# Exact Bethe-ansatz thermodynamics for the sine-Gordon model in the classical limit: Effect of long strings

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We solve the finite-temperature Bethe-ansatz (BA) integral equations with coupling constant $\mu=\pi(1-n^{-1})+0$, in the semiclassical limit $n=$ integer$\rightarrow\infty$. After eliminating long strings and then an exact bosonization, transforming $n-2$ fermion strings into a single boson, the BA integral equations are reduced to two, coupling bosons and two fermions (kinks and antikinks). In the classical limit these iterate exactly to the sine-Gordon free energy found by the transfer integral method and by other methods recently reported. We suggest that our result can be extended to general values of $\mu$.

The classical sine-Gordon (SG) model is completely integrable[1] with Hamiltonian

$$H[\phi]=\gamma_0^{-1}\int[\tfrac{1}{2}\gamma_0^2\Pi^2+\tfrac{1}{2}\phi_x^2+m^2(1-\cos\phi)]dx, \quad (1)$$

Poisson bracket $\{\Pi,\phi\}=\delta(x-x')$, coupling constant $\gamma_0>0$. As a (normally ordered) quantum system it is integrable by the quantum inverse method.[2,3] The Bose quantum SG is of particular interest because it is equivalent[4,5] to the Fermi massive Thirring model (MTM) solved by the Bethe ansatz (BA)[2,3] method, because both are equivalent by canonical transformation to a continuum limit of the quantum spin-$\frac{1}{2}$ $XYZ$ model,[5] and because this relates directly[6] to the statistical mechanics of the eight-vertex model[6,7] in 2+0 Euclidean dimensions. References 8 and 9 derive coupled integral equations for the quantum free energy of SG by mapping from the BA formulation of the quantum *statistical mechanics* (SM) of the spin-$\frac{1}{2}$ $XYZ$ model.[10] In this note we show how this otherwise largely intractable system of equations reduces, in classical limit, to results identical to those recently found[11,12] by other methods. Because of the theoretical background to quantum SG mentioned, and because forms of it, classical and quantum, appear in many low-dimensional physical problems,[13] the SM of SG has been brought almost to the status of a "test bed" for the SM of all the integrable models. The results of this note thus provide one more check[11,12] on the SM of these.

References 11 and 12 briefly report two wholly new and fundamental methods for calculating the quantum or classical free energies of large classes of integrable models. But they could only note how the conventional BA was subsumed within these and simply remark (cf. Ref. 19 in Ref. 12) that the conventional BA SM of SG[9,10] reduced in classical limit to certain classical integral equations reported there.[12] These were important because they iterated to yield the exact low-temperature asymptotic expansions for the free energy of classical SG. These expansions could be derived otherwise, but in exactly the same form, by the transfer-integral method (TIM)[12] on the classical functional integral for the partition function $Z$ of SG. Thus, a number of different ideas come together in the classical SM of SG.

Chen, Johnson, and Fowler[14] are reporting more or less the same result as ourselves—namely that the conventional BA SM of SG can be reduced in classical limit to coupled integral equations iterating to the TIM result. However, Chen *et al.*[14] solve this classical limit problem in one particular case—when a certain number $p_0$ (see below) is an exact integer $p_0=n$. Independently we have solved this problem when $p_0=n+0$ [i.e., when $p_0=n+\varepsilon$, $\varepsilon(>0)\rightarrow 0$]. It is known[9,10,12,15] that these two cases, so far the only tractable ones, are significantly different: For $p_0=n$ there are merely $n-1$ coupled integral equations; for $p_0\neq n$ (including $p_0=n+0$) there is an infinite set. Because of current interest in integrable models, and the test-bed status of SG amongst these, we use this note to report in more detail than our simple reference (Ref. 19 in Ref. 12) could our solution for the classical limit of the BA SM for SG when $p_0=n+0$.

First we quote again results by TIM for the free energy per unit length $FL^{-1}$ for classical SG. They are found by methods of matched asymptotic expansion[12] and include part results found by a number of different workers.[12] Our collected result, to the terms we have now also derived by the two methods of Ref. 12 *and* from the classical limit of conventional quantum BA as reported here, is $FL^{-1}=F_{KG}+F^{(1)}+F^{(2)}+\cdots$ where $F_{KG}=\beta^{-1}a^{-1}$ $(\ln\beta a^{-1}+\frac{1}{2}ma)$: The last is the contribution of a harmonic (linear) lattice, spacing $a$, and this necessarily diverges as $a\rightarrow 0$ (in classical limit); $\beta^{-1}\equiv T$, the temperature. The $F^{(q)}$ are

$$F^{(1)}=-\beta^{-1}m(8/\pi t)^{1/2}e^{-1/t}(1-\tfrac{7}{8}t-\tfrac{59}{128}t^2-\tfrac{897}{1024}t^3-\cdots)-\beta^{-1}m[\tfrac{1}{4}t+\tfrac{1}{8}t^2+\tfrac{3}{16}t^3+\tfrac{53}{128}t^4+\cdots], \quad (2a)$$

 

$$F^{(2)}=\frac{8m}{\pi}Me^{-2/t}\left[\ln\frac{4C}{t}-\frac{5t}{4}\left(\ln\frac{4C}{t}+1\right)-\frac{t^2}{32}\left(\ln\frac{4C}{t}+2\right)+\cdots\right], \tag{2b}$$

with $F^{(q)}=O(e^{-q/t})$, $q=3,4,\ldots$; $t\equiv(M\beta)^{-1}$ and $M\equiv 8m\gamma_0^{-1}$ is the SG soliton (kink or antikink) mass from (1). In (2b), $\ln C=0.5771\ldots$, Euler's constant. The TIM involves periodic boundary conditions for a finite density thermodynamic limit.[11,12] In principle terms at any order can now be calculated by the TIM[12], so we can take the results quoted as definitive expressions for $FL^{-1}$ for classical SG in this thermodynamic limit and check all other calculations for SG against them—as is done for the conventional BA SM of SG now.

Since $p_0=n+0$ we start from the expressions[10] for $FL^{-1}$, Eq. (6) next, valid for this case. In the conventional BA[2,3,8–10,14–16] $FL^{-1}$ is expressed in terms of energies ($\tilde{E}_j$ say) and coupled integral equations for $\tilde{E}_j$ must be solved to calculate $FL^{-1}$. Reference 15 notes that, for $p_0=n$, there are $n$ such energies (for $n-1$ "strings" plus "holes") $\tilde{E}_{n-1}$ being a free kink energy, $\tilde{E}_n$ the free antikink energy (hole in the Dirac sea), and $\tilde{E}_1,\ldots,\tilde{E}_{n-2}$ "breather" energies. For $p_0=n+0$, $\tilde{E}_{n-1}$ becomes a bound particle-hole (kink-antikink) pair, half of the holes in the sea correspond to free antikinks, and the other half combine with the "long" strings (length $>n-1$) to form free kinks. An otherwise stimulating paper by Maki[17] does not distinguish the two situations.

We have also to understand the coupling constants of the theory: for MTM the coupling constant $g=2\mu-\pi$, and $\mu\equiv\pi(p_0-1)/p_0$—related to the renormalized coupling constant $\gamma_0''=\gamma_0/(1-\gamma_0/8\pi)$ of SG by $8\pi\gamma_0^{-1}=p_0$.[2,3,5,11,18] Note that the number of breather levels of SG at $\beta^{-1}=0$ is[2,3,5,18] the integral part $[8\pi/\gamma_0'']$ of $8\pi/\gamma_0''$: This is $[p_0-1]=n-1$. When $p_0=2$ the system is a free fermion system.[5]

Exact identifications between classical solutions of SG and terms in $FL^{-1}$ for quantum SG is not possible. Still we must make the point that our procedure reported in this note chooses to reduce the quantum BA problem to a semiclassical form interpretable in terms of "phonons" (bosons) and "solitons" (kinks and antikinks—both fermions) alone. In a previous attack on the same problem[19] we tried to keep the classical SG breathers in mind throughout—so working with solitons and "breathers" but no phonons: $F_{KG}$ was the only phonon contribution and was extracted from the very low-lying breather excitations. The procedure was tedious, otherwise yielding only the leading terms of $F^{(1)}$, Eq. (2a). Even so, these two significantly different points of view must eventually be brought together. Note that the classical energy spectrum of SG involves all of solitons, breathers, and phonons.[1]

We first simplify the quantum BA equations[10] by defining a physical rapidity $x=\pi\alpha/2\mu$ in terms of the original[10] rapidity $\alpha$. Conveniently, we express the "dressed" phase shifts between BA strings in terms of BA shifts $\theta_j(x)$ defined, for $j=1,\ldots,n-1$, by

$$\theta_j(x)=-i\ln\frac{\sinh x-i\sin[\pi j/2(n-1)]}{\sinh x+i\sin[\pi j/2(n-1)]}. \tag{3}$$

We write $n$ for $p_0$, but no confusion will arise since we ultimately replace $p_0=n+0$ by $8\pi\gamma_0^{-1}$. The essential step for a semiclassical limit is $\gamma_0\to 0(n\to\infty)$. This is then followed by a classical limit at (13) and (14) below.

For $p_0=n+0$ the BA integral equations become[10]

$$\begin{aligned}\tilde{E}_j(x)=E_j(x)&+\frac{1}{2\pi\beta}\sum_{l=1}^{n-1}\frac{d}{dx}B_{jl}*\ln(1+e^{-\beta\tilde{E}_l})\\&+\frac{1}{4\pi\beta}\frac{d}{dx}B_{j,n-1}*\ln(1+e^{\beta\tilde{E}_n})\\&+\frac{1}{2\beta}\delta_{j,n-1}\ln(1+e^{\beta\tilde{E}_n}),\end{aligned} \tag{4a}$$

for $j=1,\ldots,n-1$, while for $j\geq n$ the integral equations of the general case reduce, by the action of certain $\delta$ functions, to the purely algebraic relations

$$\tilde{E}_n=\frac{1}{2\beta}\ln[(1+e^{\beta\tilde{E}_{n+1}})/(1+e^{\beta\tilde{E}_{n-1}})], \tag{4b}$$

$$\tilde{E}_j=\frac{1}{2\beta}\ln[(1+e^{\beta\tilde{E}_{j+1}})(1+e^{\beta\tilde{E}_{j-1}})],\quad j\geq n+1. \tag{4c}$$

The dressed phase shifts $B_{jl}$ are given by

$$B_{jl}=\theta_{|j-l|}(x)+\theta_{j+l}(x)+2\sum_{k=1}^{\min(j,l)-1}\theta_{j+l-2k}(x), \tag{5}$$

and $E_j(x)\equiv M_j\cosh x$, $M_j\equiv 2M\sin[\pi j/2(n-1)]$, while $f*g\equiv\int_{-\infty}^{\infty}dx'f(x-x')g(x')$. In the same notation the free energy is then

$$\begin{aligned}FL^{-1}=&-\frac{1}{2\pi\beta}\sum_{j=1}^{n-1}\int dx\,E_j(x)\ln(1+e^{-\beta\tilde{E}_j(x)})\\&-\frac{1}{2\pi\beta}\int dx\,E_s(x)\ln(1+e^{\beta\tilde{E}_n(x)})\end{aligned} \tag{6}$$

with $E_s(x)\equiv M\cosh x$.

The semiclassical approximation expands the $\theta_j(x)$ up to $O(n^{-1})$: It corresponds to expanding to $O(\gamma_0)$ in the quantum coupling constant $\gamma_0$.[11] We find

$$\begin{aligned}B_{jk}(x)=&\pi[2\min(j,k)-\delta_{jk}]\theta(x)\\&-2\pi jk/(n-1)\sinh x;\ j,k<n-1,\end{aligned} \tag{7a}$$

$$B_{j,n-1}(x)=B_{n-1,j}(x)=4j\tan^{-1}\sinh x, \tag{7b}$$

$$\frac{d}{dx}B_{n-1,n-1}(x)=\frac{4(n-1)}{\pi}\ln\frac{\cosh x+1}{\cosh x-1}, \tag{7c}$$

in which $\theta(x)=1(x>0),=-1(x<0)$. We subsequently interpret $4(n-1)\pi^{-1}$ in both (7a) and (7c) as $32\gamma_0^{-1}$ since $\gamma_0''\sim\gamma_0$ as $\gamma_0\to 0(n\to\infty)$. But, in the semiclassical "limit," we need a new semiclassical $\gamma_0>0$ for consistency with (1). The procedure of extracting this from the quantum $\gamma_0$ at (7a) and (7c) corresponds to reversing the renormalization of mass $m$ (and $\gamma_0\to\gamma_0''$) in the quantum theory from (1). Here we *define* $m=\frac{1}{8}\gamma_0 M$, $<\infty$ as $\gamma_0\to 0$.

Reference to our previous work[11,12] now shows that the combination of the term in $\theta(x)$ and the second term with singularity at $x=0$ in (7a) corresponds to a bosonization of $n-2$ of the fermions in the problem: This becomes evident too from the solution of Eqs. (4) which we now describe in semiclassical limit.

First we solve (4c) for long strings to give

$$\tilde{E}_j=[j-n+(1+e^{\beta\tilde{E}_n})^{1/2}]^2,\ j\geq n+1\,. \tag{8a}$$

Combined with (4b) this yields

$$\tilde{E}_n=\beta^{-1}\ln[1+2(1+e^{\beta\tilde{E}_{n-1}})^{1/2}]-\beta^{-1}\ln(1+e^{\beta\tilde{E}_{n-1}})\,. \tag{8b}$$

Under the same condition $p_0=n+0$ the result (8a) is given in Ref. 8 while (8b) is already found in Ref. 10. We remark on it here because, below, we shall both generalize the argument, and use it in a different context, namely for the exact bosonization just referred to. We now see from (8b) (and cf. Ref. 15) interdependence of holes ($\tilde{E}_n$) and breather labeled $n-1$. We need a physical soliton energy $\tilde{E}_s$: This cannot usefully be defined by either $-\tilde{E}_n$ or $\frac{1}{2}\tilde{E}_{n-1}$ alone and we choose a relation between "densities"

$$1+e^{-\beta\tilde{E}_s}\equiv(1+e^{-\beta\tilde{E}_{n-1}})(1+e^{\beta\tilde{E}_n})^{1/2}\,. \tag{9a}$$

This has the property that it introduces no discontinuity in soliton mass between the cases $p_0=n$, $p_0=n+0$ —contrary to Ref. 16 which defines $\tilde{E}_s$ through $\tilde{E}_n$ alone. Equation (8b) now means (9a) is equally

$$\tilde{E}_{n-1}-(2\beta)^{-1}\ln(1+e^{\beta\tilde{E}_n})=2\tilde{E}_s+\beta^{-1}\ln(1+e^{-\beta\tilde{E}_s})\,, \tag{9b}$$

and substitution of this for $\tilde{E}_{n-1}$ in (4a) induces, from the second term on the right side, extra soliton-soliton phase shifts $O(n^{-1})$ with respect to (7c). These are consistently neglected in the semiclassical limit, so (9b) in effect sets its left side equal to $2\tilde{E}_s$. We note in passing that definition (9a) was also introduced in Ref. 15 for $p_0=2+0$. We also note how (9a) immediately simplifies (6).

In these ways we find (4a) becomes (for $j=1,\ldots,n-2$)

$$\tilde{E}_j(x)=j\tilde{\varepsilon}(x)+\beta^{-1}\sum_{l=1}^{n-2}[2\min(j,l)-\delta_{jl}]\times\ln(1+e^{-\beta\tilde{E}_l(x)})\,, \tag{10a}$$

with

$$\tilde{\varepsilon}(x)=\omega(x)+\frac{\gamma_0}{8\pi\beta}\int\frac{dx'}{\sinh(x'-x)}\frac{d}{dx'}\Sigma(x')+\frac{2}{\pi\beta}\int\frac{dx'}{\cosh(x'-x)}\ln(1+e^{-\beta\tilde{E}_s(x')})\,, \tag{10b}$$

and

$$\tilde{E}_s(x)=E_s(x)+\frac{1}{\pi\beta}\int\frac{dx'}{\cosh(x'-x)}\Sigma(x')+\frac{8}{\pi\gamma_0\beta}\int dx'\ln\frac{\cosh(x'-x)+1}{\cosh(x'-x)-1}\ln(1+e^{-\beta\tilde{E}_s(x')}). \tag{10c}$$

We use $\omega(x)\equiv m\cosh x$, $M_j=2M\sin(j\gamma_0''/16)\sim jm$ as $\gamma_0\to 0$, and

$$\Sigma(x)\equiv\sum_{j=1}^{n-2}j\ln(1+e^{-\beta\tilde{E}_j(x)})\,.$$

Consistently, for new semiclassical $\gamma_0$, $M$, $E_s(x)\equiv M\cosh x$ and $M\equiv 8m\gamma_0^{-1}$. The support is $-\infty<x'<\infty$. Thus, we can go on to find from (6) in the same way that

$$FL^{-1}=-\frac{1}{2\pi\beta}\int dx\,\omega(x)\Sigma(x)-\frac{1}{\pi\beta}\int dx\,E_s(x)\ln(1+e^{-\beta\tilde{E}_s(x)})\,. \tag{11}$$

Equation (10a) is solved exactly for $n<\infty$ as

$$1+e^{\beta\tilde{E}_j}=\left[\frac{bc^j-b^{-1}c^{-j}}{c-c^{-1}}\right]. \tag{12}$$

The two parameters $b$ and $c$ are fixed by boundary conditions.[20] Note that for $n\to\infty$, when $b=c\to e^{\tilde{\varepsilon}/2}$, that (12)→ Fowler's[21] solution. It also means $\Sigma(x)=-\ln(1-e^{-\beta\tilde{\varepsilon}(x)})$ exactly in the semiclassical limit, the exact bosonization referred to. By using this in (10b) and (10c) and (11) we therefore reach

$$\tilde{\varepsilon}(x)=\omega(x)-\frac{\gamma_0}{8\pi\beta}\int\frac{dx'}{\sinh(x'-x)}\frac{d}{dx'}\ln(1-e^{-\beta\tilde{\varepsilon}(x')})+\frac{2}{\pi\beta}\int\frac{dx'}{\cosh(x'-x)}\ln(1+e^{-\beta\tilde{E}_s(x')})\,, \tag{13a}$$

$$\tilde{E}_s(x)=E_s(x)-\frac{1}{\pi\beta}\int\frac{dx'}{\cosh(x'-x)}\ln(1-e^{-\beta\tilde{\varepsilon}(x')})+\frac{8}{\pi\gamma_0\beta}\int dx'\ln\left[\frac{\cosh(x'-x)+1}{\cosh(x'-x)-1}\right]\times\ln(1+e^{-\beta\tilde{E}_s(x')})\,, \tag{13b}$$

$$FL^{-1}=\frac{1}{2\pi\beta}\int dx\,\omega(x)\ln(1-e^{-\beta\tilde{\varepsilon}(x)})-\frac{2}{2\pi\beta}\int dx\,E_s(x)\ln(1+e^{-\beta\tilde{E}_s(x)})\,. \tag{14}$$

These semiclassical results couple phonons, bosons mass $m$, energies $\tilde{\varepsilon}(x)$, with (two) fermions mass $8m\gamma_0^{-1}$, energies $\tilde{E}_s(x)$. The strictly classical results follow from $\ln(1-e^{-\beta\tilde{\varepsilon}(x)})\to\ln[\beta\tilde{\varepsilon}(x)]$ and $\ln(1+e^{-\beta\tilde{E}_s(x)})\to e^{-\beta\tilde{E}_s(x)}$ in all places. Although iteration of the resultant system introduces some awkard integrals all of these can be done.[22] The results are precisely those exhibited close to and at Eqs. (2) first found by TIM. Note that this means iteration of the classical integral equations is only asymptotic.

Our results (2) from conventional BA for SG[10] when $p_0=n+0$ bring into line yet one more aspect of the SM of quantum and classical integrable models.[2,3,11,12] The analysis will presumably apply for arbitrary $p_0$ by analytic continuation.

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