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Ensemble and temperature averaging of quantum oscillations in normal-metal rings

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The Landauer conductance G between two appropriate contacts of a one-dimensional (1D, single-channel) ring with elastic scatterers is considered as function of an enclosed Aharonov-Bohm flux ϕ through the ring's opening. It is demonstrated analytically and numerically that upon averaging over an ensemble of different microscopic systems prepared under the same macroscopic conditions, the basic period of $G(\phi)$ changes from $\phi_0=hc/e$ to $\phi_0/2$. This agrees with existing experiments. It is also shown that self-averaging out of the ϕ_0-periodic component is obtained at temperatures higher than a suitable energy-correlation range which is discussed in several regimes. Special effects due to the broad distribution of resistances are found numerically and understood theoretically. These may lead to a non-Ohmic size dependence of the conductance even at finite temperatures when a classical addition of many rings is valid.

I. INTRODUCTION AND SUMMARY

One of the important insights gained by the recent advances in applying electron localization theory to conduction in small systems has been the elucidation of the different roles played by the elastic and inelastic scattering. The former leads via the interference of the electron waves to the various effects associated with localization. Only the latter scattering can really scramble the phases of the wave functions and reduce the effects of interference. It thus follows that at sufficiently low temperatures, a system which is smaller than the characteristic phase coherence length for the electrons (defined usually by the inelastic scattering) will exhibit the effects of interference even in the presence of a substantial amount of disorder (or elastic scattering).

An interesting example is that of a multiply connected system—a small ring[1–4] or a small-radius hollow cylinder with an Aharonov-Bohm—type flux ϕ through its opening. The interference of the electron waves around the opening is sensitive to the Aharonov-Bohm flux (which modifies the phase relationships for the wave functions). Thus, one expects some sensitivity of the physical properties of such systems to the flux ϕ. In fact, ϕ is equivalent, via a well-known gauge transformation, to a change in boundary conditions of the wave function around the hole where the phase of the wave function changes by

$$\Delta\varphi=2\pi\frac{\phi}{\phi_0} \tag{1.1}$$

whenever each electronic coordinate is taken once around the opening.[5] Here $\phi_0=hc/e\simeq 4\times 10^{-15}$ mks is the (single-electron) flux quantum. It follows that *all* the physical properties of such a system are exactly periodic in ϕ with a period ϕ_0. This is a rigorous theorem[5] whose domain of validity includes static disorder and all electron-electron and electron-static ion interactions. In fact, calculations on both equilibrium properties and the conductivity[1,2,4] between two points on one-dimensional (1D) models of such systems, exhibit sizeable oscillations with a period ϕ_0. The model used to calculate these conductivity oscillations is shown in Fig. 1. Without loss of generality the elastic scattering in the two branches of the ring may be represented by effective scatterers (1 and 2 in Fig. 1). Equilibrium properties (e.g., the energy) can be studied[1] for the isolated ring. Transport properties of the system can be calculated by coupling it to reservoirs at different electrochemical potentials and applying the Landauer formula.[6] Further discussion of the model is presented in the next section.

Notwithstanding the periodicity with period ϕ_0, one of the most interesting predictions for the case of weak disorder called "weak localization theory" has been the one by Al'tshuler, Aronov, and Spivak[7] on periodic oscillations of the (Kubo-type) conductance of small doubly connected samples (such as rings or cylinders) as function of the Aharonov-Bohm flux ϕ through their opening. One surprising aspect of the calculation has been that the fundamental period of the oscillations was not ϕ_0, as the general theorem referred to above would predict, but $\phi_0/2$. The $\phi_0/2$ period is the "first harmonic" of the ϕ_0 one, thus, this periodicity does not contradict the above theorem. The question is only why the fundamental ϕ_0 period does not appear.

Before answering this question we mention that the prediction of the $\phi_0/2$ oscillation following the pioneering experimental work by Sharvin and Sharvin,[8] has received very convincing experimental support. In the more recent experiments[8,9] on long cylinders, an almost quantitative agreement with the full theory (taking into account the non-Aharonov-Bohm magnetic field inside the material) was achieved. The $\phi_0/2$ oscillation has also been clearly seen in experiments on large arrays of many "rings."[10] In all those experiments, the fundamental ϕ_0 period has not been seen. Preliminary experiments[11] on single rings were inconclusive, but did show traces of perhaps both ϕ_0 and $\phi_0/2$ oscillations, with additional aperiodic structure which appears to be due to the non-Aharonov-Bohm portion of the magnetic field.[12] More recent experiments on single rings, reported during the preparation of this paper[13] have shown clear distinct oscillations with a ϕ_0 periodicity.

The answer to the dilemma of why the fundamental oscillation of period ϕ_0 has not been seen in some of the experiments, might have been that its amplitude is large only in the unrealistic 1D (or "single channel" case). One might have expected that in a more realistic "multichannel" situation, appropriate to an experimentally feasible fine line, the ϕ_0 component averages out. To check this, Buttiker *et al.*,[14] estimated the size of the ϕ_0-periodic component of the conductivity and found that it was of the order of $1/n$, where n was the number of channels. This is of the same order as the $\phi_0/2$ weak localization contribution. Thus the ratio of the two contributions is finite for large n. Recent results of Lee and Stone,[15] on the magnitude of the fluctuations and above oscillations can be interpreted[16] by defining n as the effective number of active independent channels. Quantitative results on many channels, incorporating averaging ideas mentioned below were very recently obtained by Stone and Imry.[17]

Thus, the explanation of the above dilemma has to be sought elsewhere. To do that, we note that both the theory of Ref. 7 and the experiments on cylinders and arrays involve effectively an ensemble averaging over many microscopically distinct systems prepared with the same overall macroscopic conditions. Thus, all rings in the array have similar impurity concentrations but the precise configuration of the impurities is obviously different from sample to sample. In the theoretical calculations, one ensemble averages from the very beginning in order to use propagators that depend only on relative distances (apart from boundary-effects). In the cylinder experiments, the resistance is measured *along* a ~ 1 cm long cylinder which consists of around 10^4 pieces of length l_ϕ added classically, l_ϕ being some phase breaking length. Now, the work[2] on rings with contacts suggests that the Fourier coefficient corresponding to the ϕ_0-periodic part of the oscillation does not have a definite phase. On the other hand, the $\phi_0/2$ Fourier coefficient does have a definite phase [for example, $G(\phi)$ is minimal (maximal) at $\phi=0$ for systems without (with) spin-orbit scattering[18]]. This is due to the part following from backscattering around the loop at the origin (see, however, Ref. 19). Thus the ensemble averaging may eliminate the ϕ_0-periodic component but the $\phi_0/2$ component should survive. Similar conclusions were also reached independently in Ref. 3, although their statement that the oscillation with period ϕ_0 vanishes like the inverse length of the system is valid only for quantities averaged over the whole band of states. The low-temperature conductivity exhibits before ensemble averaging ϕ_0 and $\phi_0/2$ periodic contributions that are of the same order of magnitude.[13,17] This idea is, of course, in agreement with the fact that very recent experiments[13] on *single rings,* have in fact displayed *large* distinct ϕ_0 periodicities. Ensemble averaging was not appropriate for these experiments.

These experiments have in fact been triggered to some extent by model calculations by Imry and Shiren[19] on the Kubo conductivity of closed 1D rings and by the results we report in this paper. The inappropriateness of ensemble averaging to display the properties of specific "disordered" (or, more generally, nonperiodic) systems has been thoroughly discussed in a series of papers by Azbel,[20] who has also emphasized the possibility of learning something about the specific arrangement of constituents in such systems. A particular case, related to the buildup of Griffiths singularities was discussed by one of us.[21]

All these considerations suggest that there is an important difference between a particular realization of, for an example, a small multiply-connected system and a situation where an effective averaging has to be taken. The purpose of the present paper is to study such non-self-averaging situations,[3] and the effect of various averaging procedures. We restrict ourselves to one-dimensional rings, and consider transport properties such as the total transmission and the conductance of these rings.

In Sec. II we perform ensemble averaging over various realizations of random rings. We find by analytic considerations that ensemble-averaged quantities are periodic in the flux ϕ with periodicity of $\phi_0/2$; the ϕ_0 component is indeed averaged to zero. This is confirmed by numerical study of this problem. A further question is which quantity should be averaged for various experimental situations. This is discussed briefly. The fact that the resistance (conductance) of a ring in the ensemble has a broad, non-normal distribution[6,22] has implications for the dependence of the resistance (conductance) of an array of such rings on its size. Thus, for example, the resistance of a series array of rings of length L, in the Ohmic regime (i.e., when the size of each ring is of the order of l_ϕ) increases with L at a rate which may be faster than linear. It is intriguing to investigate in the future what the implications of this are on the temperature-dependent resistance of such arrays, or of long, narrow wires.

In Sec. III we study a different source for averaging. We consider the two reservoirs, coupled to the leads of the ring, to be at finite temperatures. Thus, there is a whole range of electron energies (or k vectors) which contribute to the current. The higher the temperature the broader is this energy window. In general, the total scattering amplitudes of the ring depend on k. Eventually, at sufficiently high temperatures, the existence of many k vectors results in an effective averaging, and similarly to the ensemble averaging situation, leads to a $\phi_0/2$ periodicity. The details of the crossover from ϕ_0 to $\phi_0/2$ periodicity are studied numerically and conditions for the appearance of a pronounced $\phi_0/2$ component are worked out. Similar consideration for the Kubo-type low-frequency conductivity of *closed* rings were recently presented by Imry and Shiren[19] and for the many-channel case, by Stone and Imry.[17] The above criteria is valid for small and intermediate ring resistance. When the elastic scattering within the branches of the ring is strong, the situation is somewhat more subtle. Due to transmission resonances,[20] only pseudodiscrete energy windows contribute significantly to the conductance. A pronounced $\phi_0/2$ harmonic may appear, but generally a complete smearing of the ϕ_0 component will not occur. The effect of the resonances on the averaging is explained and verified numerically. The results are briefly summarized in Sec. IV. We hope that our work will stimulate studies of the detailed size and temperature dependence of complex multiply-connected arrays, as well as the strong localization regime of such systems.

II. ENSEMBLE AVERAGE

As was indicated in the Introduction, the elastic scattering in the two branches of the 1D ring may be represented by effective scatterers (1 and 2 in Fig. 1). Each of them is characterized by four complex scattering amplitudes r_i,t_i,r_i',t_i' ($i=1,2$). In our notation the phases of these scattering amplitudes include both the effect of the potential barriers in the branches (i.e., the scatterers) and the "optical path" of the electron wave function along the one-dimensional channels. Thus, for example, the amplitude a of that component of the wave function on the B1C branch (i.e., in the direction of 1, see Fig. 1) near B is related to b (the amplitude of the component on B1C near C with momentum in the 1C direction) by $b=at_1$. In general primed (unprimed) quantities denote scattering amplitudes for waves coming from the left (right). Current-conservation and time-reversal requirements imply some relations among the amplitudes, which lead to a simple parametrization of the amplitudes in the absence of flux,[2]

$$\begin{aligned} t_l&=t_l'=|t_l|e^{i\alpha_l} ,\\ r_l&=|r_l|e^{i\beta_l} ,\\ r_l'&=|r_l|e^{i\gamma_l} , \end{aligned} \tag{2.1}$$

with $|t_l|^2+|r_l|^2=1$, $\gamma_l=2\alpha_l-\beta_l+\pi$, $l=1,2$. Nonzero magnetic flux confined in the center of the ring may be described by gauging the amplitudes according to

$$\begin{aligned} &t_1\to t_1e^{i\pi\phi/\phi_0},\quad t_1'\to t_1'e^{-i\pi\phi/\phi_0} ,\\ &t_2\to t_2e^{-i\pi\phi/\phi_0},\quad t_2'\to t_2'e^{i\pi\phi/\phi_0} ,\\ &r_l\to r_l,\quad r_l'\to r_l' . \end{aligned} \tag{2.2}$$

In order to calculate the total reflection and transmission *amplitudes* of the ring, r and t, respectively, and also has to determine the 3×3 scattering matrices[4] of the splitters at B and C. Here we follow the choice of Ref. 2. However, we believe that the results presented here do not depend qualitatively on the particular choice of these matrices.

The total reflection and transmission *coefficients* of the

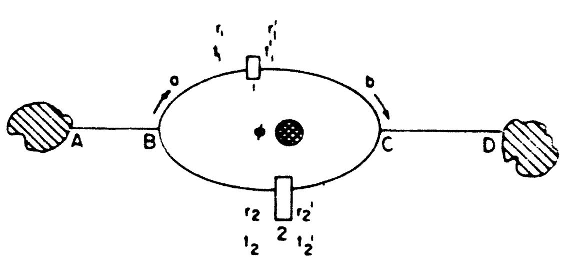

FIG. 1. One-dimensional ring confining a magnetic flux ϕ. Boxes 1 and 2 denote the elastic scattering in the two branches of the ring, with the corresponding transmission and reflection amplitudes. Points A and D are connected to reservoirs.

system are $R\equiv|r|^2$ and $T\equiv|t|^2$, respectively. The dimensionless Landauer conductance for spinless electrons is given by[5,6]

$$\frac{G}{e^2/2\pi\hbar}=\frac{T}{R} , \tag{2.3}$$

with

$$R+T=1 . \tag{2.4}$$

To perform ensemble averaging over different realizations of the ring we restrict ourselves to systems with fixed values of $|r_l|$ and $|t_l|$, and choose the phases randomly. As seen from Eq. (1.1) there are four independent phases, namely α_l and β_l ($l=1,2$). We take all points in the four-dimensional "phase" space to be distributed with uniform weight.

In order to understand why the ensemble averaging over all phases yields a periodicity of $\phi_0/2$ let us consider, for example, the backward reflection of an electron coming from A (Fig. 1). One can view the total reflected part of the wave function as an infinite sum of partial contributions to r, arising from multiple-scattering processes. Thus the electron coming from A may be scattered back at the splitter B. This process is denoted, symbolically by ABA. Similarly, the electron may perform a complete loop and then be scattered back (such a path is, for example, AB1C2BA) etc. For a given realization of $\{\alpha_l,\beta_l\}$ and a given value of the flux ϕ, the probability amplitude of the mth path has a well-defined phase $\delta_m(\{\alpha_l,\beta_l\},\phi)$. We shall show now that to each realization of random scatterers, characterized by the phases $\{\alpha_l,\beta_l\}$, corresponds another realization with phases $\{\alpha_l',\beta_l'\}$, such that

$$\delta_m(\{\alpha_l,\beta_l\},\phi)=\delta_m(\{\alpha_l',\beta_l'\},\phi+\phi_0/2) . \tag{2.5}$$

This immediately implies that the ensemble averaged reflection amplitude is periodic in $\phi_0/2$, and thus the reflection coefficient has this periodicity. One may consider some other scattering and transport quantities (the transmission coefficient, the conductance, etc.) and verify along the same lines the $\phi_0/2$ periodicity.

To satisfy Eq. (2.5) we define the following mapping

$$\begin{aligned} &\alpha_1'=\alpha_1+\pi,\quad \beta_1'=\beta_1 ,\\ &\alpha_2'=\alpha_2,\quad \beta_2'=\beta_2 . \end{aligned} \tag{2.6}$$

Note that this transformation leaves paths that do not include complete turns around the ring untouched. These transformations, however, imply a sign change of the amplitude for every complete turn, which is identical to the effect of varying ϕ by $\phi_0/2$ [cf. Eq. (2)]. This completes our proof.

We also studied numerically the ensemble average of various quantities. The ensemble average of the transmission coefficient $\langle T\rangle$ is computed and the quantity $\tilde{G}\equiv\langle T\rangle/(1-\langle T\rangle)$ is plotted in Fig. 2 versus the normalized flux $\theta=\pi\phi/\phi_0$, for various values of the branch conductances, $G_1=|t_1|^2/|r_1|^2$; $G_2\equiv|t_2|^2/|r_2|^2$. The averages were computed by using 20^4 data points in the phase space. For some θ values, we performed a much finer sampling (using up to 60^4 points) and noticed

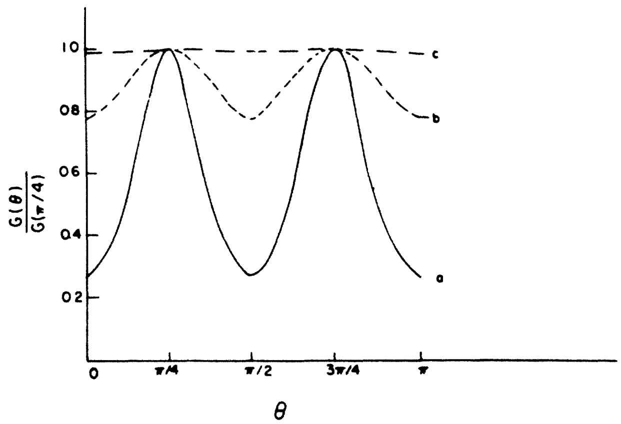

FIG. 2. $\tilde{G}(\theta)/\tilde{G}(\theta=\pi/4)$ plotted for various values of branch conductances, G_1 and G_2. (a) $G_1=100$, $G_2=50$, $G(\pi/4)=5.1$; (b) $G_1=1.0$, $G_2=0.5$, $G(\pi/4)=0.85$; (c) $G_1=0.001$, $G_2=0.0005$, $G(\pi/4)=0.000\,15$. The conductances are measured in units of e^2/h.

that the averages did not change by more than a few percent. As is seen, the periodicity is $\phi_0/2$. The sensitivity to the flux is larger the weaker is the elastic scattering.

Next we calculated the ensemble average of $\ln(T/R)$. One expects this quantity to have a well-behaved distribution.[22,23] The behavior for intermediate scattering is shown in Fig. 3, again showing periodicity of $\phi_0/2$. Physically, considering $\ln(T/R)$ is appropriate, e.g., when we have a series array of rings with l_ϕ being much larger than the ring size and the interring distance. In that case the quantum-mechanical addition rule for the resistance implies that (for sufficiently long systems) the logarithms of the resistances should be added.[22]

When l_ϕ is of the order of a ring size, the resistances of the rings should be added according to Ohm's law. We are then interested in $\langle R/T\rangle$. However, our numerical studies indicate that this quantity diverges. In order to

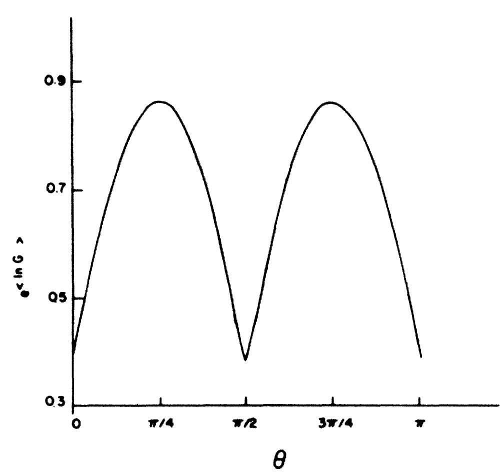

FIG. 3. $\exp(\langle \ln T/R\rangle)$ plotted versus the normalized magnetic flux θ. The dimensionless branch conductances are $G_1=1.0$, $G_2=0.5$.

study the details of this divergence we divided our four-dimensional phase space into a grid of $n\times n\times n\times n$ points, the distance between neighboring grid points being $2\pi/n$.

Defining an arbitrary cutoff $\mathscr{R}_c$ we have found numerically the number M of grid points at which the resistance $\mathscr{R}$ is larger than $\mathscr{R}_c$. We found for large $\mathscr{R}_c$, the power-law behavior

$$M\sim\mathscr{R}_c^{-\lambda}\;, \tag{2.7}$$

where $\lambda=0.5-0.6$. This indicates a divergence of $\mathscr{R}$ at special points in the phase space proportional to a power, $-\mu$, of the distance from these points, where

$$\mu=4/\lambda\cong 7.3\pm0.7\;, \tag{2.7$'$}$$

i.e, a nonintegrable divergence. Heuristic arguments for such a divergence will be given below. To see what this implies on the dependence of the resistance on the system's length let us consider a configuration of the N rings in series. Each ring is a given realization of the ensemble, and is represented by a point in a p-dimensional phase space where $p=4$ in our case. The typical distance between nearest-neighbor points is $\sim N^{-1/p}$. From the above divergence, we obtain that the average resistance per ring for a system of N rings, $\mathscr{R}_N$, scales as

$$\mathscr{R}_N\sim N^{\mu/p-1} \tag{2.8}$$

with $\mu=7.3\pm0.7$ and $p=4$ in our case. The anomalous dependence of $\mathscr{R}_N$ on N arises from the non-normal distribution of ring resistances and the larger probability to find a large resistance for larger N.

The plausible reason for this behavior is that the transmission vanishes strongly at special points in phase space. Clearly, there is sufficient freedom in the four phases to satisfy a condition of total blocking of the incoming wave by the ring. The seemingly surprising result is that the large value of μ implies that T vanishes like a high power of the distance from the special point. For $\mu\geq p$ this singularity is nonintegrable and the above effects follow. It is, in fact, straightforward to understand why μ turns out to be so large. Let us write the transmission amplitude as $F=G+iH$ where the real functions G and H each depends on the four phases α_i,β_i ($i=1,2$). In general, G and H vanish each on a three-dimensional subspace of the four-dimensional phase space. *Both* vanish on the intersection of the above three-dimensional subspaces which is a two-dimensional manifold (a surface). The transmission amplitude vanishes on the above surface. We now can consider the expansion of the analytic function F in the deviation from the above surface. There exist two lines on the latter on which the first and second order coefficients in this expansion vanish. Thus, near the intersection point of those two lines, F has a third-order zero and $T=|F|^2$ a sixth-order one. Thus, special points where T vanishes with $\mu=6$ may exist. The same applies, of course, also for the reflection coefficients R, but at different points. Note again that the result (2.8) is related to the case where the phase breaking length l_ϕ is of the order of a ring's size.[24] For parallel addition of rings, with the usual, Ohmic, addition law (for l_ϕ comparable to the size of each ring) one rather discusses the conductance $\langle G \rangle = \langle T/R \rangle$. In a way analogous to the above discussion we find numerically the average conductance of N rings in parallel, under these circumstances scales as

$$G_N \sim N^{\mu'/p - 1} \tag{2.9}$$

with $\mu' = 5.0 \pm 0.7$ and $p = 4$. Our numerical values for the exponents of the vanishing of T and R near their respective special points are in fair agreement with the dimensional counting argument presented above.

We emphasize that (2.8) and (2.9), while having to do with the special broad distribution of resistances in 1D are of a different nature than the by now well-known properties in the quantum regime.[22] They apply for classical, Ohmic, addition of many quantum single-ring units. More work is needed to find out whether similar effects may exist in the more realistic multichannel ("quasi-1D") case. In any case the results of Ref. 15, as interpreted in Ref. 16, imply that around the transition to strong localization (i.e., total ring resistance of the order of $\pi\hbar/e^2$) real systems should behave like single channel ones. We expect the results (2.8) and (2.9) to be applicable to such cases.

III. TEMPERATURE AVERAGING

A different type of averaging arises when the reservoirs (A and D in Fig. 1) are held at a finite temperature τ. The contributions of the electrons at various energies, which participate in the conduction, have to be weighed now with the factor $-df/dE$, where f is the Fermi-Dirac distribution

$$f(E) = \frac{1}{e^{(E-E_F)/k_B\tau} + 1} . \tag{3.1}$$

Here E is the energy, k_B is Boltzmann's constant, and E_F is the Fermi energy. The average dimensionless conductance $G_{\rm av}(\tau)$ is given by[25,14]

$$G_{\rm av}(\tau) = \frac{\int \left| \frac{df}{dE} \right| T(E) dE}{1 - \int \left| \frac{df}{dE} \right| T(E) dE} . \tag{3.2}$$

Strictly speaking[26] this conductance, while being the usually measured quantity, i.e., the ratio of current to voltage, might contain a thermoelectric contribution which is ignored here, for simplicity. In order to find the dependence of T on E, we express the phase of each scatterer as the sum of two contributions: (i) α_m^0 (β_m^0), $m = 1,2$, intrinsic part related to the scattering process; (ii) kl_{α_m} (kl_{β_m}), which is related to the optical length of the electron. Hereafter we assume for simplicity that α_m^0 (β_m^0) do not depend on k. Since $k_F l_{\alpha_m} \gg \alpha_m^0$ ($k_F l_{\beta_m} \gg \beta_m^0$), k_F being the Fermi wave vector, we do not expect our results to depend strongly on the above assumption. Thus a strong dependence of T on E arises from the dependence of the phases of the scattering amplitudes on the k vectors. Using $E \sim k^2$, the variation of the phases with the energy, defined by

$$\Delta\alpha_m(E) \equiv \alpha_m(E) - \alpha_m(E_F) , \tag{3.3}$$

is given by

$$\frac{\Delta\alpha_m(E)}{\alpha_m(E_F)} = \frac{1}{2} \frac{E - E_F}{k_B\tau} \frac{k_B\tau}{E_F} . \tag{3.4}$$

Similar relations hold for $\Delta\beta_m(E)$.

Equation (3.2) may thus be written as

$$G_{\rm av}(\tau) = \frac{\int_{-\infty}^{\infty} \frac{e^x}{(e^x+1)^2} T[\alpha_m,\beta_m] dx}{1 - \int_{-\infty}^{\infty} \frac{e^x}{(e^x+1)^2} T[\alpha_m,\beta_m] dx} , \tag{3.5}$$

where

$$\begin{aligned} \alpha_m(x,\tau) &= \alpha_m(E_F) \left[1 + \tfrac{1}{2} x \frac{k_B\tau}{E_F} \right] , \\ \beta_m(x,\tau) &= \beta_m(E_F) \left[1 + \tfrac{1}{2} x \frac{k_B\tau}{E_F} \right] , \end{aligned} \tag{3.6}$$

and $x = (E - E_F)/k_B\tau$. The integrals extending from $-\infty$ to $+\infty$ in Eq. (3.5), with the weight function $e^x/(e^x+1)^2$, may be approximated by introducing cutoffs at $x_{\rm min}$ and $x_{\rm max}$ that do not depend on τ. Nevertheless, the range of phases contributing to the integral increases with increasing temperature. Thus for sufficiently high temperatures (although possibly still much lower than E_F/k_B), we may have an effective averaging of the transmission over the phases, similarly to the case of the preceding section [except that Eq. (3.6) constrains the averaging in this case to be on a line in the $(\alpha_1,\beta_1,\alpha_2,\beta_2)$ four-dimensional phase space]. Note that since in this case the transmission is the averaged quantity, the divergence problem discussed in Sec. II does not arise.

This is indeed the case for weak and intermediate elastic scattering. The dependence of $\mathscr{R}(\phi) \equiv G_{\rm av}{}^{-1}(\phi)$ on τ for intermediate branch resistances $\mathscr{R}_1$ and $\mathscr{R}_2$($\mathscr{R}_i = |r_i|^2/|t_i|^2$) is shown in Fig. 4. At low temperatures the resistance is periodic in ϕ with a periodicity ϕ_0. As the temperature is increased a pronounced $\phi_0/2$ harmonic appears. At a sufficiently high temperature ($k_B\tau \simeq 0.1 E_F$ in this case) $\mathscr{R}$ acquires a fundamental $\phi_0/2$ periodicity. "High" temperature acts in this case as an effective ensemble averaging. For different realizations of the ring (i.e., with the same $\mathscr{R}_1$ and $\mathscr{R}_2$ of the system depicted in Fig. 4 but with different phases) the low-temperature behavior of the resistance is different, cf. Fig. 5. Thus at low τ we have a minimum in $\mathscr{R}$ at $\phi = 0$ [compared with a maximum in Fig. 4(a)]. At high temperatures, the choice of the zero-temperature phases of the scatterers $[\alpha_m(E_F),\beta_m(E_F)]$ is irrelevant, and Figs. 4(d) and 5(d) are practically identical. As a general rule for the cases we studied when an ensemble averaging (and a periodicity of $\phi_0/2$) is established, $\mathscr{R}$ is maximal at $\phi = 0$.

It is of interest to have estimates for the temperature at which a pronounced $\phi_0/2$ harmonic becomes relatively large. For the model considered, this temperature should decrease with decreasing branch resistances. The reason is the following. The typical phase spreading of the incoming electrons at temperature τ is given by Eq. (3.4) with $(E - E_F \sim k_B\tau)$. Therefore $\Delta\alpha \sim \tau$, may be overly small at low τ to render effective averaging. However, electrons scattered M times in the ring before leaving it will have a phase spreading of order $M\Delta\alpha$. If the branch resistances are sufficiently low, electrons can be scattered M times without a significant loss in intensity, such that $M\Delta\alpha \sim 0(1)$. This will make averaging possible, and the $\phi_0/2$ harmonic will be dominant. The smaller the branch resistances, the lower the temperature at which this condition is satisfied. Our numerical results are consistent with this argument.

The above discussion was for the special case of a strictly one-dimensional ring which is the one thoroughly considered in this work. We remark that more generally, one may define an energy correlation range, ΔE_c, as the range of E around E_F in which the phases of t and r change significantly and averaging out of the ϕ_0 component is achieved once $k_B\tau \gg \Delta E_c$. The weak localization range is defined by the dimensionless conductance $g \equiv \pi G\hbar/e^2$ being large. According to Edwards and Thouless,[27] g is given by the ratio V/w where V is the sensitivity of the energy levels to boundary conditions and w is their separation at E_F. For $V \gg w$, the condition for thermal energy averaging should be related to the energy change needed to get a significant phase change across the sample. One might thus expect the energy correlation range to be V. One can understand this from the diffusive character of the electron's motion.[17] In fact, in the ballistic range, the energy correlation range is given by $\Delta kL \sim 1$ or

$$\Delta E_c \sim \frac{\hbar v_F}{L} \quad \text{for } L \ll l\ . \tag{3.7a}$$

In the diffusive regime the length traveled by the electron across the system is given by $v_F L^2/D$. Replacing L in Eq. (3.7a) by this length, we find

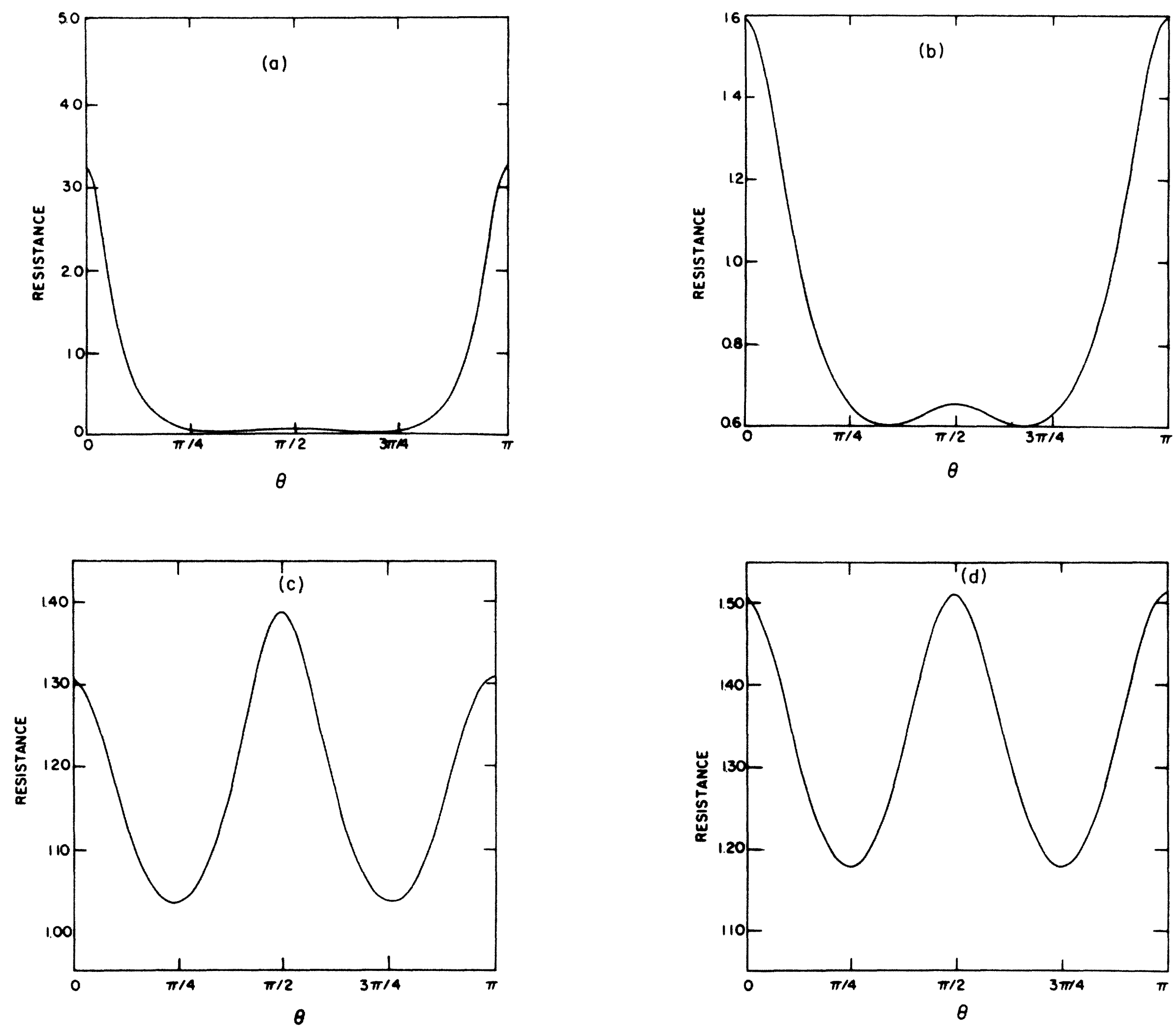

FIG. 4. Average resistance $\mathscr{R}$ as a function of the normalized flux θ for various values of $y \equiv k_B\tau/E_F$. (a) $y = 10^{-5}$, (b) $y = 10^{-4}$, (c) $y = 10^{-3}$, (d) $y = 10^{-1}$. The dimensionless resistances of the two branches are $\mathscr{R}_1 = G_1^{-1} = 1.0$, $\mathscr{R}_2 = 2.0$; the optical lengths are $k_F l_{\alpha_1} = 10\,623$, $k_F l_{\alpha_2} = 28\,710$, $k_F l_{\beta_1} = 22\,901$, $k_F l_{\beta_2} = 83\,920$.

$$\Delta E_c \sim \frac{\hbar D}{L^2} \text{ for } L \gg l \; . \tag{3.7b}$$

This energy was shown by Edwards and Thouless[27] to be identical to the parameter V, in agreement with the above expectation.

The behavior in the strong scattering regime ($\mathscr{R}_1,\mathscr{R}_2 \gg 1$) is, however, different. $\mathscr{R}(\phi)$ for a series of temperatures is shown in Fig. 6. Here, $\mathscr{R}_1=1000$, $\mathscr{R}_2=2000$. Note that complete averaging and $\phi_0/2$ periodicity are *not* achieved even at high temperatures. $\mathscr{R}(\phi=0)$ may vary from maximum to minimum as τ is varied (and vice versa in other cases). For other sets of $\{l_{\alpha_m}, l_{\beta_m}\}$ a "bump" at $\phi_0/2$ may appear at certain temperatures, but it disappears as τ is raised further. We also note that the sensitivity to the flux decreases as τ increases.

The explanation to this behavior follows from the observation that for strongly localized systems the transmission for almost all energies is exponentially small with the size of the system, except for exponentially narrow energy windows,[20,28] for which the transmission is significant. Only relatively few k vectors that correspond to these narrow transmission resonances contribute significantly to the conductance, which explains why a complete averaging may not be achieved even at relatively high temperatures. A typical structure of the resonances for $\mathscr{R}_1=500$, $\mathscr{R}_2=200$ is shown in Fig. 7. The contribution of each individual resonance[28] to the flux-dependent resistance does not have a definite phase; for some of the resonances $\mathscr{R}(\phi=0)$ is a minimum, while for the others it is a maximum. As the temperature is increased the energy interval included in the integral (3.5) increases and occasionally additional resonances contribute, which may be accompanied by changing $\mathscr{R}(\phi=0)$ from a maximum to a minimum and vice versa. We checked numerically that indeed such a change is related to the inclusion of an additional resonance in the effective energy integration

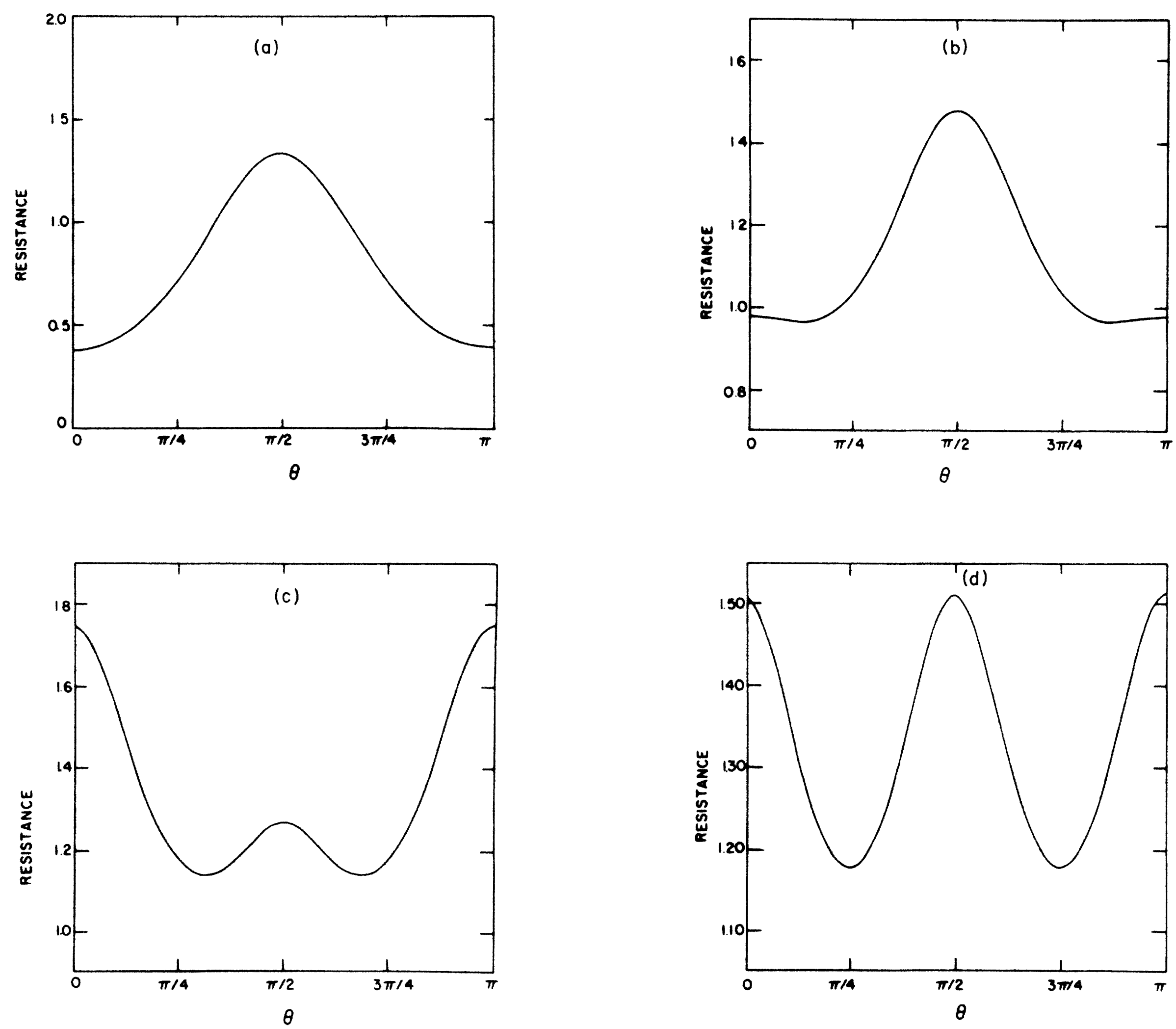

FIG. 5. Average resistance $\mathscr{R}(\theta)$ for a different ring. $\mathscr{R}_1$ and $\mathscr{R}_2$ are the same as in Fig. 4. The optical lengths are $k_F l_{\alpha_1}=10\,981$, $k_F l_{\alpha_2}=77\,122$, $k_F l_{\beta_1}=88\,721$, $k_F l_{\beta_2}=87\,230$. (a) $y=10^{-5}$, (b) $y=10^{-4}$, (c) $y=10^{-3}$, (d) $y=10^{-1}$.

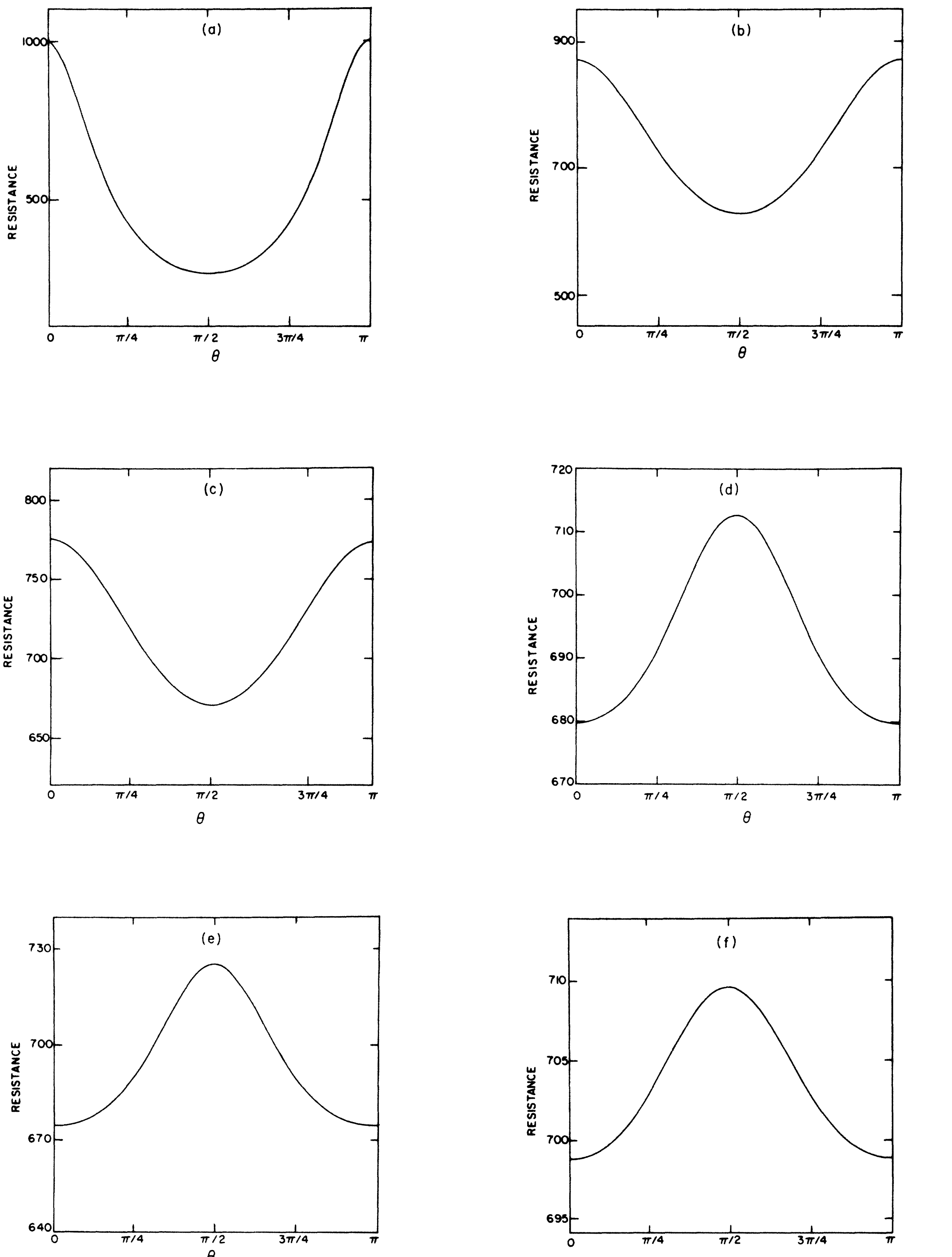

FIG. 6. Average resistance $\mathscr{R}(\theta)$ for a highly resistive ring. $\mathscr{R}_1=1000$, $\mathscr{R}_2=2000$; (a) $y=10^{-4}$, (b) $y=10^{-3}$, (c) $y=5.10^{-3}$, (d) $y=8.10^{-3}$, (e) $y=10^{-2}$, (f) $y=1.0$.

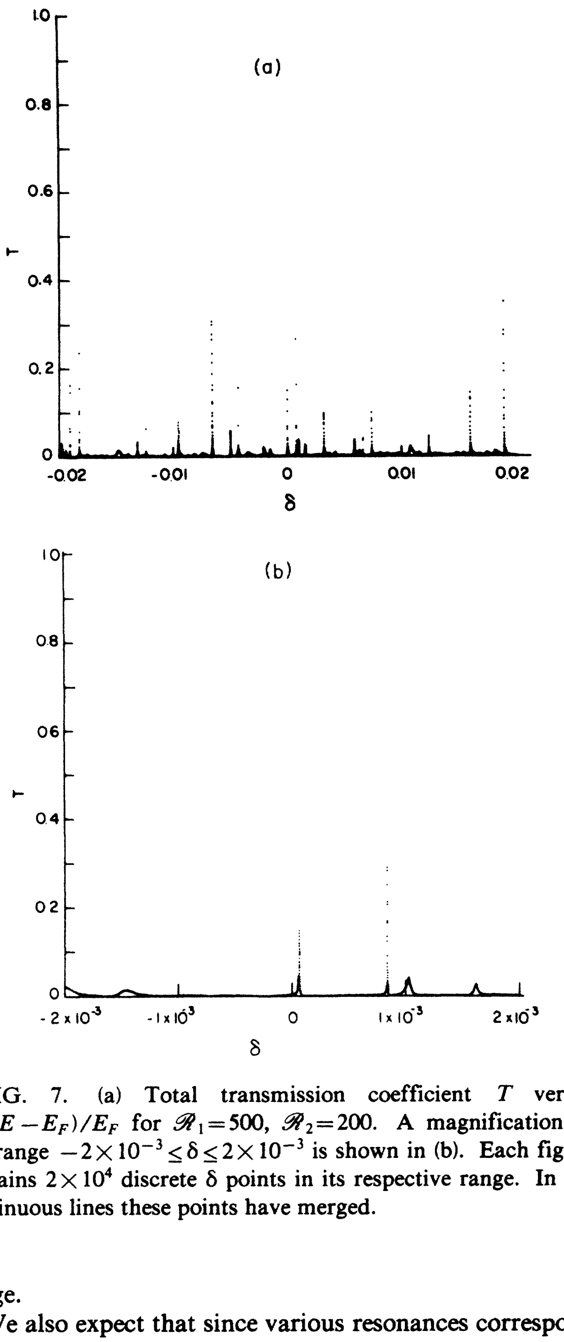

FIG. 7. (a) Total transmission coefficient T versus $\delta=(E-E_F)/E_F$ for $\mathscr{R}_1=500$, $\mathscr{R}_2=200$. A magnification of the range $-2\times10^{-3}\leq\delta\leq2\times10^{-3}$ is shown in (b). Each figure contains 2×10^4 discrete δ points in its respective range. In the continuous lines these points have merged.

range.

We also expect that since various resonances correspond to $\mathscr{R}(\phi=0)$ being maxima or minima at random, $|\mathscr{R}(\phi=0)-\mathscr{R}(\phi=\phi_0/2)|$ (which measures the sensitivity of $\mathscr{R}$ to the flux) will decrease with the inclusion of more resonances. Since the number of contributing resonances is proportional to τ, we have

$$\Delta\mathscr{R}\equiv|\mathscr{R}(\phi=0)-\mathscr{R}(\phi=\phi_0/2)|$$

$$\sim\frac{1}{\sqrt{\tau}}(\text{strong localization})\ . \qquad (3.8)$$

In Fig. 8, we plot $\Delta\mathscr{R}(\tau)$ and find $\Delta\mathscr{R}(\tau)\sim\tau^{-\nu}$, $\nu\simeq0.5$, in agreement with Eq. (3.8).

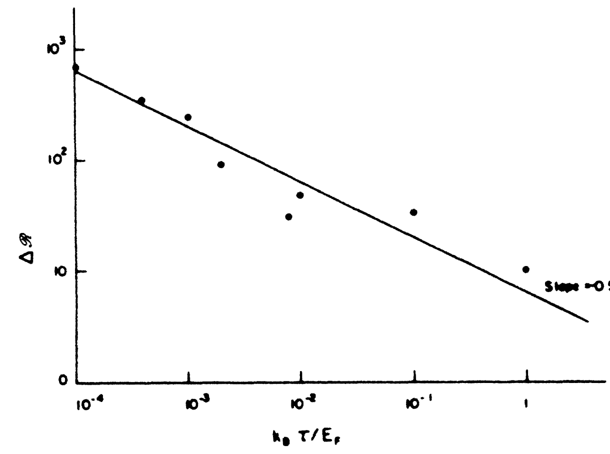

FIG. 8. $\Delta\mathscr{R}(\tau)$ on a log-log scale for a strongly localized system with $\mathscr{R}_1=1000$, $\mathscr{R}_2=2000$ [cf. Eq. (3.8)].

IV. SUMMARY

This paper is devoted to the study of various averaging processes on the flux-dependent oscillations of the Landauer-type conductance of a 1D single channel ring. The usual averaging over an ensemble of microscopically different systems prepared under the same macroscopic conditions should be equivalent to an averaging over the phases of the various scattering amplitudes. A finite temperature provides an effective averaging due to the differing optical paths of electrons with different energies. Both of these averaging processes lead in most cases, where the effective ensemble is wide enough, to a cancellation of the ϕ_0-periodic component of $G(\phi)$, changing the effective period to $\phi_0/2=hc/2e$. The strong disorder case is found to be an exception to the above, which is discussed in detail. Conditions for the temperature averaging are formulated. A novel divergence of the ensemble-averaged resistance and conductance, related to the broad distributions of those quantities is found and heuristically explained. While being of a different nature from the well-known zero-temperature anomalies, it might have measurable consequences at finite temperatures for effectively single-channel conductors.

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