Rayleigh scattering and weak localization: Effects of polarization

Michael J. Stephen and Gabriel Cwilich
Serin Physics Laboratory, Rutgers University, Piscataway, New Jersey 08854
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The scattering of light from a disordered medium is considered. The effects of polarization and the transverse nature of the light on the coherent backscattering arising from weak localization phenomena are calculated. The angular line shape of the scattered light in an infinite medium and the reflected light from a half-space and a slab is found. The scattered light contains several components and the line shape is different from that found for scalar waves. The results are in qualitative agreement with recent experiments.

I. INTRODUCTION

The scattering of light by an inhomogeneous medium is a problem of importance and one that has a long history. Recently this problem has become one of renewed interest because of the possibility of observing weak localization effects associated with light waves. The case of light is interesting because it provides the opportunity of looking in detail at the elastic scattering and interference effects associated with weak localization.[1] For electrons in a disordered metal, the details of the scattering of the electrons by impurities is less accessible and inelastic effects are generally more important. Localization is essentially a one-body problem and the statistics of the particles or waves is not important. Thus the same theory that applies for electrons[2] should also apply to other wave phenomena. For light, as we show below, there are interesting effects associated with the polarization and transverse nature of the light waves.

In most treatments of the scattering of light from a disordered medium, the scattered light waves arising from different multiple-scattering paths are treated as being incoherent. However, it is now well known that even in a disordered medium a wave scattered through a certain multiple-scattering path can interfere coherently with another wave which follows the time-reversed path. The interference is most important for light scattered in the backward direction. This interference effect was recognized some time ago[3] and presumably is the only important interference effect remaining in a truly disordered system. A recent detailed discussion for the case of electrons in metals has been given by Bergmann.[1] In recent theories of weak localization of electrons in metals[2] this interference effect is the important physical effect.

It is interesting to note that enhanced backscattering of light can also occur from a sufficiently rough surface. If there is a path for the light into the surface, then there is also a path out in the backward direction. This effect should be enhanced by the coherent interference of time-reversed waves. In this paper the scattering of the light is assumed to occur in the bulk.

In the case of light, early work concentrated on the diffusion constant.[4,5] More recently, the coherent backscattering of light from a disordered medium has been observed by van Albada and Lagendijk,[6] Wolf and Maret,[7] and Etemad.[8] In the case of scalar waves the line shape of the coherent back scattering from a half-space has been considered by Golubentsev[9] and by Akkermans, Wolf, and Maynard.[10] However, it is clear from the experiments[8] that there are interesting polarization effects and that the line shape is more complicated and interesting than that found for scalar waves. In this paper we consider the scattering of polarized electromagnetic waves from a disordered medium and find the angular distribution and polarization dependence of the light coherently backscattered from a disordered dielectric medium. It is found that for polarized scattering the backscattered light has a narrow component (in angle) of the form found in Ref. 10 superposed on a broader, roughly Lorentzian component. The depolarized scattered light consists of two roughly Lorentzian components, the weaker of which is destructive. Some preliminary results were given recently.[11] These results are in good qualitative agreement with the recent experiments of Etemad.[8]

The paper is organized as follows. In Sec. II, in order to establish a theoretical foundation, we discuss the effects of weak localization on scalar waves, and in Sec. III we determine the line shape for the scattering from a half plane. The results obtained agree with those of Ref. 10. In Sec. IV we generalize the theory in order to take into account polarization effects and the transverse nature of the light waves. In Sec. V we determine the effects of polarization on the line shape for the backscattering from a half-space. These calculations are made for isotropically polarizable scatterers. The theory is generalized to the case of anisotropic scatterers in Appendix C. In Sec. VI we discuss the results and make a comparison with experiment. In Appendix D we give an expression for the light reflected from a slab of thickness L.

II. SCALAR WAVES

The field E of the radiation of frequency ω and velocity c satisfies the wave equation

$$[\nabla^2+k^2(1+\epsilon'(\mathbf{r})]E(\mathbf{r})=j(\mathbf{r}) , \tag{2.1}$$

where $k=\omega/c$, j is a source, and ϵ' is the random part of the dielectric constant with zero mean and correlation function

$$k^4\langle \epsilon'(\mathbf{r})\epsilon'(\mathbf{r}')\rangle=\Delta\delta(\mathbf{r}-\mathbf{r}') . \tag{2.2}$$

Equation (2.1) can be rewritten

$$E(\mathbf{r})=\int D'(\mathbf{r},\rho)j(\rho)d\rho\ , \tag{2.3}$$

where D' is the retarded Green's function of (2.1). The coherent part of the field, $\bar{E}$, is obtained by averaging (2.3) (the bar indicates an average over ϵ'):

$$\bar{E}(\mathbf{r})=\int D(\mathbf{r}-\rho)j(\rho)d\rho\ . \tag{2.4}$$

The Fourier transform of the average Green's function D in the effective medium or coherent potential approximation is $D^{-1}(q)=k^2-q^2-\Sigma$ where (V is the volume)

$$\Sigma=\Delta/V\sum_q D(q)\ . \tag{2.5}$$

In this approximation the mean free path is $l=k/\Sigma_i=4\pi/\Delta$, where Σ_i is the imaginary part of Σ. The mean free path is of the Rayleigh form, i.e., $l\sim\omega^{-4}$. From (2.5) the identity

$$1=\frac{\Delta}{V}\sum_q |D(q)|^2 \tag{2.6}$$

is obtained.

The correlation function of the field is defined by

$$\Gamma(\mathbf{R},\mathbf{r})=\overline{E(\mathbf{R}+\mathbf{r}/2)E^*(\mathbf{R}-\mathbf{r}/2)}\ . \tag{2.7}$$

The Fourier transform of this quantity with respect to $\mathbf{r}$, $\Gamma(\mathbf{R},\mathbf{q})$, is proportional to the intensity of radiation at $\mathbf{R}$ with wave vector $\mathbf{q}$. We also define a related quantity, $J(\mathbf{R},\mathbf{s})$, proportional to the intensity of radiation at $\mathbf{R}$ traveling in the direction of the unit vector $\mathbf{s}$, by

$$J(\mathbf{R},\mathbf{s})=2\int q^2dq\,\Gamma(\mathbf{R},q\mathbf{s})\ . \tag{2.8}$$

The correlation function (2.7) satisfies the equation

$$\Gamma(\mathbf{R},\mathbf{r})=\bar{E}(\mathbf{R}+\mathbf{r}/2)\bar{E}^*(\mathbf{R}-\mathbf{r}/2)+\int D(\mathbf{R}+\mathbf{r}/2-\rho_1)D^*(\mathbf{R}-\mathbf{r}/2-\rho_2)\times P(\rho_1\rho_2\rho_3\rho_4)\Gamma\left[\frac{\rho_3+\rho_4}{2},\rho_3-\rho_4\right]\ , \tag{2.9}$$

where P is the vertex part. In the weak scattering limit we can calculate P by a perturbation method. According to the theory of weak electron localization,[2] in this limit we must include the ladder and maximally crossed graphs.[12] These graphs give

$$P(\rho_1\rho_2\rho_3\rho_4)=\Delta\delta(\rho_1-\rho_2)\delta(\rho_3-\rho_4)[\delta(\rho_1-\rho_3)+\Delta|D(\rho_1-\rho_3)|^2+\cdots]+\Delta\delta(\rho_1-\rho_4)\delta(\rho_2-\rho_3)[\Delta|D(\rho_1-\rho_3)|^2+\cdots]\ . \tag{2.10}$$

In the case of an infinite space we can sum these graphs by taking the Fourier transform of (2.9) (omitting the coherent terms)

$$\Gamma(\mathbf{K},\mathbf{q})=\Delta D(\mathbf{q}+\mathbf{K}/2)D^*(\mathbf{q}-\mathbf{K}/2)\times\frac{1}{V}\sum_p\left[\frac{1}{1-Q(K)}+\frac{Q(\mathbf{q}+\mathbf{p})}{1-Q(\mathbf{q}+\mathbf{p})}\right]\Gamma(\mathbf{K},\mathbf{p})\ , \tag{2.11}$$

where

$$Q(K)=\frac{\Delta}{V}\sum_p D(\mathbf{p}+\mathbf{K}/2)D^*(\mathbf{p}-\mathbf{K}/2)\ . \tag{2.12}$$

Owing to the identity (2.6), for small K, $Q(K)\sim 1-K^2l^2/3$. Thus the first term in (2.11) has the diffusion form and the second term representing the coherent back scattering peaks strongly in the backward direction $\mathbf{q}+\mathbf{p}=0$. When the field is slowly varying in space we can neglect the $\mathbf{K}$ dependence of the prefactor in (2.11) and write

$$\Gamma(\mathbf{K},\mathbf{q})=\frac{1}{k}f(q)J(\mathbf{K},\mathbf{s})\ , \tag{2.13}$$

where $f(q)=|D(q)|^2\Sigma_i/\pi$. Then, integrating (2.11) over the magnitudes of q and p

$$J(\mathbf{K},\mathbf{s})=\frac{3}{4\pi k^2l^2}\int ds'\left[\frac{k^2}{K^2}+f_1(\mathbf{s}\cdot\mathbf{s}')\right]J(\mathbf{K},\mathbf{s}')\ , \tag{2.14}$$

where[11]

$$f_1(\mathbf{s}\cdot\mathbf{s}')=\frac{1}{\sqrt{2(1+\cos\theta)}[\beta+\sqrt{2(1+\cos\theta)}]}\ , \tag{2.15}$$

where $\beta=1/kl$ and $\mathbf{s}\cdot\mathbf{s}'=\cos\theta$. The cross section f_1 diverges in the backward direction $1+\cos\theta=0$ and has a width $\theta\sim\beta$. This divergence is a consequence of the infinite space and the corresponding infinite number of diffusion paths. In reflection from a half-space or a finite geometry the extent of the diffusion paths gets cut off and the cross section becomes finite.

III. REFLECTION FROM A HALF-SPACE

We suppose that the scattering medium occupies the half-space $z>0$ and that the light is incident normally on the half-space from $z<0$. In (2.9), we omit the coherent terms and evaluate $\Gamma(\mathbf{R},\mathbf{r})$ at $\mathbf{R}=0$. In the case of weak scattering, we can replace Γ on the right by the coherent part of the field, i.e.,

$$\Gamma\left[\frac{\rho_3+\rho_4}{2},\rho_3-\rho_4\right]\sim\bar{E}(\rho_3)\bar{E}^*(\rho_4)\ .$$

The change of variables

$$\rho_{1,2}=\mathbf{R}_1\pm\rho/2,\quad \rho_{3,4}=\mathbf{R}_2\pm\rho'/2\ ,$$
$$\rho_{1,4}=\mathbf{R}_1\pm\rho/2,\quad \rho_{2,3}=\mathbf{R}_2\mp\rho'/2\ , \tag{3.1}$$

in the two sets of terms (2.10), respectively, brings them into the diffusion form, and substituting in (2.9) gives the intensity of the light reflected from the plane $z=0$ in the direction $\mathbf{s}$

$$J(\mathbf{s})=\Delta \int d\mathbf{R}_1 d\mathbf{R}_2 \left[F(\mathbf{s},\mathbf{R}_1)P^{(L)}(\mathbf{R}_1-\mathbf{R}_2)\,|E(\mathbf{R}_2)|^2+F\left[\mathbf{s},\frac{\mathbf{R}_1+\mathbf{R}_2}{2}\right]P^{(C)}(\mathbf{R}_1-\mathbf{R}_2)e^{ik\mathbf{s}\cdot(\mathbf{R}_2-\mathbf{R}_1)}\bar{E}(\mathbf{R}_2)\bar{E}^*(\mathbf{R}_1)\right] , \tag{3.2}$$

where $J(\mathbf{s})=J(R=0,\mathbf{s})$, and

$$F(\mathbf{s},\mathbf{R})=2\int q^2 dq \int d\mathbf{r}\, e^{-i\mathbf{q}\cdot\mathbf{r}} D\left[\frac{\mathbf{r}}{2}-\mathbf{R}\right] D^*\left[\frac{\mathbf{r}}{2}+\mathbf{R}\right] . \tag{3.3}$$

The quantities $P^{(L,C)}$ in (3.2) are both of the diffusion form (L and C denote ladder and crossed graphs, respectively) with Fourier transforms

$$\begin{aligned} P^{(L)}(K)&=\frac{1}{1-Q(K)}\sim\frac{3}{K^2l^2} ,\\ P^{(C)}(K)&=\frac{Q(K)}{1-Q(K)}\sim\frac{3}{K^2l^2} . \end{aligned} \tag{3.4}$$

The solution to the half-space problem is obtained from the infinite-space case by the method of images. We restrict the integrations in (3.2) to the half-space $z>0$. The diffusion propagators P are required to vanish on the plane $z=0$. A slightly better approximation is to require these propagators to vanish on a plane $z=-z_0$ with $z_0\sim l$. In the interest of simplicity we will ignore this detail, and thus P in (3.2) are replaced by

$$P^{(L,C)}(\mathbf{R}_1-\mathbf{R}_2)\to[P^{(L,C)}(x_1-x_2,y_1-y_2,z_1-z_2)-P^{(L,C)}(x_1-x_2,y_1-y_2,z_1+z_2)] . \tag{3.5}$$

The coherent field in (3.2) is $\bar{E}(R)=E_0 e^{i\mathbf{k}_i\cdot\mathbf{R}-z/2l}$ where $\mathbf{k}_i$ is the incident wave vector (along z). The function $F(\mathbf{s},\mathbf{R})$ is well approximated by (see Appendix A)

$$F(-\hat{\mathbf{z}},\mathbf{R})=\pi e^{-z/l}\delta(x)\delta(y) , \tag{3.6}$$

where $\hat{\mathbf{z}}$ is a unit vector along z. Substituting the above results in (3.2), we obtain for the light reflected close to the backward direction

$$J(\mathbf{s})=\frac{3\Delta}{4l^2}E_0^2\int dz_1dz_2d^2\mathbf{r}\exp[-(z_1+z_2)/l]P(\mathbf{r},z_1,z_2)(1+\exp\{ik[\mathbf{s}\cdot\mathbf{r}+(1-\mathbf{s}\cdot\hat{\mathbf{z}})(z_2-z_1)]\}) \tag{3.7}$$

where

$$P(\mathbf{r},z_1,z_2)=\frac{1}{[r^2+(z_1-z_2)^2]^{1/2}}-\frac{1}{[r^2+(z_1+z_2)^2]^{1/2}} , \tag{3.8}$$

and $\mathbf{r}$ is a two-dimensional vector in the x,y plane. This result agrees with that of Ref. 10. The term

$$k(1-\mathbf{s}\cdot\hat{\mathbf{z}})\,(z_2-z_1)\sim(1+\cos\theta)kl$$

is small for $\pi-\theta\sim 1/kl$. Neglecting it and carrying out the integrals (see Appendix B)

$$J(s)=\frac{3\pi\Delta l}{2}E_0^2\left[1+\frac{1}{(\eta l+1)^2}\right] , \tag{3.9}$$

where $\eta=k\sin\theta$ and θ is the angle between $\mathbf{s}$ and the z axis. The intensity in the backward direction ($\eta=0$) is increased by a factor 2 over the background and the coherent backscattering is confined to a cone of angular width $\theta\sim 1/kl$. This result was obtained by Akkermans, Wolf, and Maynard,[10] who also include the effects of the more accurate boundary condition. This line shape only applies in the scalar case, and we now consider the effects of polarization.

IV. POLARIZED LIGHT

The previous results are easily generalized to include the effects of polarization of the light. The vector field $E_i(\mathbf{r})$ of the light satisfies the wave equation (2.1) with $\nabla\cdot\mathbf{E}=0$. A term $\nabla[\mathbf{E}\cdot\nabla\ln(1+\epsilon')]$ has been omitted from (2.1) which is small if the disorder is slowly varying. We begin by assuming that the scattering is due to isotropically polarizable particles, i.e., ϵ' is a scalar. In Appendix C we consider the case of anisotropic particles. The average Green's function of (2.1) is now of the form

$$D_{ij}(q)=\delta_{ij}(q)D(q) , \tag{4.1}$$

where $\delta_{ij}(q)=\delta_{ij}-\hat{q}_i\hat{q}_j$, $\hat{\mathbf{q}}$ is a unit vector, $D^{-1}(q)=k^2-q^2-\Sigma$ and

$$\Sigma=\frac{2\Delta}{3V}\sum_q D(q) , \tag{4.2}$$

and (2.6) is replaced by $1=2\Delta/3V\sum_q|D(q)|^2$. The mean free path is $l=6\pi/\Delta$. The correlation function of the field is a tensor

$$\Gamma_{ij}(\mathbf{R},\mathbf{r})=\overline{E_i(\mathbf{R}+\mathbf{r}/2)E_j^*(\mathbf{R}-\mathbf{r}/2)} . \tag{4.3}$$

We first examine the case of scattering in an infinite medium. Equation (2.11) now becomes (repeated indices are to be summed)

$$\Gamma_{ij}(\mathbf{K},\mathbf{q})=\Delta D_{im}(\mathbf{q}+\mathbf{K}/2)D^*_{jn}(\mathbf{q}-\mathbf{K}/2)\frac{1}{V}\sum_p \left[\left\langle mn\left|\frac{1}{1-Q(K)}\right|rs\right\rangle+\left\langle ms\left|\frac{Q(\mathbf{q}+\mathbf{p})}{1-Q(\mathbf{q}+\mathbf{p})}\right|rn\right\rangle\right]\Gamma_{rs}(\mathbf{K},\mathbf{p}) , \quad (4.4)$$

where the matrix Q is given by

$$Q_{mnrs}(K)=\frac{\Delta}{V}\sum_p D_{mr}(\mathbf{p}+\mathbf{K}/2)D^*_{ns}(\mathbf{p}-\mathbf{K}/2) . \quad (4.5)$$

In order to evaluate (4.4), we require the eigenvectors and eigenvalues of the matrix Q, and these can be found by expanding Q in powers of K^2. The eigenvectors can be found from

$$Q_{mnrs}(0)=\tfrac{1}{10}(6\delta_{mr}\delta_{ns}+\delta_{mn}\delta_{rs}+\delta_{ms}\delta_{nr}) , \quad (4.6)$$

and using these eigenvectors, the eigenvalues can be found to order K^2 from the expansion of $Q(K)$. The normalized eigenvectors and eigenvalues λ are given by

$$\delta_{rs}/\sqrt{3},\quad \lambda_1=1-K^2l^2/3 , \quad (4.7)$$

$$\delta_{rs}(\delta_{r,x}+\omega\delta_{r,y}+\omega^2\delta_{r,z})/\sqrt{3},\quad \lambda_2=\tfrac{7}{10}(1-\tfrac{1}{3}K^2l^2) , \quad (4.8)$$

and the complex conjugate ($\omega=e^{2\pi i/3}$),

$$(\delta_{ra}\delta_{sb}\pm\delta_{rb}\delta_{sa})/\sqrt{2} ,$$

$$\lambda_+=\frac{7}{10}-\frac{l^2}{70}(23K^2-10K_a^2-10K_b^2) , \quad (4.9)$$

$$\lambda_-=\frac{1}{2}-\frac{l^2}{10}(3K^2-2K_a^2-2K_b^2) ,$$

where $a,b=x,y,z$ and $a\neq b$. This gives a total of nine eigenvectors as required.

We use these eigenvectors and eigenvalues in (4.4) and write

$$\Gamma_{ij}(\mathbf{K},q)=\frac{1}{k}f(q)J_{ij}(\mathbf{K},\mathbf{s})$$

as in (2.13). After integration over the magnitudes of q and p we obtain the generalization of (2.14)

$$J_{ij}(\mathbf{K},\mathbf{s})=\frac{3}{8\pi}\int d\mathbf{s}'[\sigma^{(L)}_{ijrs}(K)+\sigma^{(C)}_{ijrs}(\mathbf{s},\mathbf{s}')]J_{rs}(\mathbf{K},\mathbf{s}') . \quad (4.10)$$

Each of the cross sections in (4.10) can be written as the sum of four parts corresponding to the four types of eigenvectors in (4.7)–(4.9). Thus

$$\sigma^{(L,C)}_{ijrs}=\sum_{i=1}^{4}\sigma^{(L,C,i)}_{ijrs} ,$$

where

$$\sigma^{(L1)}_{ijrs}=\delta_{ij}(s)\delta_{rs}/K^2l^2 , \quad (4.11)$$

$$\sigma^{(L2)}_{ijrs}=[\delta_{ij}(3\delta_{ir}-1)-s_is_j(3\delta_{ir}+3\delta_{jr}-1-3s_r^2)]\times\delta_{rs}f^{(L2)}(K) , \quad (4.12)$$

$$\sigma^{(L3,4)}_{ijrs}=\sum_{ab}[\delta_{ia}(s)\delta_{jb}(s)\pm\delta_{ib}(s)\delta_{ja}(s)]\delta_{ra}\delta_{sb}f^{(L3,4)}_{ab}(K) , \quad (4.13)$$

$$\sigma^{(C1)}_{ijrs}=\frac{\delta_{is}(s)\delta_{jr}(s)}{l^2k^2}f^{(1)}(\mathbf{s}\cdot\mathbf{s}') , \quad (4.14)$$

$$\sigma^{(C2)}_{ijrs}=\delta_{is}(s)\delta_{jr}(s)(3\delta_{rs}-1)f^{(C2)}(k(\mathbf{s}+\mathbf{s}')) , \quad (4.15)$$

$$\sigma^{(C3,4)}_{ijrs}=\sum_{ab}[\delta_{ia}(s)\delta_{jb}(s)\delta_{bs}\delta_{ar}\pm\delta_{ia}(s)\delta_{ja}(s)\delta_{sb}\delta_{rb}]\times f^{(C3,4)}_{ab}(k(\mathbf{s},\mathbf{s}')) , \quad (4.16)$$

where

$$f^{(L2)}(K)=\frac{10}{9+7K^2l^2} , \quad (4.17)$$

$$f^{(L3)}_{ab}(\mathbf{K})=\frac{35(1-\delta_{ab})}{21+(23K^2-10K_a^2-10K_b^2)l^2} , \quad (4.18)$$

$$f^{(L4)}_{ab}(\mathbf{K})=\frac{5(1-\delta_{ab})}{5+(3K^2-2K_a^2-2K_b^2)l^2} , \quad (4.19)$$

and $f^{(C2)}=\frac{7}{10}f^{(L2)}$, $f^{(C3)}=\frac{7}{10}f^{(L3)}$, and $f^{(C4)}=\frac{1}{2}f^{(L4)}$. The upper and lower signs in (4.13) and (4.16) apply to the indices 3 and 4, respectively.

The form of the scattering in (4.10) is simply illustrated by considering light being scattered from $\mathbf{s}'$ to $\mathbf{s}$ (see Fig. 1) so that Eq. (4.10) is

$$J_{ij}(K,\mathbf{s})=\frac{3}{8\pi}\sigma_{ijrs}(\mathbf{s},\mathbf{s}')J'_{rs}(\mathbf{K},\mathbf{s}') . \quad (4.20)$$

We choose the x axis perpendicular to the scattering plane defined by $\mathbf{s}'$ and $\mathbf{s}$ and the y axis in this plane and perpendicular to $\mathbf{s}$. Thus for the incident light $J'_{xx}=J'$, $J'_{yy}=\cos^2\theta J'$, $J'_{zz}=\sin^2\theta J'$, where θ is the scattering angle. For the scattered light $J_{xx}=J_\perp$ and $J_{yy}=J_{\parallel}$. Equation (4.20) can now be written

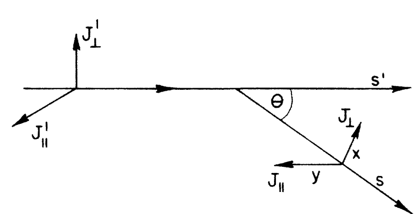

FIG. 1. Scattering of light from $\mathbf{s}'$ to $\mathbf{s}$. $J'_\perp, J_\perp$ and $J'_{\parallel}, J_{\parallel}$ are the incident and scattered intensities perpendicular and in the scattering plane, respectively.

$$J_{\parallel}=\frac{3}{8\pi}\left[\frac{1}{K^2l^2}+(3\cos^2\theta-1)f^{(L2)}(K)+\cos^2\theta\left[\frac{f^{(1)}(\mathbf{s}\cdot\mathbf{s}')}{k^2l^2}+2f^{(C2)}\right]+\sin^2\theta(f_{yz}^{(C3)}-f_{yz}^{(C4)})\right]J'_{\parallel}$$
$$+\frac{3}{8\pi}\left[\frac{1}{K^2l^2}-f^{(L2)}(K)+f_{xy}^{(C3)}-f_{xy}^{(C4)}\right]J'_{\perp} \tag{4.21}$$

$$J_{\perp}=\frac{3}{8\pi}\left[\frac{1}{K^2l^2}-f^{(L2)}(K)+\cos^2\theta(f_{xy}^{(C3)}-f_{xy}^{(C4)})+\sin^2\theta(f_{xz}^{(C3)}-f_{xz}^{(C4)})\right]J'_{\parallel}$$
$$+\frac{3}{8\pi}\left[\frac{1}{K^2l^2}+2f^{(L2)}(K)+\frac{f^{(1)}(\mathbf{s}\cdot\mathbf{s}')}{k^2l^2}+2f^{(C2)}\right]J'_{\perp}\ . \tag{4.22}$$

The first two terms in each of the large square brackets are the Rayleigh scattering terms and the last two terms in each bracket arise from weak localization. The argument of f is $f(k(\mathbf{s}+\mathbf{s}'))$ if it is not explicitly included. We discuss these latter terms. The most important backscattering term is $f^{(1)}(\mathbf{s}\cdot\mathbf{s}')$ [Eq. (2.15)] which preserves the polarization. This term was discussed in Ref. 11. The most important depolarizing term of the backscattering terms is

$$f_{xy}^{(C3)}-f_{xy}^{(C4)}=\frac{49}{21+k^2l^2[46(1+\cos\theta)-10\sin^2\theta]}-\frac{5}{5+k^2l^2[6(1+\cos\theta)-2\sin^2\theta]}\ . \tag{4.23}$$

The line shape of the depolarized light (4.23) is approximately Lorentzian and very different from the polarized component (2.15). These results apply in the case of the infinite medium. In order to compare with experiment we now consider scattering from a half-space.

V. REFLECTION OF POLARIZED LIGHT FROM A HALF-SPACE

We consider the same geometry as in Sec. III in which light polarized along x is incident normally on the half plane $z>0$ occupied by the scattering medium. Equation (3.2) for the light intensity reflected from the plane $z=0$ generalizes to

$$J_{ij}(\mathbf{s})=\Delta\int d\mathbf{R}_1 d\mathbf{R}_2\left[F_{ijmn}(\mathbf{s},\mathbf{R}_1)P_{mnrs}^{(L)}(\mathbf{R}_1-\mathbf{R}_2)\bar{E}_r(\mathbf{R}_2)\bar{E}_s^*(\mathbf{R}_2)\right.$$
$$\left.+F_{ijmn}\left[\mathbf{s},\frac{\mathbf{R}_1+\mathbf{R}_2}{2}\right]P_{mnrs}^{(C)}(\mathbf{R}_1-\mathbf{R}_2)e^{ik\mathbf{s}\cdot(\mathbf{R}_2-\mathbf{R}_1)}\bar{E}_r(\mathbf{R}_2)\bar{E}_s^*(\mathbf{R}_1)\right]\ , \tag{5.1}$$

where

$$F_{ijmn}(\mathbf{s},\mathbf{R})=2\int q^2dq\int d\mathbf{r}\,e^{-i\mathbf{q}\cdot\mathbf{r}}D_{im}(\mathbf{r}/2-\mathbf{R})D_{jn}^*(\mathbf{r}/2+\mathbf{R}) \tag{5.2}$$

and the kernels $P^{(L,C)}$ each consists of four parts

$$P^{(L,C)}=\sum_{i=1}^{4}P^{(L,C,i)}\ .$$

The Fourier transforms of these quantities are

$$P_{mnrs}^{(L1)}(K)=\delta_{mn}\delta_{rs}/K^2l^2\ , \tag{5.3}$$

$$P_{mnrs}^{(L2)}(K)=\delta_{mn}\delta_{rs}(3\delta_{mr}-1)f^{(L2)}(K)\ , \tag{5.4}$$

$$P_{mnrs}^{(L3,4)}(K)=\sum_{ab}(\delta_{ma}\delta_{nb}\pm\delta_{mb}\delta_{na})\delta_{ra}\delta_{sb}f_{ab}^{(L3,4)}(K)\ . \tag{5.5}$$

$P_{mnrs}^{(Ci)}$ is obtained from $P_{mnrs}^{(Li)}$ by interchanging the indices n and s on the right-hand side of (5.3)—(5.5) and replacing $f^{(Li)}$ by $f^{(Ci)}$.

The light scattered from the half-space is now obtained as before from (5.1) by restricting the integrals to the half space $z>0$. The kernels (5.3)—(5.5) are calculated in real space and the boundary conditions satisfied by the method of images as in Eq. (3.5). The coherent field on the right in (5.1) is $\bar{E}_r(\mathbf{R})=E_0\delta_{rx}e^{i\mathbf{k}_i\cdot\mathbf{R}-z/2l}$ with $\mathbf{k}_i$ along z. In the above geometry F in (5.1) is

$$F_{ijmn}(-\hat{\mathbf{z}},\mathbf{R})=\pi e^{-z/l}\delta_{im}\delta_{jn}\delta(x)\delta(y)\ . \tag{5.6}$$

The above results are substituted in (5.1) and the intensities of the light reflected close to the backward direction polarized along x and y, respectively, are

$$J_{xx}(\mathbf{s})=\frac{\Delta E_0^2}{4l^2}\int dz_1dz_2d^2\mathbf{r}\,e^{-(z_1+z_2)/l}[P_0^{(L)}(r,z_1,z_2)+e^{ik\mathbf{s}\cdot\mathbf{r}}P_0^{(C)}(\mathbf{r},z_1,z_2)]\ , \tag{5.7}$$

$$J_{yy}(\mathbf{s})=\frac{\Delta E_0^2}{4l^2}\int dz_1dz_2d^2\mathbf{r}\,e^{-(z_1+z_2)/l}[P_1(r,z_1,z_2)+e^{ik\mathbf{s}\cdot\mathbf{r}}P_2(r,z_1,z_2)]\ . \tag{5.8}$$

As in (3.7), we have neglected the small term $k(1-\mathbf{s}\cdot\mathbf{z})(z_1-z_2)$. The four kernels in (5.7) and (5.8) are each of the form

$$P_i(r,z_1,z_2)=P_i(r,z_1-z_2)-P_i(r,z_1+z_2) \tag{5.9}$$

where

$$P_0^{(L)}(r,z)=\frac{1}{L_1}(1+\tfrac{20}{7}e^{-L_1/l_1}) , \tag{5.10}$$

$$P_0^{(C)}(r,z)=\frac{1}{L_1}(1+2e^{-L_1/l_1}) , \tag{5.11}$$

$$P_1(r,z)=\frac{1}{L_1}(1-\tfrac{10}{7}e^{-L_1/l_1}) , \tag{5.12}$$

$$P_2(r,z)=\frac{49}{26L_2}e^{-L_2/l_2}-\frac{5}{2L_3}e^{-L_3/l_3} , \tag{5.13}$$

where $L_1^2=r^2+z^2$, $L_2^2=\frac{23}{13}r^2+z^2$, $L_3^2=3r^2+z^2$, and $l_1^2=\frac{7}{9}l^2$, $l_2^2=\frac{23}{21}l^2$, $l_3^2=\frac{3}{5}l^2$.

We substitute (5.10)–(5.13) in (5.9) and then in (5.7) and (5.8) and carry out the integration (see Appendix B). The reflected intensities are

$$J_{xx}(\mathbf{s})=C\left[1+\frac{1}{(1+\eta l)^2}+\frac{20}{7}\left[\frac{1}{(1+l/l_1)^2}+\frac{7/10}{\left[1+\frac{l}{l_1}(1+\eta^2 l_1^2)^{1/2}\right]^2}\right]\right] , \tag{5.14}$$

$$J_{yy}(\mathbf{s})=C\left[1-\frac{10}{7}\frac{1}{(1+l/l_1)^2}+\frac{49/46}{\left\{1+\frac{l}{l_2}[1+\eta^2(l_2')^2]^{1/2}\right\}^2}-\frac{5/6}{\left\{1+\frac{l}{l_3}[1+\eta^2(l_3')^2]^{1/2}\right\}^2}\right] . \tag{5.15}$$

where $(l_2')^2=\frac{13}{21}l^2$, $(l_3')^2=l^2/5$, $\eta=k\sin\theta$, and $C=\pi\Delta E_0^2 l/2$. These results give the diffusely reflected light from a half-space close to the backward direction $\theta\sim\pi$. The incident light is polarized along x and thus (5.14) gives the polarized component and (5.15) the depolarized component.

VI. DISCUSSION

We now discuss the form of the coherent backscattering from the half-space (5.14) and (5.15) and compare with the experimental data of Etemad.[8] For polarized scattering (5.14) the intensity in the backward direction ($\eta=0$) is increased by a factor 1.9 over the background ($\eta l>1$). The backscattered light has two peaks: the second term in (5.14) which was given by Akkermans, Wolf, and Maynard,[10] and the fourth term which is approximately Lorentzian. The relative heights of these two peaks are 1 and $2(1+l/l_1)^{-2}=0.44$ and their widths at half maximum are $\theta_{1/2}=0.414/kl$ and $\theta_{1/2}=1.67/kl$, respectively. These results are in good qualitative agreement with the results of Etemad. A plot of the angular dependence of the backscattering is given in Fig. 2.

For the depolarized scattering (5.15) the intensity in the backward direction ($\eta=0$) is increased over the background ($\eta l>1$) by a factor of 1.2. The backscattered light again has two peaks, the third and fourth terms in (5.15). The negative sign of the fourth term indicates destructive interference. The relative heights of these two peaks are 0.28 and -0.16 and their widths are $1.97/kl$ and $3.17/kl$, respectively. The fourth term in (5.15) is thus quite broad and may be difficult to distinguish from the background. The angular dependence of the depolarized backscattering is shown in Fig. 2. The sharp peak present in the polarized scattering is absent in the depolarized part again in qualitative agreement with Etemad's results.

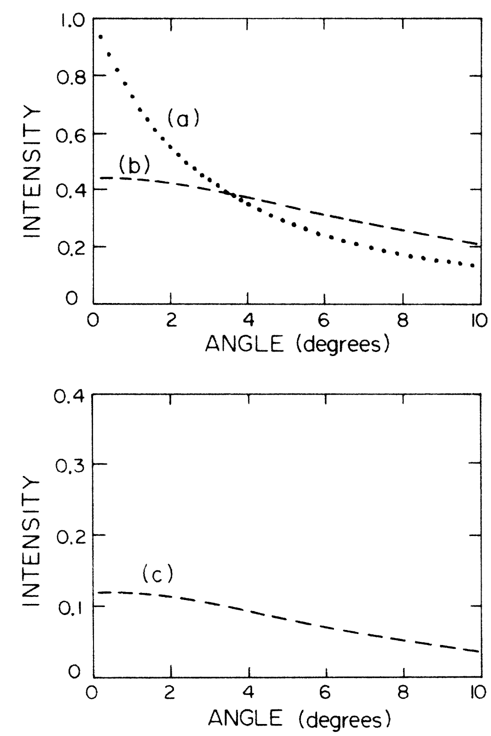

FIG. 2. The angular dependence of the coherent backscattering for $kl=10$. Curves a and b are the two polarized components [the second and fourth terms in Eq. (5.14)] and c the depolarized component [the third and fourth terms in Eq. (5.15)].

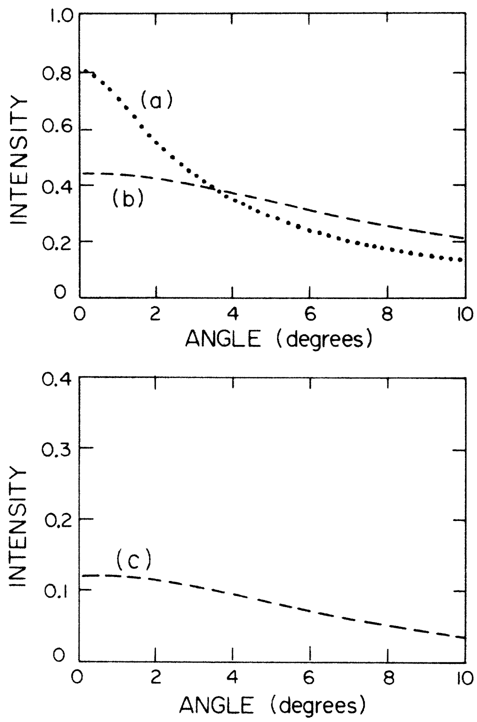

FIG. 3. The angular dependence of the coherent backscattering for $kl=10$ for a slab of finite width $L=10l$. Curves a and b are the two polarized components [the second and fourth terms in Eq. (D5)] and curve c is the depolarized component [the third and fourth terms in Eq. (D6)].

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APPENDIX A

We calculate $F(\mathbf{s},\mathbf{R})$ in Eq. (3.3) with

$$D(R)=\frac{\exp\left[ik-\frac{1}{2l}\right]R}{4\pi R} .$$

Then for $r<R$,

$$D\left[\frac{r}{2}-R\right]D^*\left[\frac{r}{2}+R\right]\sim\frac{\exp(-ik\mathbf{R}\cdot\mathbf{r}/R-R/l)}{(4\pi R)^2} , \tag{A1}$$

and substituting in (3.3) gives

$$F(\mathbf{s},\mathbf{R})=\frac{\pi}{R^2}e^{-R/l}\int q^2dq\,\delta(\mathbf{q}+k\mathbf{R}/R) . \tag{A2}$$

For $\mathbf{q}$ along $-z$ we get

$$F(-\hat{\mathbf{z}},\mathbf{R})=\pi e^{-z/l}\delta(x)\delta(y) . \tag{A3}$$

APPENDIX B

The integrals required to evaluate (5.7) and (5.8) are all of the form

$$\int_0^\infty dz_1dz_2\int d^2\mathbf{r}\exp(i\mathbf{q}\cdot\mathbf{r})\exp[-(z_1+z_2)/l]\left\{\frac{1}{[r^2+(z_1-z_2)^2]^{1/2}}\exp(-[r^2+(z_1-z_2)^2]^{1/2}/l_1)\right.$$
$$\left.-\frac{1}{[r^2+(z_1+z_2)^2]^{1/2}}\exp(-[r^2+(z_1+z_2)^2]^{1/2}/l_1)\right\} \tag{B1}$$

Introducing a new variable $u=z_1\pm z_2$ in the two terms in the square bracket respectively and integrating over the remaining z variable (B1) becomes

$$\int_0^\infty du\,(l-u)e^{-u/l}\int d^2\mathbf{r}\,e^{i\mathbf{q}\cdot\mathbf{r}}\frac{1}{(r^2+u^2)^{1/2}}\exp[-(r^2+u^2)^{1/2}/l_1] . \tag{B2}$$

This reduces to[13]

$$\frac{2\pi l_1}{(1+q^2l_1^2)^{1/2}}\int_0^\infty du\,(l-u)\exp[-u(1+q^2l_1^2)^{1/2}/l_1-u/l]=\frac{2\pi l^3}{\left[1+\frac{l}{l_1}(1+q^2l_1^2)^{1/2}\right]^2} . \tag{B3}$$

The remaining integrals can be obtained by setting $q=0$ and/or $l_1=\infty$ in (B3).

APPENDIX C

In this Appendix we generalize the previous results to the case where the particles responsible for the scattering have an anisotropic polarizability. The dielectric constant fluctuations in (2.1) are replaced by a tensor $\delta\epsilon'_{mr}(\mathbf{r})$ with correlation function

$$k^4\langle\delta\epsilon_{mr}(\mathbf{r})\delta\epsilon_{ns}(\mathbf{r}')\rangle=\Delta\delta(\mathbf{r}-\mathbf{r}')C_{mnrs} , \tag{C1}$$

where

$$C_{mnrs}=\frac{1}{1+2\gamma}[(1-2\gamma)\delta_{mr}\delta_{ns}+\gamma(\delta_{mn}\delta_{rs}+\delta_{ms}\delta_{nr})] . \tag{C2}$$

γ is a measure of the anisotropy and is given by $\gamma=(a-b)/(3a+2b)$ where $a=\alpha_1^2+\alpha_2^2+\alpha_3^2$, $b=\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1$ and the α are the principal polarizabilities of the scattering particles. For a low density n of scatterer $\Delta\sim 16\pi^2 nk^4a/3$. The eigenvectors of C_{mnrs} are the same as those of $Q_{mnrs}(0)$ given in (4.7)–(4.9). The corresponding eigenvalues denoted by λ' are

$$\lambda_1'=1,\ \lambda_2'=\frac{1-\gamma}{1+2\gamma},\ \lambda_+'=\frac{1-\gamma}{1+2\gamma},\ \lambda_-'=\frac{1-3\gamma}{1+2\gamma}\ . \quad \text{(C3)}$$

In (4.4), $1/(1-Q)$ and $Q/(1-Q)$ get replaced by $C/(1-CQ)$ and $CQC/(1-CQ)$, respectively. The fact that C and Q have the same eigenvectors means that the cross sections (4.17)–(4.19) get multiplied by certain scale factors depending on (C3). Thus the f functions in (4.12) and (4.13) are now replaced by

$$f'^{(L2)}(K)=\frac{1-\gamma}{1+9\gamma}f^{(L2)}(K,l_a)\ ,$$
$$f'^{(L3)}(K)=\frac{1-\gamma}{1+9\gamma}f^{(L3)}(K,l_a)\ , \quad \text{(C4)}$$
$$f'^{(L4)}(K)=\frac{1-3\gamma}{1+7\gamma}f^{(L4)}(K,l_b)\ ,$$

where $l_a^2=(1-\gamma)l^2/(1+9\gamma)$ and $l_b^2=(1-3\gamma)l^2/(1+7\gamma)$. The notation on the right-hand side of (C4) means for example that in $f^{(L2)}(K)$ of (4.17) l is replaced by l_a and the whole function multiplied by a factor $(1-\gamma)/(1+9\gamma)$. In a similar way the f functions in (4.15) and (4.16) are replaced by

$$f'^{(C2)}(K)=\left[\frac{1-\gamma}{1+2\gamma}\right]\left[\frac{1-\gamma}{1+9\gamma}\right]f^{(C2)}(K,l_a)\ ,$$
$$f'^{(C3)}(K)=\left[\frac{1-\gamma}{1+2\gamma}\right]\left[\frac{1-\gamma}{1+9\gamma}\right]f^{(C3)}(K,l_a)\ , \quad \text{(C5)}$$
$$f'^{(C4)}(K)=\left[\frac{1-3\gamma}{1+2\gamma}\right]\left[\frac{1-3\gamma}{1+7\gamma}\right]f^{(C4)}(K,l_b)\ .$$

These same factors then carry over into (4.21), (4.22), (5.4), and (5.5).

The reflected intensities from the half-space (5.14) and (5.15) now become

$$J_{xx}(s)=C\left[1+\frac{1}{(1+\eta l)^2}+\frac{20}{7}\left[\frac{1}{(1+l/l_{1a})^2}+\frac{1-\gamma}{1+2\gamma}\frac{7/10}{[1+l/l_{1a}(1+\eta^2 l_{1a}^2)^{1/2}]^2}\right]\right] \quad \text{(C6)}$$

$$J_{yy}(s)=C\left[1-\frac{10}{7}\frac{1}{(1+l/l_{1a})^2}+\frac{49}{46}\frac{1-\gamma}{1+2\gamma}\frac{1}{[1+l/l_{2a}(1+\eta^2 l_{2a}'^2)^{1/2}]^2}\right.$$
$$\left.-\frac{5}{6}\frac{1-3\gamma}{1+2\gamma}\frac{1}{[1+l/l_{3b}(1+\eta^2 l_{3b}'^2)^{1/2}]^2}\right] \quad \text{(C7)}$$

where $l_{1a}^2=\frac{7}{9}l_a^2$, $l_{2a}^2=\frac{23}{21}l_a^2$, $l_{3b}^2=\frac{3}{5}l_b^2$, and $(l_{2a}')^2=\frac{13}{21}l_a^2$, $(l_{3b}')^2=l_b^2/5$. We thus see that polarizability anisotropy of the scattering particles does not qualitatively change the results.

APPENDIX D

In this Appendix we give the expressions for the reflected intensity from an infinite slab of thickness L. We consider the same geometry as in Sec. V in which light polarized along x is incident normally on the slab which occupies the region $0<z<L$. The boundary conditions that we use are that the diffusion propagators (3.5) and (5.10)–(5.13) vanish on both surfaces $z=0$ and $z=L$. This can be achieved by the method of images but an infinite set of images is required. Thus (3.5) and (5.9) are replaced by

$$P(r,z_1,z_2)=\sum_{n=-\infty}^{\infty}[P(\mathbf{r},z_1-z_2+2Ln)$$
$$-P(\mathbf{r},z_1+z_2+2Ln)]\ . \quad \text{(D1)}$$

The expressions (3.7), (5.7), and (5.8) can be evaluated as before. In the case of scalar waves

$$J(s)=\frac{3\pi\Delta l}{2}E_0^2[I(0)+I(\eta l)]\ , \quad \text{(D2)}$$

where $(L'=L/l)$

$$I(x)=\frac{1}{x(1+x)^2}\left[x-e^{-2L'(1+x)}+(1-x)e^{-2L'}+\frac{4x}{1-x}e^{-2L'}(1-e^{L'(1-x)})\right]$$
$$-\frac{1}{x}\frac{e^{-2L'x}}{1-e^{-2L'x}}\left[\frac{1-e^{-L'(1+x)}}{1+x}-\frac{1-e^{-L'(1-x)}}{1-x}\right]^2 \quad \text{(D3)}$$

and

$$I(0)=1-4e^{-L'}+3e^{-2L'}+2L'e^{-2L'}-\frac{2}{L'}(1-e^{-L'}-L'e^{-L'})^2 \, . \tag{D4}$$

In the case of polarized waves, (5.14) and (5.15) are replaced by

$$J_{xx}(\mathbf{s})=C\{I(0)+I(\eta l)+\tfrac{20}{7}I(l/l_1)+2I[(1+\eta^2 l_1^2)^{1/2}l/l_1]\} \, , \tag{D5}$$

$$J_{yy}(\mathbf{s})=C(I(0)-\tfrac{10}{7}I(l/l_1)+\tfrac{49}{46}I\{[1+\eta^2(l_2')^2]^{1/2}l/l_2\}-\tfrac{5}{6}I\{[1+\eta^2(l_3')^2]^{1/2}l/l_3\}) \, . \tag{D6}$$

A plot of the angular dependence of the backscattering is given in Fig. 3.

[1]G. Bergmann, Phys. Rep. **107**, 1 (1984).

[2]L. P. Gorkov, A. I. Larkin, and D. E. Khmelnitskii, Pis'ma Zh. Eksp. Teor. Fiz. **30**, 248 (1979) [JETP Lett. **30**, 228 (1979)]; E. Abrahams and T. V. Ramakrishnan, J. Non-Cryst. Solids **35**, 15 (1980).

[3]For references see K. M. Watson, J. Math Phys. **10**, 688 (1969).

[4]S. John, H. Sompolinsky, and M. J. Stephen, Phys. Rev. B **27**, 5592 (1983).

[5]E. Akkermans and R. Maynard, Phys. Rev. B **32**, 7850 (1985).

[6]M. P. van Albada and A. Lagendijk, Phys. Rev. Lett. **55**, 2692 (1985).

[7]P. E. Wolf and G. Maret, Phys. Rev. Lett. **55**, 2696 (1985).

[8]S. Etemad, R. Thompson, and H. J. Andrejco, Phys. Rev. Lett. **57**, 575 (1986); S. Etemad (unpublished).

[9]A. A. Golubentsev, Zh. Eksp. Teor. Fiz. **86**, 47 (1984) [Sov. Phys.—JETP **59**, 26 (1984)].

[10]E. Akkermans, P. E. Wolf, and R. Maynard, Phys. Rev. Lett. **56**, 1471 (1986).

[11]M. J. Stephen, Phys. Rev. Lett. **56**, 1809 (1986).

[12]J. S. Langer and T. Neal, Phys. Rev. Lett. **16**, 984 (1966).

[13]I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1980), pp. 721 and 6646.