PHYSICAL REVIEW B VOLUME 34, NUMBER 9 1 NOVEMBER 1986

Effects of the phase periodicity on the quantum dynamics of a resistively shunted Josephson junction

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(Received 28 April 1986)

A phenomenological model is introduced for the dissipative quantum dynamics of the phase ϕ across a current-biased Josephson junction. The model is invariant under $\phi\rightarrow\phi+2\pi$. This enables us to restrict ϕ to the interval 0 to 2π, equating $\phi+2\pi$ with ϕ, and study the role played by the resulting nontrivial topology. Using Feynman's influence functional theory it is shown that the dissipation suppresses interference between paths with different winding numbers. For Ohmic dissipation this interference is completely destroyed, and ϕ can effectively be treated as an extended coordinate. This justifies the use of the usual washboard potential description of a current-biased junction even in the quantum case, provided an Ohmic dissipation mechanism is present.

In the past several years there has been much theoretical[1,2] and experimental[3-5] interest in the possibility of observing "secondary" macroscopic quantum effects in small-capacitance Josephson junctions. In these junctions the charging energy is comparable to the Josephson coupling energy, and quantum-mechanical fluctuations of the phase difference ϕ across the junction become important. This necessitates treating the phase as a quantum operator $\hat{\phi}$, which is canonically conjugate to the operator $\hat{n}$ which transfers n Cooper pairs across the junction;[6] $[\hat{\phi},\hat{n}]=i$. Since ϕ is a phase variable, ϕ and $\phi+2\pi$ are *physically identical* states.[7] This fact is of little consequence for macroscopic quantum tunneling since the important changes in ϕ are small compared to 2π. Indeed ϕ is usually treated as an extended coordinate,[1,8] $\phi\in[-\infty,\infty]$.

Recently a new type of macroscopic quantum phenomena has been discussed in the literature,[9-11] which is the possible existence of "Bloch" oscillations in current-biased Josephson junctions. Under appropriate conditions, an applied dc current is predicted to cause voltage oscillations across the junction. In terms of the phase ϕ, these oscillations are analogous to Bloch oscillations of an electron in a periodic potential in the presence of an applied field.[12] The oscillations involve motions of ϕ which are large compared to 2π. This necessitates treating with great care the indistinguishability of the states ϕ and $\phi+2\pi$. Likharev and Zorin[10] suggested that due to the environmental coupling these states could be treated as though they were distinguishable. However, conditions were not given as to the form or strength of the dissipation required.

The purpose of this work is to introduce and study a model for the quantum dynamics of a damped Josephson junction, which *respects* the indistinguishability of ϕ and $\phi+2\pi$. The damping is assumed to be due to a frequency-dependent conductor shunting the junction.[13] As in the original Caldeira-Leggett model,[1] this dissipative element is described by a harmonic-oscillator heat bath. However, the oscillators are coupled to the angular velocity $\dot{\phi}(t)$, rather than the phase itself. In this way the Lagrangian for the full system is invariant under $\phi\rightarrow\phi+2\pi$, so that ϕ can be restricted to the interval $[0,2\pi]$. By using Feynman's influence functional theory,[14] we study the role played by the nontrivial topology of ϕ (i.e., $\phi+2\pi=\phi$) on the junction's quantum dynamics. As our main result we derive a condition on the frequency dependence of the shunting conductor for which the effects of the nontrivial topology are entirely washed out. At zero temperature this condition is that the shunting conductance have a nonzero limit as $\omega\rightarrow 0$, which includes the case of Ohmic dissipation. Under such circumstances it is valid to treat ϕ as an extended coordinate ($\phi\in[-\infty,\infty]$) provided one is only interested in the distribution of $\dot{\phi}$, rather than of ϕ itself. In particular, this justifies the use of the common washboard-potential description of a current-biased Josephson junction even in the quantum case, provided an Ohmic dissipation mechanism is present.

We start by motivating the junction Hamiltonian. Consider an ideal Josephson junction of capacitance C shunted by a normal conductor and biased by an external current $I(t)$. The normal current through the conductor I_n is assumed to be linear in the junction voltage V,

$$I_n(t)=\int_{-\infty}^{t}dt'G(t-t')V(t') \ . \tag{1}$$

If we ignore this conductance for the moment, the total charge on the junction plates at time t is

$$\hat{Q}_0=\int^{t}I(t')dt'+2e\hat{n} \ ,$$

where $2e\hat{n}$ is the charge transferred across the junction due to Cooper-pair tunneling (the carets denote quantum operators). Since the charging energy is $\hat{Q}_0^2/2C$ the Hamiltonian is[15]

$$\hat{H}=\frac{1}{2C}\left[2e\hat{n}+\int^{t}I(t')dt'\right]^2-E_J\cos\hat{\phi} \ , \tag{2}$$

where the second term in (2) is the Josephson locking energy[6] and $[\hat{\phi},\hat{n}]=i$ as previously mentioned. Equation (2) is invariant with respect to 2π translations in ϕ.

Next, we want to modify the above description in order to include the conductor shunting the junction. To do this we must account for the charge being transported through this element; the total charge on the plates at time t is then $\hat{Q}=\hat{Q}_0-\int^{t}\hat{I}_n(t')dt'$, where $\hat{I}_n$ is the operator for the normal current. Since the current response of the shunting conductor is assumed to be linear [Eq. (1)] it can be modeled by a heat bath of harmonic oscillators. The normal current is expressed as $\hat{I}_n = -\sum_j \lambda_j d\hat{x}_j/dt$, where the $\{x_j\}$ are oscillator coordinates and the $\{\lambda_j\}$ are a set of coupling constants to be specified. The new Hamiltonian is then

$$\hat{H} = \frac{1}{2C}\left[2e\hat{n} + \int^t I(t')dt' + \sum_j \lambda_j \hat{x}_j\right]^2 - E_J\cos\hat{\phi} + \hat{H}_{\rm osc} , \tag{3}$$

where $\hat{H}_{\rm osc}$ is the Hamiltonian for the harmonic oscillator bath with masses $\{m_j\}$ and frequencies $\{\omega_j\}$. As far as the dynamics of $\hat{\phi}$ alone is concerned, the λ_j's enter only via the spectral density[1] $J(\omega)$,

$$J(\omega) = \frac{\pi}{2}\sum_j \frac{\lambda_j^2\omega_j}{m_j}\delta(\omega-\omega_j) . \tag{4}$$

This function is chosen as follows. We require that the voltage operator $\hat{V} = \hbar(d\hat{\phi}/dt)/2e$, generated from (3) via $i\hbar d\hat{\phi}/dt = [\hat{\phi},\hat{H}]$ is related to the normal current operator $\hat{I}_n$ as in Eq. (1). This is achieved by choosing

$$J(\omega) = \omega G(\omega) , \tag{5}$$

where $G(\omega)$ is the cosine transform of $G(t)$. The function $G(\omega)$ is a frequency-dependent shunting conductance; for Ohmic dissipation it is independent of frequency.[1,2] Equations (3)–(5) fully specify the model. The Lagrangian corresponding to (3) is

$$L = \tfrac{1}{2}M\dot{\phi}^2 + E_J\cos\phi - \frac{\hbar\dot{\phi}}{2e}\left[\int^t I(t')dt' + \sum_j \lambda_j x_j\right] + L_{\rm osc} , \tag{6}$$

where $M = (\hbar/2e)^2 C$.

There are a few comments worth making about the above approach. It should be emphasized that Eq. (3) is in effect a "two-fluid" description, consisting of the unpaired (normal) electrons and the superfluid (Cooper pair) electrons. Since the total number of Cooper pairs (in both superconductors) commutes with $\hat{H}$, the number of pairs is conserved; quasiparticle excitations are ignored entirely.[13] In this approach, the two "fluids" are therefore distinct and not interchangeable.

The Lagrangian (6) is very similar to the Lagrangian introduced by Caldeira and Leggett[1] in their discussion of macroscopic quantum tunneling:

$$L_{\rm CL} = \tfrac{1}{2}M\dot{\phi}^2 - U(\phi) + \sum_j \tfrac{1}{2}m_j\dot{x}_j^2 - \sum_j \tfrac{1}{2}m_j\omega_j^2\left[x_j + \frac{\hbar\phi\lambda_j}{2em_j\omega_j}\right]^2 , \tag{7}$$

$$U(\phi) = -E_J\cos\phi - (\hbar\phi/2e)I .$$

In fact, the classical dynamics generated by the two Lagrangians are identical, since (6) can be transformed into the Caldeira-Leggett form by adding a total time derivative and then performing a canonical transformation which interchanges the oscillator coordinates and momenta. In addition, as we will show below, if ϕ is treated as an *extended* coordinate ($\phi \in [-\infty,\infty]$) the *quantum* dynamics generated by the two models are also equivalent. However, $L_{\rm CL}$ is not invariant with respect to 2π translations in ϕ. For the macroscopic quantum tunneling problem this is unimportant. However, in a correct treatment of such phenomena as Bloch oscillations,[9–11] ϕ should be restricted to $\phi \in [0,2\pi]$. The advantage of the Lagrangian (6) is that it is invariant under $\phi \rightarrow \phi + 2\pi$, enabling a quantum description when ϕ is defined on this restricted interval.

We now study the quantum dynamics generated by the Lagrangian (6), paying particular attention to the nontrivial topology that arises when $\phi \in [0,2\pi]$. We employ the influence functional theory developed by Feynman and Vernon[14] which studies directly the time evolution of the reduced density matrix. It is convenient to define the following representation of the reduced density matrix:

$$\rho(\phi,\lambda,t) = {\rm Tr}_B\langle\phi|\exp[iM\lambda(d\hat{\phi}/dt)/2\hbar]\hat{\rho}_{\rm tot}(t) \times\exp[iM\lambda(d\hat{\phi}/dt)/2\hbar]|\phi\rangle , \tag{8}$$

which can be viewed as the generating function for moments of ϕ and $\dot{\phi}$. Here $\hat{\rho}_{\rm tot}(t)$ is the system-plus-bath density matrix at time t for the Hamiltonian (3), and $|\phi\rangle$ denotes an eigenstate of $\hat{\phi}$ defined on the interval 0 to 2π. The trace in (8) is over the bath degrees of freedom. The probability distribution of ϕ follows directly from $\rho(\phi,\lambda,t)$ by setting $\lambda = 0$, whereas moments of $\dot{\phi}$ are generated by differentiation with respect to λ at $\lambda = 0$ and tracing over ϕ.

We now apply the Feynman-Vernon theory to study the time evolution of ρ from a given initial condition. For convenience we assume a factored form

$$\langle\phi_0|\exp[iM\lambda(d\hat{\phi}/dt)/2\hbar]\hat{\rho}_{\rm tot}(0)\exp[iM\lambda(d\hat{\phi}/dt)/2\hbar]|\phi_0\rangle = \rho(\phi_0,\lambda_0,0)\hat{\rho}_{\rm osc}(\beta) , \tag{9}$$

where $\hat{\rho}_{\rm osc}(\beta)$ is the equilibrium density matrix of the uncoupled oscillator bath at temperature $T = (k_B\beta)^{-1}$. The long-time steady-state properties of the junction should not be influenced by this particular choice. After a tedious but straightforward calculation,the reduced density matrix $\rho(\phi,\lambda,t)$ at time t is related to $\rho(\phi_0,\lambda_0,0)$ by a propagator $K(\phi,\lambda,t;\phi_0,\lambda_0,0)$, which is represented as a double path integral over paths $\phi(t)$ and $\phi'(t)$, with $\phi, \phi' \in [0,2\pi]$. These paths are coupled by an influence functional which depends on ϕ and ϕ'. At this point it is convenient to introduce a winding number representation.[16] The amplitude to go from ϕ_1 to ϕ_2 on a ring (i.e., $0 \le \phi_1, \phi_2 \le 2\pi$) is expressed as a sum of amplitudes for an *extended* coordinate starting at ϕ_1 and ending at $\phi_2 + 2\pi n$. The winding number n runs over all integers from $-\infty$ to ∞ and corresponds to the number of times a given path cycles through 2π, i.e., winds around the ring. Two winding numbers, n and m, are needed for the forward and backward paths ϕ and ϕ'. In this way ρ may be expressed in the form

$$\rho(\phi,\lambda,t) = \int d\phi_0\, d\lambda_0 \sum_{n,m} K_{nm}(\phi,\lambda,t;\phi_0,\lambda_0,0)\,\rho(\phi_0,\lambda_0,0) , \tag{10}$$

with the propagator

$$K_{nm}=\exp\left[i2\pi(m-n)(2e)^{-1}\int_0^t I(t')dt'\right]\int_{\phi_0+\lambda_0/2}^{\phi+\lambda/2+2\pi n}D\phi\int_{\phi_0-\lambda_0/2}^{\phi-\lambda/2+2\pi m}D^*\phi'\exp\{(i/\hbar)[S_0(\phi)-S_0(\phi')]\}e^{i\Phi_{nm}}\ , \quad (11)$$

and with S_0 the action for a washboard potential in the absence of damping,

$$S_0=\int_0^t dt'[\tfrac{1}{2}M\dot{\phi}^2+E_J\cos\phi+(\hbar/2e)I(t')\phi]\ . \quad (12)$$

The integration in (10) is over the range $0\le\phi_0+\frac{1}{2}\lambda_0\le 2\pi$. The path integration in (11) is for an *extended* coordinate ϕ; this was obtained at the expense of introducing the additional summation over winding numbers. The influence phase Φ_{nm} represents the effect of the dissipative environment. It is given by $[\theta=(\phi+\phi')/2, \chi=\phi-\phi']$

$$i\Phi_{nm}=i\Phi_{\rm CL}+\frac{2\pi}{\hbar}(n-m)\int_0^t dt'A(t-t')\chi(t')$$
$$+\frac{i}{\hbar}2\pi(n-m)\int_0^t dt'B(t-t')[\theta(t')-\phi_0]$$
$$-(n-m)^2F\ , \quad (13a)$$

with

$$A(t)=\left[\frac{\hbar}{2e}\right]^2\frac{1}{\pi}\int_0^\infty d\omega\,G(\omega)\sin(\omega t)\coth(\beta\hbar\omega/2)\ , \quad (13b)$$

$$F=\frac{2\pi}{\hbar}\left[\frac{\hbar}{2e}\right]^2\int_0^\infty\frac{d\omega}{\omega}G(\omega)\coth(\beta\hbar\omega/2)\ , \quad (13c)$$

and $B(t)$ is the $\beta\to 0$ limit of $\beta\hbar\,\partial_t A(t)$. The first term in (13a) is the Caldeira-Leggett influence phase[17]

$$i\Phi_{\rm CL}=-\left[\frac{i}{\hbar}\right]\int_0^t dt'\int_{t'}^t ds\,\dot{\theta}(t')\chi(s)B(s-t')$$
$$-\left[\frac{1}{\hbar}\right]\int_0^t dt'\int_0^{t'} ds\,\chi(t')A(t'-s)\chi(s)\ , \quad (14)$$

which is the result for an extended coordinate described by the Lagrangian (7). The remaining terms depend explicitly on the difference between the winding numbers and reflect the nontrivial topology of the phase ϕ when restricted to the interval 0 to 2π.

There are several important points to notice about the results (10)–(14). If ϕ had been treated as an extended coordinate from the start, the sum over winding numbers would be replaced by the $n=m=0$ term alone. The resulting dynamics is then equivalent to that generated by the Caldeira-Leggett Lagrangian (7). Thus for an *extended* coordinate the two Lagrangians (6) and (7) generate identical quantum dynamics.

When $\phi\in[0,2\pi]$, the last term in (13a) suppresses those configurations of the double path integral in which the two paths have different winding numbers, $n\ne m$. The dissipation tends to destroy the interference between these paths. The factor $\exp(-F)$ is the well-known Franck-Condon factor which describes the reduction of an overlap between two states of ϕ (separated by 2π) due to the environmental degrees of freedom.[2] However, it did not arise from an adiabatic treatment of the fast oscillator variables as is usually the case, but rather as a consequence of an exact elimination of the bath degrees of freedom. There are now two possibilities:

(i) The Franck-Condon factor is finite. This is the case for shunting conductances $G(\omega)$ which vanish sufficiently rapidly at low frequencies so that the integral in (13c) is convergent. In particular, for $G(\omega)\sim\omega^\nu$ for $\omega\to 0$, F is finite for $\nu>0$ at $T=0$ and $\nu>1$ for nonzero temperature. In this case, although configurations with $n\ne m$ are suppressed, they nevertheless contribute to the functional integral (11) and may give rise to nontrivial dynamical effects.

(ii) The Franck-Condon factor vanishes. This occurs for $\nu\le 0$ at $T=0$ and $\nu\le 1$ for $T\ne 0$, since then $F=\infty$. In particular, this case includes Ohmic dissipation ($\nu=0$) where $G(\omega)$ approaches a nonzero constant at low frequencies. Under these circumstances, configurations with different winding numbers are suppressed completely and have zero weight. The dissipative environment destroys *completely* the interference between the two paths.

The difference between these two cases (at $T=0$) can be understood heuristically in simple physical terms. Consider the charge transferred through the resistor when the phase cycles through 2π. The motion of the phase introduces a nonzero voltage which drives the normal current as in Eq. (1). As the phase cycles a net nonzero charge is transferred only for $\nu\le 0$. In this case the system ends up in a state orthogonal to the one it started in; the overlap between two paths with different winding numbers ($n\ne m$) should vanish. In contrast, for $\nu>0$ no net charge is transferred, the final state is not orthogonal, and configurations with $n\ne m$ contribute.

Enormous simplifications occur in case (ii) since the double summation over winding numbers in (10) reduces to a single sum over $n=m$. The time evolution of ρ may then be written

$$\rho(\phi,\lambda,t)=\int d\phi_0\,d\lambda_0\sum_n K_{\rm ext}(\phi+2\pi n,\lambda,t;\phi_0,\lambda_0,0)\times\rho(\phi_0,\lambda_0,0)\ , \quad (15)$$

where $K_{\rm ext}\equiv K_{n=m}$ is the propagator for an extended coordinate, with dynamics described by *either* the Lagrangian (6) or (7). A further simplification occurs if we restrict attention to the moments of $\dot{\phi}$ (the junction voltage) which follows from

$$\rho(\lambda,t)=\int_0^{2\pi}d\phi\,\rho(\phi,\lambda,t)\ , \quad (16)$$

by differentiating with respect to λ at $\lambda=0$. Integrating on (15) gives

$$\rho(\lambda,t)=\int d\phi_0\,d\lambda_0\,\rho(\phi_0,\lambda_0,0)\times\int_{-\infty}^{\infty}d\phi\,K_{\rm ext}(\phi,\lambda,t;\phi_0,\lambda_0,0)\ , \quad (17)$$

which is precisely the generating function for the moments of $\dot{\phi}$ for an *extended* coordinate.

Thus we arrive at our main result. When the frequency-dependent shunting conductance $G(\omega)$ gives a vanishing Franck-Condon factor [i.e., $F=\infty$ in (13c)] the nontrivial effects which arise from restricting ϕ to $[0,2\pi]$ are entirely suppressed. As far as the moments of $\dot{\phi}$ are concerned, the results are entirely equivalent to those obtained for an *extended* coordinate, with dynamics generated by either (6) or (7). In particular, this justifies the use of the Caldeira-Leggett Lagrangian[1] for calculating the voltage response in a current driven Josephson junction, provided an Ohmic dissipation mechanism is present, $G(\omega\rightarrow 0)\neq 0$. On the contrary, in cases where F is finite this equivalence to an extended description does not hold. Nontrivial dynamical effects may arise from the interference between paths with different winding numbers.

We would like to acknowledge helpful discussions with Professor A. J. Leggett. M.P.A.F. expresses gratitude for support by AT&T Bell Laboratories, and A.T.D. expresses gratitude for support from National Science Foundation Grant No. NSF-DMR-15550. This research was supported in part by the John D. and Catherine T. MacArthur Foundation at the University of Illinois under Grant No. 0-6-40129.

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