_c)^2_{(2)}=\int d^dR\int d^dR'\int d^dr\int d^dr'\,G_4^0(\mathbf{R}-\mathbf{R}';\mathbf{r},\mathbf{r}')\Theta_2\left[\frac{\mathbf{R}}{\xi'};\frac{\mathbf{r}}{a}\right]\Theta_2\left[\frac{\mathbf{R}'}{\xi};\frac{\mathbf{r}'}{a}\right]\Big/(\xi a)^{2d}(k_BT_c)^4\,, \tag{4.20}$$

in which, on using translational invariance, one has

$$G_4^0(\mathbf{R}-\mathbf{R}',\mathbf{r},\mathbf{r}')=\langle\langle w_0(\mathbf{R}+\tfrac{1}{2}\mathbf{r})w_0(\mathbf{R}-\tfrac{1}{2}\mathbf{r})w_0(\mathbf{R}'+\tfrac{1}{2}\mathbf{r}')w_0(\mathbf{R}'-\tfrac{1}{2}\mathbf{r}')\rangle\rangle-G^0(\mathbf{r})G^0(\mathbf{r}')\,. \tag{4.21}$$

This four-point randomness correlator will, like $G^0(\mathbf{r})$, be supposed to decay rapidly on the scale b_0 so that contributions to (4.20) with $|\mathbf{R}-\mathbf{R}'|\gg b_0$ will be unimportant. Furthermore, the Θ_2 factors will cut off the integration when $|\mathbf{r}|,\ |\mathbf{r}'|\gg a$. Consequently, the leading contributions to $(\Delta t_c)^2_{(2)}$ when $\xi\rightarrow\infty$ can be estimated by replacing $\mathbf{R}'$ by $\mathbf{R}$ in the second Θ_2 factor in (4.20). If we then put

$$\tilde{G}_4\left[\frac{\mathbf{r}}{b_0},\frac{\mathbf{r}'}{b_0}\right]=\int\frac{d^dR}{b_0^d}G_4^0(\mathbf{R}-\mathbf{R}';\mathbf{r},\mathbf{r}')\,, \tag{4.22}$$

$$\tilde{\Theta}_2\left[\frac{\mathbf{r}}{a},\frac{\mathbf{r}'}{a}\right]=\int\frac{d^dR}{\xi^d}\Theta_2\left[\frac{\mathbf{R}}{\xi};\frac{\mathbf{r}}{a}\right]\Theta_2\left[\frac{\mathbf{R}}{\xi};\frac{\mathbf{r}'}{a}\right]\,, \tag{4.23}$$

we finally obtain

$$(\Delta t_c)^2_{(2)}\approx\int\frac{d^dr}{a^d}\int\frac{d^dr'}{a^d}\left[\frac{b_0}{\xi}\right]^d\tilde{\Theta}_2\left[\frac{\mathbf{r}}{a},\frac{\mathbf{r}'}{a}\right]\tilde{G}_4\left[\frac{\mathbf{r}}{b_0},\frac{\mathbf{r}'}{b_0}\right]\times(k_BT_c)^{-4}\,. \tag{4.24}$$

We conclude[23] that the second-order contribution to $(\Delta t_c)^2$ is of order $(b_0/\xi)^d$ whatever the value of θ (≥ 0); by contrast, the first-order contribution varies as $(b_0/\xi)^{d+\theta}$, as seen in (4.6). Consequently, when the randomness is constrained the second-order term dominates and the original Harris criterion for irrelevance, namely, $\alpha<0$, is recaptured. However, the strength of the randomness is now measured by $(\bar{w}_0/k_BT_c)^4(b_0/a)^d$, whereas previously, for $\theta=0$, it was measured by $(\bar{w}_0/k_BT_c)^2(b_0/a)^d$ (where we have used $\xi\propto a/t^{\nu}$). The appearance of the factor $(b_0/a)^d$ for constrained randomness, rather than its square, will be significant when we return to consider helium in Vycor.

F. Scaling revisited

Let us lastly return to the scaling argument: where was it defective? Two features were oversimplified to a degree that was damaging. *First,* randomness was represented by a single scaling argument, whereas there should be many arguments, one for each potentially relevant or irrelevant variable. Thus, if we write, in the simplest short-range situation,

$$\hat{G}_q=\hat{G}_0+\tfrac{1}{2}\hat{G}''q^2+O(q^4)\,,$$
$$\equiv\Gamma_0\bar{w}^2b^d-\Gamma_2(\bar{w}^2b^{d+2})q^2+O(q^4)\,, \tag{4.25}$$

one should expect *distinct* dimensionless randomness variables or scaling fields

$$g_{R,0}=\Gamma_0\bar{w}^2(b/a)^d,\quad g_{R,2}=\Gamma_2\bar{w}^2(b/a)^{d+2}\,, \tag{4.26}$$

etc., with distinct crossover exponents $\phi_{R,0}=\alpha$, $\phi_{R,2}=\alpha-2\nu,\ldots$.

Second, the scaled arguments should not simply involve the variables $g_{R,0}$, $g_{R,2}$, etc., but rather allowance must be made for *nonlinear scaling fields*[24] which will be of the form

$$\tilde{g}_{R,0}=g_{R,0}(1+e_{00}g_{R,0}+e_{02}g_{R,2}+\cdots)+e_2(g_{R,2})^2+\cdots\,, \quad (4.27)$$

$$\tilde{g}_{R,2}=g_{R,2}(1+e_{20}g_{R,0}+e_{22}g_{R,2}+\cdots)+\cdots\,, \quad (4.28)$$

etc., where e_{00}, $e_{02}, \ldots$ are nonuniversal coefficients. The last term shown in (4.27) will prove to be crucial in the present context; further nonlinear terms involving t, g, etc., have not been displayed. In total, the scaling form (3.7) should thus be extended to

$$\mathcal{X}(T,g,g_{R,0},g_{R,2})\approx\frac{1}{\tilde{t}^{\gamma_0}}X\left[\frac{\tilde{g}}{\tilde{t}^{\phi}};\frac{g_{R,0}+e_2(g_{R,2})^2+\cdots}{\tilde{t}^{\alpha}};\frac{\tilde{g}_{R,2}}{\tilde{t}^{\alpha-2\nu}}\right], \quad (4.29)$$

where the further irrelevant variables have been neglected.

Now if Γ_0 and hence $g_{R,0}$ does not vanish, i.e., if the randomness is unconstrained, and if $\overline{w}^2$ is regarded as measuring the strength of the randomness, one may, in fact, drop the variable $g_{R,2}$ *and* the last, irrelevant argument in (4.29). This leads back to the original formulation (3.7) with $g_R \cong g_{R,0}$ and a crossover exponent $\phi_R=\alpha$. On the other hand, if $\Gamma_0\equiv 0$ so that the randomness is constrained, matters change drastically: the last argument may again be dropped as it is still irrelevant; however, the second argument becomes, in leading order, $e_2(g_{R,2})^2/\tilde{t}^{\alpha}$. This can be written in the original form (3.7) but with $\phi_R=\frac{1}{2}\alpha$, as in (1.3), and

$$g_R=(e_2)^{1/2}g_{R,2}=\Gamma_2(e_2)^{1/2}\overline{w}^2(b/a)^{d+2}\,. \quad (4.30)$$

The nonlinear mixing coefficient e_2 must come from more detailed calculations: it is a pure number but since it depends intimately on the randomness it could be a function of the ratio b/a. Indeed, reference back to (4.24) and the subsequent discussion (with the correspondences $b_0\cong b$, $\overline{w}_0/k_BT_c\cong\overline{w}$) suggests that $e_2(g_{R,2})^2$ should correspond to $\overline{w}^4(b/a)^d$ which implies $e_2\propto(a/b)^d$. This leads to the conclusion

$$g_R=\overline{\Gamma}_2\overline{w}^2(b/a)^{(d/2)+2}\,, \quad (4.31)$$

in which $\overline{\Gamma}_2$ is a numerical coefficient.

The power of b/a appearing here differs from both the anticipated forms in (4.26). This proves to be significant when considering a Bose fluid in a random potential. Accordingly, it is of interest to check the result by an alternative, more definitive route. We can accomplish this by carrying through a scaling-cum-perturbation calculation, analogous to that sketched in Sec. II B, on the basis of the replicated Hamiltonian (2.10) for the case $\Gamma_0=0$ with $\Gamma_2\neq 0$ [in (4.25)]. Note, however, that a *second derivative* with respect to the randomness strength $\overline{w}^2$ is now required. If, following Ref. 7(b), we work with the susceptibility $\mathcal{X}$, we must seek a term in $\partial^2\mathcal{X}/(\partial\overline{w}^2)^2$ diverging as

$$\mathcal{A}(a,b)t^{-\gamma_0-2\phi_R}=\mathcal{A}(a,b)t^{-(\gamma_0+\alpha)}\,, \quad (4.32)$$

where we have used $\phi_R=\frac{1}{2}\alpha$ which leads to $\gamma_0+\alpha=1+\frac{1}{2}(4-d)=\frac{1}{2}(6-d)$: in terms of the amplitude we then have $g_R^2=(\overline{w}^2)^2\mathcal{A}(a,b)/X_2$ in which X_2 is an unimportant numerical coefficient [deriving in part from the scaling function X in (4.29)]. The variable t here is to be identified as ra^2 in (2.10) with $q_\Lambda=\pi/a$.

The appropriate term in the perturbation theory corresponds to the graph shown in Fig. 2 which yields a contribution to $\mathcal{X}$ of

$$I(r;a,b)=\frac{a^{2d-10}}{r^2}\int_{\mathbf{q}}\int_{\mathbf{q}'}\frac{\hat{G}_{\mathbf{q}}\hat{G}_{\mathbf{q}'-\mathbf{q}}}{(r+q^2)^2(r+q'^2)}\,, \quad (4.33)$$

where the factor a^{2d-10} ensures the dimensionless character of I. Repeated use of the identity

$$1/(r+k^2)=1/k^2-r/k^2(r+k^2) \quad (4.34)$$

isolates the required powers of r in the various terms; when one substitutes $\hat{G}_q=\overline{w}^2(b/a)^d\gamma(bq)$, as in (2.21), and uses $\gamma(y)\approx\Gamma_2y^2$ as $y\to 0$, one obtains[25]

$$\Delta I=-J_d\frac{\overline{w}^4b^{2d}a^{-d-4}}{(ra^2)^{(6-d)/2}}\int_{\mathbf{q}}\left[\frac{\gamma^2(bq)}{q^4}+\tfrac{1}{2}d\Gamma_2b^2\frac{\gamma(bq)}{q^2}\right],$$

$$=-J_d'\frac{\overline{w}^4(b/a)^{d+4}}{t^{\gamma_0+2\phi_R}}\times\int_0^{\pi b/a}\frac{\gamma(x)}{x^2}\left[\frac{\gamma(x)}{x^2}+\tfrac{1}{2}d\Gamma_2\right]x^{d-1}dx\,, \quad (4.35)$$

in which J_d and J_d' are numerical constants. Note that the integral converges at the origin for all $d>0$ since $\Gamma_0\equiv 0$; in addition, it converges at the upper limit when $b/a\to\infty$ since $\hat{G}_{\mathbf{q}}\propto\gamma(bq)$ is integrable. Thus, (4.35) confirms the surmise (4.31) for the dependence of g_R on b/a for $b\gtrsim a$.[26]

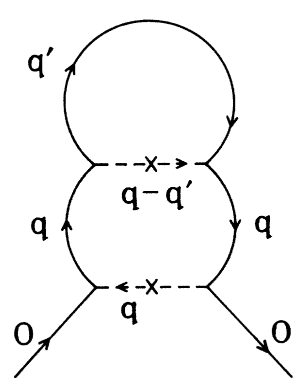


FIG. 2. The irreducible second-order perturbation diagram in the randomness which yields the dominant crossover correction to the susceptibility.

In summary, a careful scaling argument for constrained randomness yields the reduced crossover exponent value $\phi_R = \frac{1}{2}\alpha$ for a dimensionless randomness variable given by (4.31).

G. Near the ideal Bose limit

Finally, let us bring the analysis to bear on the scaling of randomness near the ideal Bose fluid limit as discussed generally in Sec. III. To complete the picture we need, in addition to the result $\phi_R = \frac{1}{2}\alpha$, an expression for the exponent ψ_R, defined in (3.8), for the case of constrained randomness: this must correct the naive result (3.12).

If we accept (4.31) and make the substitutions $\overline{w} = \overline{w}_0 / k_B T$ and $a = \Lambda_T$, which is the result suggested for the cutoff by the Bose-fluid to spin-system matching developed in Ref. 7, we obtain

$$g_R \propto \frac{\overline{w}_0^2}{(k_B T)^2} \left[\frac{b_0}{\Lambda_T} \right]^{(d/2)+2} \propto \overline{w}_0^2 \left[\frac{b_0^2 m^*}{\hbar^2} \right]^{(d/4)+1} T_c^{(d/4)-1} , \quad (4.36)$$

for $T \simeq T_c$. This corresponds to $\psi_R = \frac{1}{4}d - 1$ and hence, by (3.10), to the result

$$\omega = \tfrac{1}{4}d - 1 - \tfrac{1}{2}(d-2)(\phi_R/\phi) = -\tfrac{1}{2} \ . \quad (4.37)$$

Because this is negative the effective irrelevance criterion found in Sec. III for $T_c \to 0$ is *violated*, i.e., the randomness remains *relevant* at ideal-Bose-gas criticality. Indeed, the scaling combination $z = g_R / g^{1/2} \propto \overline{w}_0^2 v_0^{-1/2} T_c^{\omega}$ diverges when $T_c \to 0$ (although more weakly than for unconstrained randomness). Thus even though constrained randomness leads to a smaller crossover exponent (and a smaller amplitude) it is still a relevant perturbation in crossover from ideal Bose behavior.

It remains, however, to check the validity of the substitution of Λ_T for a in (4.31) since this would actually yield a different result if made directly in the perturbation integral in (4.35) because b/a vanishes when $T_c \to 0$.[26] To this end one may perform the scaling-cum-perturbation analysis directly for a weakly interacting Bose fluid in a random field. The replica approach goes through quantum mechanically and yields,[25] in leading order, an interaction term

$$\overline{\mathcal{U}}^{(p)} = -\tfrac{1}{2}\beta \sum_{i,j=1}^{p} \int d^d r \int d^d r' \int^{\beta} d\tau \int^{\beta} d\tau' \hat{\psi}_i^{\dagger}(\mathbf{r},\tau)\hat{\psi}_i(\mathbf{r},\tau)\hat{\psi}_j^{\dagger}(\mathbf{r}',\tau')\hat{\psi}_j(\mathbf{r}',\tau')[v(\mathbf{r}-\mathbf{r}')\delta_{ij}\delta(\tau-\tau') - G^0(\mathbf{r}-\mathbf{r}')] , \quad (4.38)$$

in the analog of the replicated spin Hamiltonian displayed in (2.10). In this expression $v(\mathbf{r})$ and $G^0(\mathbf{r})$ denote, as before, the particle pair potential and the correlator of the random potential; the variable τ is the time-ordering parameter and $\psi_i^{\dagger}(\mathbf{r},\tau)$ and $\psi_j(\mathbf{r},\tau)$ are the corresponding replicated second-quantized wave-function operators. Note that there is no lattice structure or corresponding momentum space cutoff; however, the inverse propagator in momentum space takes the form[7] $(ik_n - \varepsilon_{\mathbf{k}} + \mu)$ where μ is the chemical potential, $\varepsilon_k = \hbar^2 k^2 / 2m^*$, and the $k_n = 2\pi n / \beta$ with $n = 0, \pm 1, \pm 2, \ldots$ are the Matsubara frequencies, conjugate to the variable τ, over which a sum must be taken. The effective interaction in (4.38) arising from the interparticle interactions evidently carries a factor $\delta(\tau - \tau')$ and so is "local" in τ: Fig. 3(a) shows the corresponding Fourier-space vertex. When the perturbation theory is implemented and compared with the corresponding lattice spin theory one finds, as a result, that the coupled Matsubara sums (or restricted τ integrals) yield an *effective* momentum cutoff $q_{\Lambda} = \pi / a$ with $a \propto \Lambda_T$. On the other hand, the random potential is "static" and hence "nonlocal" in τ, i.e., the τ and τ' integrals over the correlator $G^0(\mathbf{r}-\mathbf{r}')$ in (4.38) are unrestricted: in momentum space, therefore, the randomness vertex carries only the Matsubara frequency $k_n \equiv 0$: see Fig. 3(b). The only cutoff that then operates in the perturbation expansion is that supplied directly by the finite range b_0 of $G^0(\mathbf{r}-\mathbf{r}')$ or, in other words, by the integrability of $\hat{G}^0(\mathbf{k})$. The required divergent contribution to the (off-diagonal[7]) susceptibility χ then arises only from the $k_n \equiv 0$ terms in the Matsubara sums and the relevant integral is precisely analogous to (4.33) with *no cutoff* imposed on the $\mathbf{q}$ and $\mathbf{q}'$ integrals. It follows that the integral in (4.35) carries an infinite upper limit; however, the prefactors of a are correctly reproduced by $a \propto \Lambda_T$.[25] This confirms the validity of (4.26) and (4.37) and hence the relevance of constrained randomness even as $T_c \to 0$.

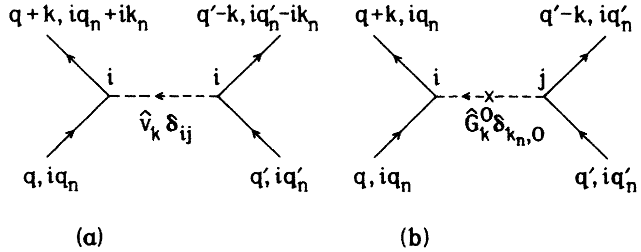


FIG. 3. Vertices for Bose fluid perturbation theory: (a) for the particle-particle interaction with pair potential $v(\mathbf{r})$, (b) for the randomness correlator $\hat{G}_k^0$. Factors of $\beta = 1/k_B T$ are not shown; i and j denote the replica indices. Note that in (a) the vertex carries a conserved Matsubara frequency ik_n, whereas in (b) only the frequency $ik_n \equiv 0$ is carried.

V. DISCUSSION

The experiments of Reppy and co-workers[8,9] on the superfluidity of helium adsorbed in Vycor exhibit an unambiguous crossover in critical behavior as the transition temperature T_c is depressed towards zero by reducing the overall adsorbed density: for moderately large T_c, the critical behavior of the superfluid density closely matches that of ordinary bulk helium, the Vycor apparently doing little more than changing T_c; as $T_c \to 0$, however, a new critical behavior is observed which is seen to be close to that of an ideal Bose fluid. The experimental data are well described by a crossover scaling formulation, as would be expected on general grounds whatever the actual physical nature of the crossover. An unbiased fit for the crossover scaling exponent, ϕ_T (for T_c regarded as the control variable), suggests[7] $\phi_T \simeq 2$ or perhaps somewhat smaller, but a fit with $\phi_T = 2$ provides an excellent "collapse of data."[7-9]

If one assumes that an interacting-to-ideal crossover in criticality is being observed, one may apply the detailed theory developed in Ref. 7. This predicts $\phi_T = 2$. Furthermore, the scaling function calculated to first order in $\epsilon = 4 - d$ fits the scaled data rather well.[7,9] The reasonableness of such an interpretation is confirmed by the fitted values, $m^*/m \simeq 1.5$ and $a^*/a \simeq 1.3$, of the effective-mass and effective-scattering-length ratios, for the "mobile" helium in Vycor.[7] Nevertheless, this interpretation accounts explicitly neither for the "immobile" helium, "frozen" in a monomolecular layer on the Vycor, nor for the irregular, random character of Vycor glass.

Two separate physical questions are entailed; to see this, suppose first that Vycor glass was actually a completely regular crystalline system resembling, say, a zeolite. Randomness would then play no role but there would still be a competition between a "diagonal" (i.e., positional) order of the helium condensed in some more-or-less regular fashion on the crystalline surface, and an "off-diagonal" order, leading to superfluidity. Thus, consider helium condensed on high-quality graphite: as a function of coverage various positionally ordered surface phases are seen. Superfluidity, such as seen in the experiments of Crooker *et al.*,[8] is to be expected only at coverages somewhat in excess of a monolayer. The *onset* of superfluidity at $T_c = 0$ in such a system should, presumably, be viewed as a special multicritical phenomenon controlled by an appropriate renormalization-group fixed point.[27] It is possible, perhaps even likely, that the $T_c > 0$ critical behavior springing from this multicritical onset point is controlled in the limit $T_c \to 0$ by an ideal Bose fixed point; in that case, the theory of Ref. 7 should be truly applicable. It would be interesting to try to test this conjecture by finding a zeolite type of system to replace Vycor.

Real Vycor, however, is irregular, which introduces the second question. In principle, this randomness can cause crossover from the "*uniform-onset*" multicritical point to a new, "*random-onset*" multicritical point and, likewise, for the superfluid critical behavior as $T_c \to 0$. In this full random onset phenomenon, however, the competition with diagonal (or real-space) ordering would still seem to play a significant role. However, it is natural to split the problem by focusing attention first only on off-diagonal (or superfluid) ordering and the effects of randomness on that.

This approach has motivated the present work. We have seen that the nonrandom or pure ideal-Bose-gas criticality considered in Ref. 7 is strongly unstable to a generic random potential. However, as explained, the randomness in Vycor is probably better described by a model with constrained randomness. The simple Harris and scaling criteria suggest that such constrained randomness is actually irrelevant as $T_c \to 0$, so that the theory of Ref. 7 should be applicable. But, as explained in detail, the simple arguments are too naive, although they can be corrected. Instead, ideal Bose criticality is found to be unstable also to constrained random perturbations even though the crossover exponent is reduced (from $\phi_R = \frac{1}{2}$ to $\phi_R = \frac{1}{4}$ for $d = 3$) and the effective strength of the randomness is significantly diminished.

What does this finding imply for the fits of the data to the theory of crossover to ideal behavior without randomness? One logical possibility is that the success is essentially accidental in that the data truly pertain to some distinct, randomness-dominated crossover for which the crossover exponent merely happens to be close to $\phi_T = 2$, while equally, the limiting behavior of $\rho_s(T)$ resembles somewhat but does not coincide with ideal Bose behavior. We cannot rule this out, but more plausible, we believe, is an alternative possibility, namely, that the existing data explore a regime in T_c and t in which the scaled randomness (as measured by the variables $y_R \propto \overline{w}_0^2/t^{\phi_R}$ or $z \propto \overline{w}_0^2/(v_0)^{1/2}$: see Secs. III and IV) remains sufficiently small and slowly varying as to be ineffective, while the crossover in the scaled interaction strength, measured by T_c/t^{ϕ_T} or v_0/t^{ϕ}, is broadly scanned. This view would explain the good fits to the theoretically predicted scaling function and exponents and the reasonable values found for m^*/m and a^*/a. It would follow, however, that one should see a buildup of deviations from the previous fits if the experiments could be pushed to still smaller values of T_c. Indeed, if one takes seriously the slight "rounding" or smearing" of the transition already seen in the data[8,9] (rather than interpreting these features as artifacts resulting from macroscopic inhomogeneities in the Vycor), one might believe that some evidence for such deviations was already available.[28]

We have not here explored the onset multicritical phenomena *per se* but we have, through the discussion in Sec. I, seen that criticality in an ideal Bose fluid in a random potential must be unusually sensitive to interactions owing to Bose condensation into localized regions in which the particle density is unbounded. Crossover directly to such a limit thus seems unlikely to be observable. Rather, one presumes that the introduction of interactions will lead first to an interacting but frozen, or "localized," nonsuperfluid phase and then, as some "mobility threshold" is passed, to the onset of superfluidity and a nonzero superfluid transition temperature T_c.

Hertz, Fleishman, and Anderson[29] have applied localization-theory ideas to a system of interacting bosons in a strong random external potential. They use a spin-glass formulation and take, as an initial basis, the random noninteracting system: it is supposed that the corresponding spectrum exhibits a mobility edge. The interactions are then treated within a Hartree approximation. The effective density of states is found to be discontinuous at the renormalized mobility edge. This corresponds to a standard two-dimensional density of states and thus leads to the prediction that Bose condensation cannot occur.

Hertz *et al.* admit to being unsure of the domain of validity of their analysis and recognize that standard renormalization-group arguments lead to contrary conclusions, namely, that $n \geq 2$ criticality is stable since $\alpha < 0$. Certainly, their analysis does not seem appropriate to helium in Vycor.

Furthermore, Bray and Moore[30] have reconsidered the same problem, but focusing on the fully interacting susceptibility matrix $\chi_{ij}^{\mu\lambda} \propto \langle s_i^\mu s_j^\lambda \rangle$ rather than on the random exchange matrix. They show, in contradistinction to Hertz *et al.*, that the effective density of states must vanish (continuously) at the instability point, which is identified as the transition, if this occurs at nonzero temperature. This casts further doubts on the conclusions of Hertz *et al.*

More recently, Ma, Halperin, and Lee[31] have specifically addressed the problem of the destruction of superfluidity at zero temperature by strong randomness, i.e., they consider the random-onset transition. With the aid of various heuristic mappings, they convert the description of the transition to that in a $(d+1)$-dimensional system with long-range correlated disorder. By appealing to finite-size scaling they then obtain an estimate of the exponent of variation of T_c as the overall density ρ approaches ρ_0, the onset density. The numerical value found for this exponent is close to but not the same as that for ideal Bose behavior. However, Ma *et al.* do not discuss the nature of any crossover in *critical* behavior, i.e., for small but fixed $t=(T-T_c)/T_c$ as $T_c \to 0$, which is what is observed in the helium in Vycor experiments. It is conceivable that this could still be ideal or else rather close to ideal. Another concern, as regards applicability to the onset phenomenon in helium, relates to the competition with diagonal ordering in addition to randomness: as alluded to above, this should play a role in the nonrandom case and could still be important when randomness acts. Nevertheless, this new attack is promising and one may hope that further work along such lines will lead to fuller elucidation of the behavior of helium in Vycor and to a clarification of the apparent agreement with the description in terms of a crossover to ideal behavior.

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[22]A careful analysis also reveals singularities of the form $t^{1-\alpha}$ (arising from $\langle |\vec{\sigma}(\mathbf{x})|^2 \rangle$ and from an operator product expansion of $|\vec{\sigma}(\mathbf{x})|^2 |\vec{\sigma}(\mathbf{x}')|^2$). Such terms represent a *shift* in T_c proportional to g_R and, within scaling, arise from a *linear mixing* of scaling fields which replaces t by $\tilde{t}=t+e_R g_R$ where e_R is a nonuniversal constant: see the scaling analysis of Ref. 7(b). It should also be noted that when $G(\mathbf{x})$ decays rapidly (for $\theta=0,2,4,\ldots$), the use of the simple scaling form for $C_{\mathscr{G}}(x)$ in (4.10) requires fuller justification than presented.

[23]One might be concerned that the vanishing of $\hat{G}_{\mathbf{q}=0}=\int d^d r\, G^0(\mathbf{r})$ might somehow dictate the vanishing of $\tilde{G}_4$. That this does not happen in general can be seen, for example, by considering the case of Gaussian randomness for

which G_4 can be expressed entirely in terms of $G^0(\mathbf{r})$ and $\tilde{G}_4$ is easily seen to remain nonzero.

[24]See F. J. Wegner, Phys. Rev. B **5**, 4529 (1972) for the general renormalization-group derivation of nonlinear scaling fields; and A. Aharony and M. E. Fisher, Phys. Rev. Lett. **45**, 679 (1980) and Phys. Rev. B **27**, 4394 (1983) for a demonstration of their practical significance.

[25]See P. B. Weichman, Ph.D. thesis, Cornell University, 1986.

[26]If, formally, one considers $b \ll a$ in (4.35) one finds that g_R varies as $(b/a)^{d+2}$, in agreement with $g_{R,2}$ as given by (4.26) and in contrast to (4.31). However, it is not physically meaningful to posit a randomness range b much less than the inverse cutoff or lattice spacing a. For application to helium, however, the issue needs further consideration as given below.

[27]We are indebted to Daniel S. Fisher (private communication) for emphasizing this point.

[28]V. Kotsubo and G. A. Williams [Phys. Rev. B **33**, 6106 (1986)] argue that Vycor should be modeled as a system of *independent,* approximately 100 Å diameter, spherical pores, on the inner surfaces of which a two-dimensional interacting superfluid or Kosterlitz-Thouless transition takes place. The observed "rounding" of the ρ_s vs T curves near T_c is then interpreted as a finite-size rounding of the bulk two-dimensional transition. However, a more realistic model for Vycor would have to contain some physical connections and hence coupling *between* neighboring spheres. These couplings, *however weak,* represent a *relevant* perturbation to the $d=2$ transition, and must induce a *dimensional crossover,* from $d=2$ finite-size behavior to $d=3$ infinite-size, bulk behavior. This crossover is known to be very strong, being controlled in magnitude by the ordering susceptibility in the lower dimensionality; thus the crossover exponent from $(d-1)$ to d for normal criticality is $\phi=\gamma_{d-1}$: see A. Aharony, Ref. 4(b), pp. 417–418. It hence seems likely to us that the couplings would have to be extremely weak in order that $d=2$ critical behavior dominate over an experimentally accessible temperature range. Thus, even though, in principle, the postulated effect should be observable in a "sintered glass" or a "powder" composed of large, uniform spheres of radius, say 1 micron or larger, with very few interparticle contacts, we do not find the Kotsubo-Williams arguments at all persuasive as applied to the Crooker-Reppy experiments. Recall, indeed, that clear evidence of three-dimensional-like behavior, with critical exponent $\zeta \simeq 0.67$ for the superfluid density, is already seen in the experiments at rather low incremental fillings above onset: Further, we find the related graphical fits offered by Kotsubo and Williams quite unconvincing.

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