Gapless superfluidity in ^{3}He films

Zlatko Tešanović
Department of Physics, Harvard University, Lyman Laboratory, Cambridge, Massachusetts 02138

Oriol T. Valls
School of Physics and Astronomy, University of Minnesota, Minneapolis, Minnesota 55455
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The effects of diffusive boundary scattering in the superfluid properties of thin ($k_F^{-1} \ll d < \xi$) ^{3}He films are studied. We assume that surface scattering arises from random, uncorrelated irregularities in the substrate. For a specified surface roughness there is a critical thickness d^c below which superfluidity disappears. For d larger than d^c superfluidity is only moderately suppressed and certainly within observational limits. As the thickness decreases toward d^c we find that the A-phase-like behavior is followed by an intermediate, experimentally accessible region in which the excitation spectrum is gapless. Thus, "dirty" anisotropic superfluidity is predicted to be realizable in ^{3}He films. We present calculations of the density of states in the gapless regime, and the zero-temperature phase diagram. We briefly discuss the specific heat and other properties of the gapless state.

I. INTRODUCTION

Since its discovery nearly fifteen years ago,[1] superfluidity in ^{3}He has been a very rewarding field of study, both theoretically and experimentally. Progress in our understanding of this anisotropic superfluid has had widespread implications in many areas of condensed-matter physics.[2] Similarly, superfluid ^{4}He films have provided[3] a convenient testing ground for the theoretical ideas developed in recent years in the context of order in two-dimensional systems.[4] It is in this context that the recent experimental report[5] of superflow in ^{3}He films must be viewed as particularly exciting. Due to the complexity of the order parameter, two-dimensional superfluidity may prove to be a far richer and more extensive phenomenon in ^{3}He films than in their boson counterpart.

From the theoretical point of view, some studies of the properties of clean, ideal ^{3}He films have appeared. Possible phases were discussed in Refs. 6 and 7. In the physically relevant regime where only the superfluid component is two-dimensional (i.e., the thickness d satisfies $k_F^{-1} \ll d \lesssim \xi$, where ξ is the coherence length and k_F the Fermi wave vector) we have[8] studied the NMR response of ^{3}He films and, recently,[9] the detailed behavior of properties such as the transition temperature as the number of layers in the film changes. Our results, however, did not include the effects of boundary roughness. Since it is well known that any type of disorder has a pair-breaking effect in anisotropic superfluids,[10] it is clearly very important to understand the qualitative and quantitative consequences of this type of scattering. This is particularly relevant if one bears in mind that in a real-world experiment, the influence of the substrate on some properties of thin ^{3}He films may be far from negligible.

The main question that we take up in this paper deals with the effects that random scattering from the boundaries has on films of superfluid ^{3}He. Our main result is that, for experimentally relevant ranges of the surface roughness, the transition temperature is only moderately depressed, and the effects found in Ref. 9 are merely broadened. Observation of superfluidity is, therefore, not inhibited. But the nature of the superfluid state may be quite different from that in the bulk or even in the ideal film. We predict that there exists a relatively wide and experimentally accessible region in the surface roughness versus thickness phase diagram (see Sec. III) in which the excitation spectrum of superfluid ^{3}He films is gapless.[11] Thus, if our prediction is correct, a unique opportunity to observe a qualitatively different superfluid state, a "dirty" anisotropic superfluid, exists in ^{3}He films.

This paper is organized as follows: In Sec. II we show how the surface roughness problem can be mapped onto the problem of an ideal fluid in the presence of a two-dimensional random potential. This problem is in turn equivalent to that of randomly distributed scattering centers confined to a thin layer at the boundaries of an otherwise clean film. Starting from the solution for the clean limit[9] we find the equations for the diagonal and off-diagonal parts of the self-energy using a self-consistent Born approximation (SCBA) treatment of random scattering. These equations are solved in Sec. III where results are obtained for the transition temperature T_c as a function of film thickness d, and surface roughness. The calculation of the dynamical density of states then reveals that there exists, for a given thickness, a critical strength of the surface scattering (or, conversely, a critical film thickness for a specified surface roughness) for which the quasiparticle excitation spectrum becomes gapless. The range of parameters for which such a behavior arises indicates that the gapless regime could be reached in many reasonable experimental situations. We then present some additional results for various quantities which are likely to be of experimental interest. Finally, in Sec. IV we offer some concluding remarks.

Two more points are worth explaining. First we will consider the thickness regime $k_F^{-1} \ll d \lesssim \xi$. Since $\xi \sim 400$ Å at $T=0$ and it increases with temperature, this is a very wide regime. For $d \lesssim \xi$ the A phase is stable (at saturated vapor pressure, ~ 0.28 bar) in the film. If $d \gg \xi$, textures would develop in the order parameter connecting the surface favored ABM state to the bulk stable B phase (Balian-Werthamer state). The case of a semi-infinite slab of ^{3}He-B in contact with randomly rough wall has been studied by Buchholtz[12] using quasiclassical methods. Note, however, that the thickness range for which our methods are applicable is not bounded from above provided only that the A phase is stable throughout the film. For example, in the slab case ($d \gg \xi$) at high pressures ($p > 21$ bar) the results of Secs. II and III are still correct, although, of course the importance of surface scattering effects will diminish as the thickness increases. On the other hand, we will limit ourselves to thicknesses much larger than the interparticle spacing (i.e., $d \gg k_F^{-1}$) when it is safe to assume that the properties of the normal fluid are basically the same as in the bulk. The very thin film region, $d \sim k_F^{-1}$, is more difficult to study because the normal fluid itself is two-dimensional and the substrate effects may strongly influence the quasiparticle interactions. The available experimental information[13] indicates that very thin ^{3}He films do indeed have an intricate behavior, quite different than thicker ones. The second point is that the methods discussed in this paper have wider applicability. In Ref. 14 they have been applied to the question of a possible anisotropic pairing in heavy-fermion superconductors.

II. MODEL FOR SURFACE ROUGHNESS

The effects of surface roughness on the properties of superfluid ^{3}He films are overwhelmingly determined by the pair-breaking nature of random scattering. This is closely related to the pair-breaking property of nonmagnetic impurity scattering for anisotropic superfluids.[10] In the present section we will further elucidate this connection.

The first step along the way is to choose a specific model for surface roughness. In a particular experimental setup the substrate parameters will be fixed but there is a wide variety of materials which could be used as a substrate in different experiments. Therefore, we need a model of sufficient generality, yet simple enough to allow finiteness of calculational effort. In what follows, we will assume that the dominant part of the surface scattering comes from random, uncorrelated irregularities in the surface. We will also assume that the average "strength" of surface scattering is small in a sense that will become clear below. This will lead to a "white-noise" type of diffuse scattering. While the assumption of uncorrelated irregularities may not be fully justified for most realistic substrate materials, we still think it is a very reasonable one, for the following reasons: First, the main effect of the correlations will be to "renormalize" the pair-breaking field found for the uncorrelated case. Therefore, we can hope to account for this by simply choosing some "effective" uncorrelated surface to represent the real physical one. Secondly, if the correlations are included, the calculation becomes significantly more complicated with the additional angular dependence of the pair-breaking parameters. These complications seem unnecessary for our purposes, which emphasize the physical nature of the effects of surface scattering, as compared to the smooth case. It will therefore suffice to use the uncorrelated model.

In Ref. 9 we have modeled the films as an infinite square well of thickness d in the direction of z axis.[15] To simulate surface roughness we will assume that the thickness of the film is now a function of position in the plane of the film (x-y plane),

$$d(x,y)=d+w(x,y)\ , \tag{2.1}$$

where d is the nominal average thickness. Consistent with our assumptions, we will take $w(x,y)$ to be a Gaussian random function,

$$\langle w(x,y)\rangle = 0\ , \tag{2.2a}$$

$$\langle w(x,y)w(x',y')\rangle = w^2\delta(x-x')\delta(y-y')\ . \tag{2.2b}$$

Equation (2.2b) expresses our assumption of totally uncorrelated irregularities. The parameter w, which measures the overall surface roughness (the rms fluctuations in thickness) may be thought of as a product $w=ha$, where h is the average "height" of the bumps and a some scattering length. We will be assuming in what follows that $w/d^2 \ll 1$. Also, we will be considering a situation in which only one of the surfaces is rough; the one at $z=0$ presumed smooth. All the results are straightforwardly generalized to the case when both surfaces are rough.

The problem of scattering by a random surface has been considered in some detail by Chaplik and Entin.[16] There it has been shown how the boundary condition for the Green's function at a random surface can be transferred into the Schrödinger equation for motion in a two-dimensional, state-dependent random potential, assuming that $w(x,y) \ll d$. The single-particle Green's function can then be found from a perturbation expansion, quite analogous to that used by Abrikosov and Gorkov to treat impurity scattering in superconducting alloys.[17] While the procedure applied in Ref. 16 is conceptually straightforward the details are rather involved. We will, therefore, briefly sketch here how to rederive some of their results by a more direct method.

Let us assume that in an otherwise clean film of thickness d there is a thin layer of thickness $w^{1/2} \ll d$ near one of the surfaces, in which particles experience random impurity scattering given by a potential $v(x,y)$. Except within this layer, $v(x,y)$ vanishes everywhere in the film. A perturbation theory expansion in $v(x,y)$ can be worked out using the Abrokosov-Gorkov method, as we shall see below. It turns out that such a perturbation expansion is equivalent term by term to that for the surface scattering problem, as given in Ref. 16, provided only that one establishes the proper connection between $\langle v(x,y)v(x',y')\rangle$ and $\langle w(x,y)w(x',y')\rangle$.[18]

To perform a perturbation expansion in $v(x,y)$ we first need the matrix element describing scattering from state

$(\mathbf{k},\nu)$ to state $(\mathbf{k}',\nu')$ caused by this random potential. Here and below, $\mathbf{k}$ denotes a two-dimensional wave vector and ν is the principal quantum number associated with the boundary condition in the z direction. If the eigenstates of the unperturbed Hamiltonian along the z direction are taken to be those of an infinite square well potential, i.e.,

$$u_\nu(z)=\left[\frac{2}{d}\right]^{1/2}\sin\left[\frac{\pi\nu}{d}z\right],\quad \nu=1,2,\ldots . \tag{2.3}$$

This matrix element is easily found to be

$$\begin{aligned}\langle \mathbf{k}',\nu' \,|\, v(x,y)\,|\,\mathbf{k},\nu\rangle &= \frac{v(\mathbf{k}-\mathbf{k}')}{A}\frac{2}{d}\int_{d-w^{1/2}}^{d}dz\,\sin\left[\frac{\nu'\pi}{d}z\right]\sin\left[\frac{\nu\pi}{d}z\right]\\ &\cong -\frac{2\pi^2}{3}\frac{v(\mathbf{k}-\mathbf{k}')}{A}\left[\frac{w^{1/2}}{d}\right]^3\nu\nu'(-1)^{\nu+\nu'} ,\end{aligned} \tag{2.4}$$

where A is the area of the film. If we now assume that

$$\langle v(x,y)v(x',y')\rangle = v^2\delta(x-x')\delta(y-y') \tag{2.5}$$

from Eq. (2.4), it follows by direct comparison that every diagram in the perturbation expansion of Ref. 16 can be identified with the corresponding diagram in the Abrikosov-Gorkov perturbation expansion in $v(x,y)$, provided that

$$w^3v^2=\frac{9}{8m^2}w^2 . \tag{2.6}$$

Equation (2.6) enables us now to study the effect of a rough surface using perturbation theory. In particular, we are interested in finding the self-consistent expression for the normal and anomalous Green's functions. To do this, we first perform some useful manipulations very similar to those employed in Ref. 9.

First the normal and anomalous Green's functions $G_{\sigma\sigma'}(\mathbf{r},\mathbf{r}',\omega_n)$ and $F_{\sigma\sigma'}(\mathbf{r},\mathbf{r}',\omega_n)$ are Fourier transformed in the x-y plane. These Fourier transforms can then be expanded in terms of the eigenfunctions (2.3),

$$G_{\sigma\sigma'}(z,z',\mathbf{k};\omega_n)=\sum_\nu u_\nu(z)u_\nu(z')G_\nu^{\sigma\sigma'}(\mathbf{k},\omega_n) , \tag{2.7}$$

$$F_{\sigma\sigma'}(z,z',\mathbf{k};\omega_n)=\sum_\nu u_\nu(z)u_\nu(z')F_\nu^{\sigma\sigma'}(\mathbf{k},\omega_n) . \tag{2.8}$$

The three-dimensional Fermi sphere degenerates into a set of circles, and ν may be thought of as the index which enumerates these different "subbands." In the limit of perfectly smooth boundaries $G_\nu^{\sigma\sigma'}$ and $F_\nu^{\sigma\sigma'}$ are given by,[9]

$$G_\nu^0(\mathbf{k},\omega_n)=\frac{-i\omega_n-\xi_k-\lambda_\nu}{E_{k,\nu}^2} , \tag{2.9}$$

$$F_\nu^0(\mathbf{k},\omega_n)=\frac{\Delta_{\sigma\sigma'}^\nu(\hat{\mathbf{k}})}{E_{k,\nu}^2} , \tag{2.10}$$

where, for clarity, we have suppressed spin indices in G and F. In the above,

$$\lambda_\nu=\frac{1}{2m}\left[\frac{\nu\pi}{d}\right]^2 , \tag{2.11}$$

$$\xi_k=\epsilon_k-\mu , \tag{2.12}$$

$$E_{k,\nu}^2=\omega_n^2+(\xi_k+\lambda_\nu)^2+|\,\mathrm{Tr}\Delta_{\sigma\sigma'}^\nu(\hat{\mathbf{k}})\,|^2 . \tag{2.13}$$

The chemical potential μ must be found self-consistently in terms of the density and the number of layers. Only in the limit $d\to\infty$ is it given by $k_F^2/2m$.[10] In Eqs. (2.10) and (2.13), $\Delta^\nu(\hat{\mathbf{k}})$ is the coefficient in the expansion of the gap in terms of the u_ν,

$$\Delta_{\sigma\sigma'}(z,z',\mathbf{k})=\sum_\nu u_\nu(z)u_\nu(z')\Delta_{\sigma\sigma'}^\nu(\hat{\mathbf{k}}) \tag{2.14}$$

and it can be written in the form

$$\Delta_{\sigma\sigma'}^\nu(\hat{\mathbf{k}})=\Delta_{\sigma\sigma'}(\hat{\mathbf{k}})\sin\theta_\nu , \tag{2.15}$$

where $\sin\theta_\nu$ is the discrete equivalent of the usual $\sin\theta$,

$$\sin\theta_\nu=\left[1-\left[\frac{\nu}{\nu_0}\right]^2\right]^{1/2} . \tag{2.16}$$

In all the sums the index ν runs from 1 to ν_c, where ν_c is the largest integer smaller than $\nu_0\equiv 2md^2\mu/\pi^2$.

Let us now consider what happens in the presence of a random boundary. As discussed before, we can treat diffusive surface scattering by performing a perturbation theory expansion for a random potential $v(x,y)$ and using Eq. (2.6). The self-consistent Born approximation (SCBA) that we utilize leads to the following self-consistent equations for $G_\nu(\mathbf{k},\omega_n)$ and $F_\nu(\mathbf{k},\omega_n)$:

$$[i\omega_n-\xi_{\mathbf{k}}-\lambda_\nu-\Sigma_\nu(i\omega_n)]G_\nu(\mathbf{k},\omega_n)+\Delta_\nu(\hat{\mathbf{k}})F_\nu^\dagger(\mathbf{k},\omega_n)=1 , \tag{2.17}$$

$$[i\omega_n-\xi_{\mathbf{k}}-\lambda_\nu-\Sigma_\nu(-i\omega_n)]F_\nu^\dagger(\mathbf{k},\omega_n)+\Delta_\nu^\dagger(\hat{\mathbf{k}})G(\mathbf{k},\omega_n)=0 . \tag{2.18}$$

In (2.17) and (2.18) spin indices have been suppressed. The self-energy $\Sigma_\nu(i\omega_n)$ is given by:

$$\Sigma_\nu(i\omega_n)=\frac{4\pi^4}{9}\left[\frac{w^{1/2}}{d}\right]^6 v^2\nu^2\sum_{\mathbf{k},\nu'}(\nu')^2G_\nu(\mathbf{k},\omega_n) . \tag{2.19}$$

As a consequence of assumptions (2.2a) and (2.2b) there is no contribution to the self-energy from the anomalous Green function. Then, $G_\nu(\mathbf{k},\omega_n)$ is found by solving Eqs. (2.17) and (2.18) and can be written in the following form:

$$G_\nu(\mathbf{k},\omega_n)=\frac{-i\tilde{\omega}_n(\nu)-\xi_{\mathbf{k}}-\lambda_\nu}{\tilde{\omega}_n^2(\nu)+(\xi_{\mathbf{k}}+\lambda_n)^2+\Delta^2\sin^2\theta_\nu} , \tag{2.20}$$

where

$$\tilde{\omega}_n(\nu)=\omega_n+i\nu^2\Sigma_\nu(i\omega_n) . \tag{2.21}$$

The sum over $\mathbf{k}$ in Eq. (2.19) can be done using standard techniques, and one then obtains from (2.21) and (2.19) the self-consistent equation for the self-energy (from now on we drop the index n in the Matsurbara frequencies)

$$\tilde{\omega}_\nu=\omega+\Gamma_0\cos^2\theta_\nu\sum_{\cos\theta_{\nu'}}\frac{\tilde{\omega}_{\nu'}\cos^2\theta_{\nu'}}{[\tilde{\omega}_{\nu'}^2+\Delta^2\sin^2\theta_{\nu'}]^{1/2}}\ , \qquad (2.22)$$

where, with the help of (2.6), we have :

$$\Gamma_0=\frac{\pi^4}{2md^2}\left[\frac{w^2}{d^4}\right]\nu_0^5\ . \qquad (2.23)$$

Similarly, the self-consistent equation for the gap parameter is also obtained from (2.17) and (2.18),

$$1=\frac{3}{2}\frac{\pi\lambda}{k_Fd}\pi T\sum_{\tilde{\omega}}\sum_{\cos\theta_\nu}\frac{\sin^2\theta_\nu}{(\tilde{\omega}_\nu^2+\Delta^2\sin^2\theta_\nu)^{1/2}}\ , \qquad (2.24)$$

where we have assumed that the strength of the pairing interaction is not significantly affected by the surface roughness. This is a reasonable assumption, since the roughness should not affect normal state properties, under our assumptions. Equations (2.22) and (2.24) contain, in principle, all the required information about a superfluid ^{3}He film on a rough substrate.

III. PROPERTIES OF SUPERFLUID STATE

We are now in a position to calculate the effect of diffuse surface scattering on various physical quantities. First, let us find the transition temperature, T_c. This can be accomplished by expanding Eq. (2.24), assuming small Δ, and making use of (2.22). After straightforward algebra the following expression is obtained:

$$\ln\frac{T_c(d)}{T_{c0}(d)}=-\left\langle\psi\left[\frac{1}{2}+\frac{\Gamma\cos^2\theta_\nu}{2\pi T_c(d)}\right]\right\rangle+\psi(\tfrac{1}{2})\ , \qquad (3.1)$$

where $T_{c0}(d)$ is the transition temperature in the smooth boundary limit, ψ is the digamma function, Γ is defined as

$$\Gamma=\Gamma_0\frac{\nu_c(\nu_c+1)(\nu_c+\frac{1}{2})}{3\nu_0^3}\ , \qquad (3.2)$$

and the brackets denote the weighted average:

$$\sum_\nu\sin^2\theta_\nu(\cdots)\Big/\sum_\nu\sin^2\theta_\nu\equiv\langle\cdots\rangle\ . \qquad (3.3)$$

Equation (3.1) does not have the universal form characteristic of a uniform pair-breaking field.[20] This is obviously due to the state dependence of the self-energy (2.19). We will see later that this state dependence leads also a somewhat unexpected behavior in the dynamical density of states.

From Eq. (3.1) the variation of T_c as a function of the thickness and of the parameter w which determines the strength of the surface scattering can be plotted. To this end, it is useful to introduce the dimensionless parameter

$$y\equiv k_F^4w^2\ , \qquad (3.4)$$

which measures the surface roughness as given by the rms parameter w defined in (2.2) in terms of the interparticle distance. For specified values of y, the plot of T_c as a function of d is shown in Fig. 1. The most prominent feature is the existence of a critical thickness $d^c(y)$ below which superfluidity is destroyed. For thicknesses larger than this $d^c(y)$, however, T_c is only moderately suppressed, typically by 25% , and is certainly within observational limits.

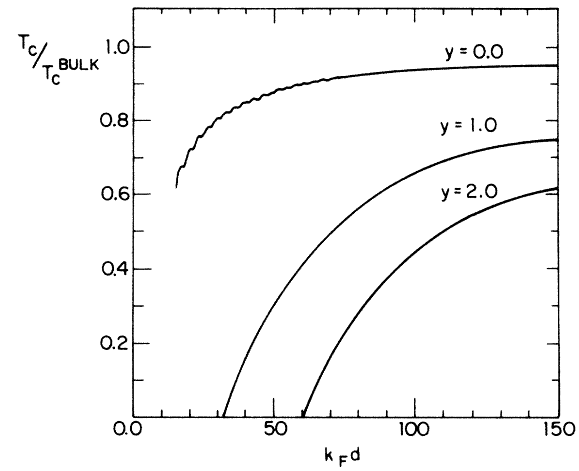

FIG. 1. Transition temperature of a superfluid ^{3}He film as a function of thickness. Surface roughness parameter y is defined in (3.4). In an ideal ($y=0.0$) film case, the wiggles in T_c are due to quantum size effects. We show results in the region $k_Fd>15$.

The quantity $d^c(y)$ can be extracted from (3.1) by expanding around $T_c\rightarrow 0$. The result is

$$\Gamma^c=\pi T_{c0}/2\gamma\eta\ , \qquad (3.5)$$

where $\gamma=1.78\ldots$ is Euler's constant and $\ln\eta\equiv\langle\ln\cos^2\theta_\nu\rangle$. Equation (3.5) is an implicit equation for d^c since T_{c0} also depends on thickness (see Ref 9). However, for $d^c\gg k_F^{-1}$ this dependence is very weak[9] and one has $T_{c0}\cong T_c^{\rm bulk}$. One can then show from (3.5) that

$$d^c(y)\cong 0.75y\eta\xi_0\ , \qquad (3.6)$$

where ξ_0 is the standard BCS coherence length.

For $y=1$ one has $k_Fd^c\cong 35$. It follows that superfluidity in ^{3}He films of moderate thickness (say twenty to several hundred layers) will not be destroyed by surface roughness except when $y\gg 1$ which would represent a very high degree of rms strength for the uncorrelated fluctuations: there would have to be, for example, a large number of bumps and holes, covering altogether a large percentage of the surface, having an average radius of two or three interparticle distances and a height or depth ten times as large.

The above has a direct bearing on the results obtained for ideal boundaries in Ref. 9, where the quantum size effects in quantities such as T_c and the transverse NMR frequency shift, Ω_A, were discussed. The quantum size effects in T_c were found to be small; the characteristic large oscillations with thickness which are found for s pairing do not appear for p wave. We concluded that these oscillations would be difficult to detect, even with an ideal boundary. The frequency Ω_A, however, exhibits[9] a reasonably strong oscillatory behavior related to the oscillations found in the density of states, and it is, therefore, the most obvious candidate for experimentally finding quantum size effects in superfluid ^{3}He films. Surface roughness has a twofold effect on Ω_A. First, its overall magnitude decreases: it scales with T_c^2 and will obviously disappear at $d=d^c$. Second, random boundary scattering will tend to smooth out the sharp jumps in the magnetic susceptibility, which are directly reflected in Ω_A, and to eventually diminish the oscillatory character of Ω_A. The oscillation will be observable so long as $\Gamma/\Delta\lambda_{v_c} < (\ll)1$, where $\Delta\lambda_{v_c}=\lambda_{v_c+1}-\lambda_{v_c}$. After a short calculation one finds $\Gamma/\Delta\lambda_{v_c}=y/6\pi^2$ which is much less than unity for y in the range of interest (i.e., $d^c<\xi_0$). As a consequence we expect that, for d larger than d^c, boundary scattering will not substantially alter the oscillator behavior of Ω_A.

Many low-temperature properties of the superfluid are determined by the character of the quasiparticle excitation spectrum. To study these properties it is first necessary to find the dynamical quasiparticle density of states. We start by writing Eq. (2.20) in the following form:

$$u_v=\frac{\omega}{\Delta}+\zeta\cos^2\theta_v\sum_{\cos\theta_{v'}}\frac{u_{v'}\cos^2\theta_{v'}}{[u_{v'}^2+\sin^2\theta_{v'}]^{1/2}}, \tag{3.7}$$

where $u_v\equiv\omega_v/\Delta$ and $\zeta\equiv3\Gamma/\Delta$. Equation (3.7) is an implicit equation from which $u_v=u_v(\omega/\Delta)$ is to be calculated. The complications again come through the v dependence. It is useful to make the following ansatz: $u_v=\omega/\Delta+\zeta U\cos^2\theta_v$. Equation (3.7) is then transformed into an equivalent equation for U,

$$U\left[\frac{\omega}{\Delta}\right]=\sum_{\cos\theta_v}\frac{(\omega/\Delta)\cos^2\theta_v+\zeta U\cos^4\theta_v}{[(\omega/\Delta+\zeta U\cos^2\theta_v)^2+\sin^2\theta_v]^{1/2}}. \tag{3.8}$$

Once (3.8) is solved and $U(\omega/\Delta)$ known, the dynamical density of states can be found from

$$\begin{aligned}N(\omega)&=-\frac{1}{\pi}\mathrm{Im}\sum_{\mathbf{k}}\sum_{v}G_v(\mathbf{k},\omega)\\&=-\frac{N(0)}{\pi}\mathrm{Im}\sum_v\int d\xi\frac{-\tilde\omega_v-\xi-\lambda_v}{-\tilde\omega_v^2+(\xi+\lambda_v)^2+\Delta^2\sin^2\theta_v},\end{aligned}$$

that is,

$$\begin{aligned}N(\omega)&=N(0)\mathrm{Im}\sum_v\frac{\tilde\omega_v}{[\Delta^2\sin^2\theta_v-\tilde\omega_v^2]^{1/2}}\\&=N(0)\mathrm{Im}\sum_{\cos\theta_v}\frac{\omega/\Delta+\zeta U\cos^2\theta_v}{[\sin^2\theta_v-(\omega/\Delta+\zeta U\cos^2\theta_v)^2]^{1/2}}.\end{aligned} \tag{3.9}$$

Equations (3.8) and (3.9) have to be solved numerically. $N(\omega)/N(0)$ is plotted in Fig. 2 for several values of ζ. Two kinds of behavior are present. For $\zeta<1.6$ the density of states vanishes at the Fermi level and the system resembles an A-phase state, with $N(\omega)\sim\omega^2$ for small ω, the coefficient in front of ω^2 being increased from its clean film value. When $\zeta>1.6$, one observes gapless behavior, i.e., there is a finite density of states at the Fermi surface, although of course it is smaller than in the normal fluid. Thus, in this case the superfluid film resembles a dirty superconductor. The transition disappears altogether as $\Gamma\to\Gamma^c$, as previously discussed. Comparison of Fig. 2 with the density of states obtained in the presence of uniform impurities[21] reveals another characteristic feature of the boundary scattering effects. While the peak in the density of states is very quickly reduced and broadened by an increase in the concentration of uniform impurities, the sharp-peaked structure survives in the case of even very rough surfaces. This can be traced back to the state dependence of a pair-breaking field; close to the equator of the Fermi "sphere," where the gap reaches its maximum the effect of boundary scattering is strongly reduced.

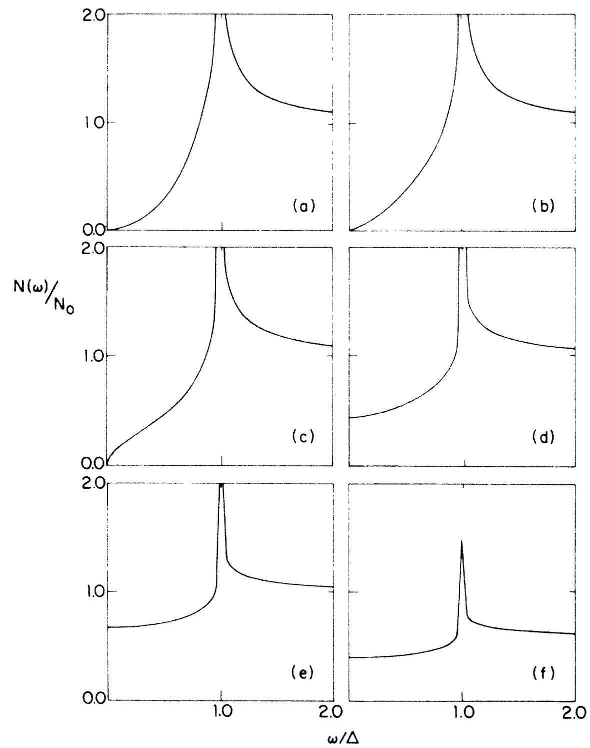

FIG. 2. The dynamical quasiparticle density of states for various values of $\zeta=(3\Gamma)/\Delta$: (a) $\zeta=0.0$, (b) $\zeta=0.5$, (c) $\zeta=1.6$, (d) $\zeta=6.0$, (e) $\zeta=18.0$, (f) generic form of the density of states in the gapless regime exhibiting the peaked structure around $\omega/\Delta=1$.

The various equilibrium states of the superfluid film are summarized on the $T=0$ phase diagram in Fig. 3. Depending on the values of d and y, three different types of states are predicted: (i) the A-phase-like state with the low-temperature quantities having algebraic temperature dependences characteristic of the A phase, but with modified numerical constants, (ii) the "gapless" regime, with finite density of states at the Fermi sphere, and, finally, (iii) the normal liquid, where the pairing correlations are completely suppressed by random scattering. Potentially the most interesting regime is the gapless one. It has not been possible up to now to prepare a dirty ^{3}He superfluid sample in which the interplay of superfluidity and re-

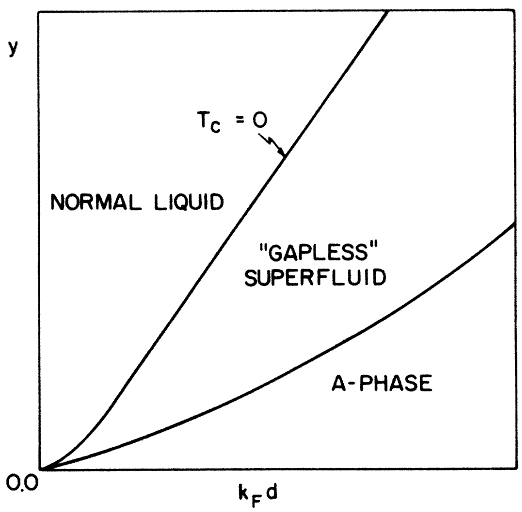

FIG. 3. The $T=0$ phase diagram of superfluid ^{3}He film. Quantitatively, boundaries between different regions will depend on the particular model for surface scattering but the qualitative features are expected to persist. By controlling the thickness, or by a careful choice of the substrate material, it should be possible to experimentally "open" the "window" with a view on properties of a dirty anisotropic superfluid.

duced mean free path could be studied. ^{3}He films on realistic, rough substrates, offer this unique opportunity, with a wide range of consequences for superflow, transport properties, and behavior in magnetic field. Particularly attractive is the possibility of effectively "tuning" the strength of surface scattering by controlling the thickness. The "window" of thickness for which the gapless behavior is obtained seems reasonably wide and there should be no unsurmountable experimental difficulties in observing it.

The experimental property in which the absence of a gap will be most obviously felt is the specific heat. At low temperatures, instead of the T^3 law characteristic of the A phase one will find linear behavior. The slope will be, as required by thermodynamics, smaller than that of the normal phase. The jump in the specific heat at T_c is a decreasing function of T_c, going to zero when $T_c \to 0$ [and $\Gamma \to \Gamma^c(d)$]. The specific heat can be calculated in a fairly straightforward way, making use of the appropriate generalization of methods used to treat gapless superconductivity.[22]

In addition, the presence of surface roughness will introduce new contributions to transport properties. Ordinarily, ^{3}He at low temperatures is the ultimately clean system, since no impurities can be dissolved in it. Transport properties are determined by quasiparticle collisions, except in the ballistic regime, (when the mean free path is very large). As the thickness of a superfluid ^{3}He film is reduced, or the roughness increases, impurity-like contributions to quantities such as the sound attenuation coefficient or thermal conductivity should become apparent. Indeed, one should be able to observe the full range of interplay between the two characteristic lengths, ξ and the "impurity" scattering length, as in ordinary superconductors.

IV. CONCLUSIONS

The main conclusions of this paper are summarized by examining the zero-temperature phase diagram of Fig. 3. We see that, depending on the thickness and the degree of surface smoothness one can have three different regimes: in the first, an ordinary, A-phase-like superfluidity is obtained and the results of Refs. 8 and 9 (for example) are applicable; the second is a normal regime; and finally, there is an intermediate region in which the excitation spectrum of the superfluid is gapless. In the last regime, the thermodynamic and transport properties of the system vary according to what may be thought of as an effective mean free path.

The actual calculations are performed assuming a "white noise" model for surface roughness, with uncorrelated irregularities in the substrate. This assumption was dictated by calculational convenience and may not be realistic for many surfaces, where irregularities may be, within some length scale, highly correlated. However, we do not believe that the qualitative conclusions as to the existence of different regimes and, in particular, of the gapless state are likely to be changed should one use a more realistic description of the surface.

Superfluidity in ^{3}He films, therefore, may exhibit a richer variety of phenomena than had been previously thought. It would be of great value if the preliminary report of Ref. 5 proved to be an inspiration for more experimental effort in this area.

Note added in proof. L. H. Kjaldman, J. Kurkijarvi, and D. Rainer [J. Low Temp. Phys. 33, 577 (1978)] pointed out the existence of critical thickness for superfluidity in narrow channels and pores. Using their *quasiclassical* approach a single critical size is obtained corresponding to fully diffusely reflecting boundary conditions (which have no clear microscopic meaning). We thank Professor D. Rainer for useful discussions. One of us (Z.T.) is grateful to Professor R. C. Richardson, Professor J. Reppy, Professor J. P. Harrison, Dr. A. Sachrajda, Dr. M. Freeman, and Dr. T. Guamilla for their patience in explaining to him some experimental aspects of ^{3}He in confined geometries.

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