Lines and domain walls in dilute ferromagnets

Anton Bovier and Jürg Fröhlich
Theoretische Physik, Eidgenössische Technische Hochschule-Hönggerberg, CH-8093 Zürich, Switzerland

Ueli Glaus
Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742
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We discuss the fluctuations of directed polymers in a random medium perturbatively. We argue that the disorder is irrelevant in dimensions $d > 3$. We also study fluctuations of domain walls in d-dimensional dilute ferromagnets. By relating this model to a random-field Ising model in $d-1$ dimensions, we find that rigid interfaces exist at low temperatures in dimensions $d \geq 4$, while for $d \leq 3$ interfaces are delocalized at all temperatures. Some new numerical results in two dimensions are presented which confirm the prediction that the fluctuation exponent has the value $\frac{2}{3}$.

I. INTRODUCTION

The problem of the roughening of domain walls in ferromagnets with random exchange couplings has received considerable attention over the last months. So far, most results concern two-dimensional dilute ferromagnets. Numerical results have been presented in Refs. 1 and 2, and analytical results in Refs. 3–5. The most reliable analytical results were based on the observation that, after some suitable approximations, this problem can be linked to the study of the noise-driven Burgers equation.[3,4,6,7] In particular, it has been shown that the average fluctuation height h_L of an interface of length L behaves, asymptotically, as

$$h_L \sim L^{2/3} \quad \text{as } L \to \infty \ , \tag{1.1}$$

for all temperatures, the exponent 2/3 being exact.

In this paper we propose to discuss generalizations in two directions. First, we consider the fluctuations of a line (or "directed polymer") in arbitrary dimensions. This question is related to properties of the spin-spin correlation function at high temperatures in dilute ferromagnets. It has previously been investigated in a continuum approximation using the Burgers equation.[4,6,7] We propose to study this problem in perturbation theory in terms of heat kernels, without taking recourse to the Burgers equation. As in the earlier treatments, we find that weak disorder is irrelevant in dimensions $d > 3$, and thus

$$h_L \sim L^{1/2}, \quad \text{for } d > 3 \ . \tag{1.2}$$

In the marginal dimension, $d = 3$, we show that perturbatively logarithmic corrections appear, at most. This does not, however, exclude the possibility that nonperturbatively they add up to a power correction, since the theory is not asymptotically free.

Another prediction of perturbation theory is that a crossover should take place in two dimensions from *entropy*-dominated fluctuations, scaling as $L^{1/2}$ on length scales $L < L_0$, to *disorder*-dominated fluctuations, scaling as $L^{2/3}$ on length scales $L > L_0$, where L_0 depends on the temperature T and on the strength of the disorder ϵ, and diverges as T goes to infinity, or ϵ goes to zero.

In the second part of this paper, we study $(d-1)$-dimensional interfaces in d dimensions. Within the conventional approximations [solid on solid (SOS), only the disorder in the bonds perpendicular to the interface relevant] we reformulate the model in terms of step contours. The resulting model is very similar to the $(d-1)$-dimensional random-field Ising model. Within the uncertainties with regard to this latter model (in particular, with regard to validity of the one-countour approximation), recent work on this problem[8–10] then implies that the rigidity of the interface in a pure ferromagnet at low temperatures is destroyed in three dimensions by the disorder, whereas in four and more dimensions, a rigid phase persists at low temperatures. We also give some heuristic estimates on the height of the fluctuations in various regimes.

The final section is devoted to the presentation of some numerical results in two dimensions, which are somewhat complementary to previous measurements.[1,2] We have measured the averaged distribution of the interfacial height with high statistics (5×10^4 realizations of the disorder) at finite temperatures. The data confirm earlier findings and theoretical predictions on the crossover from $L^{1/2}$ to $L^{2/3}$ behavior at finite temperature. Furthermore, and this is the main new result of our simulations, we find that the averaged distribution can, with good accuracy, be described as Gaussian. This implies that the effect of the disorder may be accommodated in just one parameter, the covariance of the distribution, thus justifying a very simple scaling ansatz.

II. LINES IN d DIMENSIONS

We consider dilute Ising ferromagnets with Hamiltonian

$$H_J = -\sum_{\langle i,j\rangle} J_{ij}\sigma_i\sigma_j \ , \tag{2.1}$$

where σ_i are Ising spins taking values ± 1, and the J_{ij} are independent random variables with mean 1, for example, and covariance $\epsilon \ll 1$. The specific form of their distribution is expected to be unimportant, and we will choose it to our convenience in specific situations. In the present section we take it to be

$$d\mu(J_{ij}) = \frac{1}{\sqrt{2\pi\epsilon}} \exp\left[-\frac{(J_{ij}-1)^2}{2\epsilon} \right] . \tag{2.2}$$

Using the high-temperature expansion in terms of random walks, we can write the two-point function (at high temperatures) as a sum over walks:

$$\langle \sigma_x \sigma_y \rangle_J = \sum_{\omega:\ x\to y} \prod_{\langle i,j\rangle \in \omega} (\beta J_{ij}) z_{J,\beta}(\omega) , \tag{2.3}$$

where $z_{J,\beta}(\omega) \equiv z(\omega) \to 1$ as $\beta \to 0$.

Consider a plane Π_L orthogonal to the x axis intersecting the x axis at a distance L from the origin. We may ask for the probability $\rho_L(y_\perp)$ for a walk ω, weighted as in (2.3), to hit Π_L at the point $(L,y_\perp) \in \Pi_L$, i.e.,

$$\rho_L(y_\perp) = \frac{\sum_{\omega:\ 0\to(L,y_\perp)} \prod_{\langle i,j\rangle\in\omega} (J_{ij}\beta) z(\omega)}{\sum_{y_\perp} \sum_{\omega:\ 0\to(L,y_\perp)} \sum_{\langle i,j\rangle\in\omega} (J_{ij}\beta) z(\omega)} . \tag{2.4}$$

This distribution evidently also governs the height of fluctuations of a line going from (0,0) to $(2L,0)$. [More precisely, the probability that such a line passes through $(L,y_\perp)$ is given by $\rho_L(y_\perp)\rho_L(-y_\perp)$.] The second moment of the averaged distribution, the square of the average height of the fluctuations,

$$h_L^2 \equiv \sum_{y_\perp} y_\perp^2 [\rho_L(y_\perp)]_{\rm av} = \left[\frac{\sum_{y_\perp} y_\perp^2 \langle \sigma_0 \sigma_{(L,y_\perp)} \rangle}{\sum_{y_\perp} \langle \sigma_0 \sigma_{(L,y_\perp)} \rangle} \right]_{\rm av} . \tag{2.5}$$

is the quantity studied most often.[3,5] This quantity also describes the fluctuations of an interface in a two-dimensional dilute ferromagnet; see Sec. III. It should also be noted that contrary to naive expectations, the denominator in the last expression in *not* self-averaging.

For the evaluation of h_L [or $\rho_L(y_\perp)$], one usually introduces some approximations which are generally believed not to alter the essential features of the result, and which we will also adopt in the sequel.

(i) Put $z(\omega)=1$. This corresponds to ignoring loop contributions in the high-temperature series and it becomes exact in the limit as β tends to zero.

(ii) Ignore lines with "overhangs." Again, this introduces errors that tend to zero as β becomes small.

(iii) Ignore the disorder in the "perpendicular" directions, i.e., take only the couplings on bonds parallel to the x axis to be random, while all the other ones are put equal to 1. The rational behind this is the idea that the main effect of the disorder is due to the possibility of gaining energy by making deviations from the shortest path. But only the parallel bonds contribute to this gain: A perpendicular step is always made in excess and it costs energy—how much it costs is not expected to be important.

Under these assumptions, we can describe a walk ω making L steps in the x direction by a function ϕ_t, $t=0,1,2,\ldots,L$, taking values in $\mathbb{Z}^{d-1}$, with an action

$$S(\phi) = \sum_{t=1}^{L} \left[\tfrac{1}{2}(\phi_t - \phi_{t-1})^2 + \frac{\epsilon}{\beta} \mathscr{V}(\phi_t,t) \right] , \tag{2.6}$$

where $\epsilon\mathscr{V}$ is the fluctuating part of the parallel bonds $[\epsilon\mathscr{V} = \ln(J-1),\ \bar\beta = -\ln\beta]$, i.e., a Gaussian random variable with mean 0 and covariance

$$\int \mathscr{V}(y_\perp,t)\mathscr{V}(y'_\perp,t')d\mu(\mathscr{V}) = \delta_{y_\perp y'_\perp}\delta_{tt'} . \tag{2.7}$$

$\rho_L(y_\perp)$ is then given by the expression

$$\rho_L(y_\perp) = \frac{\int D\mathscr{W}_\beta(\phi) \exp\left[-\epsilon \sum_t \mathscr{V}(\phi_t,t) \right] \delta(\phi_L - y_\perp)}{\int D\mathscr{W}_\beta(\phi) \exp\left[-\epsilon \sum_t \mathscr{V}(\phi_t,t) \right]} , \tag{2.8}$$

where

$$D\mathscr{W}_\beta(\phi) = \frac{1}{N} \prod_{t=0}^{L} d\phi_t \exp\left[-\bar\beta \sum_{t=1}^{L} \tfrac{1}{2}(\phi_t - \phi_{t-1})^2 \right] \delta(\phi_0) , \tag{2.9}$$

and the normalization is chosen such that

$$\int D\mathscr{W}_\beta(\phi) = 1 . \tag{2.10}$$

Clearly, for $\epsilon=0$ the result is just the heat kernel, i.e. (we put $\bar\beta=1$)

$$\rho_L^{\epsilon=0}(y_\perp) = [\exp(-L\Delta)]_{0y_\perp} \sim \frac{1}{(2\pi L)^{(d-1)/2}} \exp\left[-\frac{y_\perp^2}{2L} \right] , \tag{2.11}$$

for L large, and hence

$$h^{\epsilon=0}(L) \sim L^{1/2} . \tag{2.12}$$

It is tempting to try a perturbation expansion about this free theory, for the average of ρ_L. To derive it, we find it convenient to use the so-called "replica trick." [Note that here, the use of the replica trick to derive the perturbation expansion is just a convenient means to facilitate the combinatorics. The same series results if the perturbation expansion is derived for fixed $\mathscr{V}(\phi,t)$, and then averaged over the disorder]. We write

$$[\rho_L(y_\perp)]_{\rm av}=\int \frac{d}{dx}\left[\ln\int D\mathcal{W}_\beta(\phi)\exp\left[-\epsilon\sum_t \mathcal{V}(\phi_t,t)-x\delta(\phi_L-y_\perp)\right]\right]_{x=0} d\mu(\mathcal{V})$$

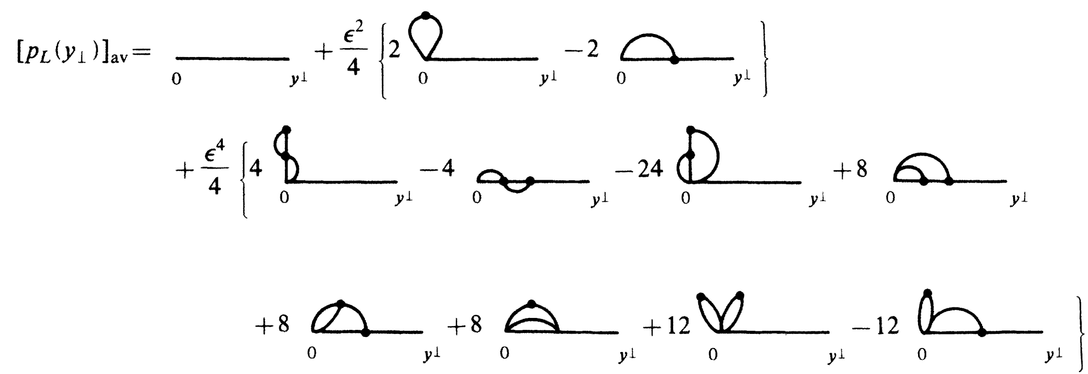
$$=\lim_{n\to 0}\int \frac{d}{dx}\left\{\frac{1}{n}\int\prod_{\alpha=1}^{n} D\mathcal{W}_\beta(\phi^\alpha)\exp\left[-\epsilon\sum_{\alpha=1}^{n}\left[\sum_t \mathcal{V}(\phi_t^\alpha,t)-x\delta(\phi_L^\alpha-y_\perp)\right]\right]-\frac{1}{n}\right\}_{x=0} d\mu(\mathcal{V})$$

$$=\lim_{n\to 0}\int\prod_{\alpha=1}^{n} D\mathcal{W}_\beta(\phi^\alpha)\exp\left[\frac{\epsilon^2}{2}\sum_{\alpha\neq\alpha'}\sum_t \delta(\phi_t^\alpha-\phi_t^{\alpha'})\right]\delta(\phi_L^1-y_\perp)\ . \tag{2.13}$$

From this expression the perturbation series is derived quite simply. In fact we have

$$[\rho_L(y_\perp)]_{\rm av}=\lim_{n\to 0}\sum_{k=0}^{\infty}\frac{(\epsilon^2/2)^k}{k!}\sum_{t_1,\ldots,t_k}\ \sum_{\substack{\alpha_1\neq\alpha_1',\\ \alpha_2\neq\alpha_2',\\ \cdots\\ \alpha_k\neq\alpha_k'}}\int\prod_{\alpha=1}^{n} D\mathcal{W}_\beta(\phi^\alpha)\prod_{t=1}^{k}\delta(\phi_{t_i}^{\alpha_i}-\phi_{t_i}^{\alpha_i'})\delta(\phi_L^1-y_\perp)\ . \tag{2.14}$$

The first few terms of this series can be represented in the following graphical form:



(2.15)

Power counting shows that a kth order term in this series behaves like

$$L^{-(d-1)/2}(L^{(3-d)/2})^k\ . \tag{2.16}$$

In a continuum approximation, this result can also be obtained from a simple rescaling argument.

Thus, in dimensions larger than three, the higher-order corrections vanish as $L\to\infty$, and the asymptotic behavior of $[\rho_L(y_\perp)]_{\rm av}$ is given by the unperturbed theory. Therefore, weak disorder is irrelevant. But in dimensions below three, the higher-order terms grow with increasing powers of L and the perturbation theory is inappropriate to extract the asymptotic behavior. However, we may expect that this series is reliable on length scales L, such that

$$\epsilon^2 L^{(3-d)/2} \ll 1$$

(for general $\beta\neq 1$, this is modified to $\beta^{1/2}\epsilon^2 L^{(3-d)/2}\ll 1$); and that thus the fluctuaions are essentially normal up to that length scale, whereas on larger scales the influence of the disorder becomes important and a modified, disorder-dominated behavior sets in.

The situation in three dimensions, the borderline case, is somewhat peculiar. All terms in the series behave like L^{-1}, up to logarithmic corrections. Therefore, one might expect only logarithmic corrections to the leading Gaussian scaling of the fluctuations, i.e.,

$$h(L)\sim L^{1/2}(\ln L)^B\ . \tag{2.17}$$

The theory not being asymptotically free, it is however conceivable that it gets driven to a nonperturbative fixpoint with power-law corrections. This question deserves further numerical study.

We have calculated explicitly, in three dimensions, all diagrams up to order ϵ^4 found that

$$h(L)^2=2L+\frac{\epsilon^2}{4\pi}L-\frac{2\epsilon^4}{(4\pi)^2}+\frac{\epsilon^4}{(4\pi)^2}L\ln L+O(\epsilon^6)\ . \tag{2.18}$$

III. INTERFACES IN d DIMENSIONS

We now turn to the dilute Ising model at *low* temperatures. We estimate the fluctuations of a $(d-1)$-dimensional interface forced into the system by applying

"+"-boundary conditions on the upper and "−"-boundary conditions on the lower half of a box of side length L. Note that in two dimensions this problem is related to the one studied in the previous section by duality.

In the unperturbed system, it is known that at small temperatures, fluctuations remain bounded for $d \geq 3$, while in two dimensions they behave as $L^{1/2}$, for all $T > 0$.

We argue that the disorder in the couplings enhances fluctuations and that the dimension in and above which stable interfaces occur at small temperatures is raised to four.

With the use of the same reasoning as in the preceding section, we make the same approximations as before: I.e., we use the SOS approximation for our interfaces and we ignore the disorder in the perpendicular direction.

In this approximation, the interface can be described by a collection, Γ, or oriented loops, γ_i, such that for any two of them we have either (i) $\text{int}\gamma_i \cap \text{int}\gamma_j = \varnothing$, (ii) $\text{int}\gamma_i \subseteq \text{int}\gamma_j$, or (iii) $\text{int}\gamma_i \supseteq \text{int}\gamma_j$.

The orientation of a loop indicates whether it represents a step "up" or "down." The energy of such an ensemble is then given by

$$E(\Gamma) = \sum_{i=1}^{n} |\gamma_i| + \sum_{i=1}^{n} \sum_{x \in \text{int}\gamma_i} [\mathscr{V}(h_{\gamma_i} + \sigma_{\gamma_i}, x) - \mathscr{V}(h_{\gamma_i}, x)] , \tag{3.1}$$

where h_γ is the "height" of γ, i.e.,

$$h_\gamma = \sum_{i:\ \text{int}\gamma \subset \text{int}\gamma_i} \sigma_{\gamma_i} , \tag{3.2}$$

and σ_γ is equal to $+1$ or -1, depending on whether the orientation of γ is up or down.

In this representation, our interface model strongly resembles the random-field Ising model in $d-1$ dimensions. In fact, if we consider only a single contour, this resemblance is exact, with $\mathscr{V}(h_\gamma + \sigma_\gamma, x) - \mathscr{V}(h_\gamma, x)$ playing the role of the magnetic fields h_x. The h_x are independent random variables, and by the central limit theorem, the "bulk" energy of a large contour is a random variable with mean zero and covariance $\epsilon\,|\,\text{int}\gamma\,|^{1/2}$. The possible energy gain of a contour due to the bulk term is thus of the order of $\epsilon\,\text{const}\,|\gamma|^{(d-1)/2(d-2)}$, while the surface energy is always $|\gamma|$. Thus, if $d > 3$, for large contours the bulk term is insignificant and the total energy behaves like $|\gamma|$, implying that for sufficiently low temperatures such a contour cannot be formed. For dimensions $d \leq 3$, however, the bulk term is always relevant. With positive probability, it will be energetically favorable to form the contour γ, and the interface will therefore be rough at all temperatures. This is essentially the Imry-Ma argument.[11]

On the level of this one-contour approximation (i.e., "no contours within contours"), this picture can be turned into a completely rigorous argument, as has been shown in Ref. 8. For an extensive discussion of the validity of the one-countour approximation, see in particular Ref. 9. For the random-field model, Imbrie[10] has shown rigorously that there is no magnetization at zero temperature in three dimensions. Presumably, his proof can be also carried over to our model. There is thus strong evidence for the conjecture that interfaces are rigid in dimension $d > 3$, while in dimensions three and below, interfaces are rough at all temperatures, *including zero.*

We would like to conclude this section with a few comments on the nature of the interface fluctuations in these systems. First of all, we want to stress the distinction between disorder- and entropy- dominated fluctuations. While the latter are a thermal effect that invariably vanishes at $T = 0$, the former are due to the structure of the ground state(s), i.e., the fact that the energetically most favorable state(s) are no longer given by the smallest interface, but by rather rough ones. Characteristically, the roughness therefore persists at zero temperature.

We note that in the systems considered here, fluctuations at low temperatures (if the interface is rough at all) are always disorder dominated. At high temperatures, entropic effects should become visible. In two dimensions, they become dominate only on small (though increasing with T) length scales: Up to some scale $L(T)$, the fluctuations are Gaussian, and due to thermal excitations on larger scales, the effect of the disorder is dominant and the fluctuations the $L^{2/3}$ behavior. Clearly $L(T)$ diverges as T goes to infinity.

In three dimensions the situation is somewhat different. We have seen that bulk energy and entropy scale in the same way, but since the entropy enters into the free energy with a factor T in front, at sufficiently high temperatures it will dominate over the surface-energy term. Therefore, while at low temperatures the disorder perturbs an otherwise rigid interface, at higher temperatures it can be considered as a perturbation of an interface that is already rough. We may thus expect a crossover.

It is an interesting question to estimate the height of fluctuations in the various regimes and to see whether such a crossover could be observed. This may be done using a heuritic ansatz for the surface free energy. Unfortunately, these ansätze always contain some *a priori* assumption on the interface structure and are therefore not always reliable. We present their prediction nevertheless, but warn the reader not to take them literally. See, e.g, Ref. 12 for discussions. Let L denote the linear size of the system and h the maximal height of the interface fluctuation. At low temperatures, assuming that the disorder is relevant (i.e., the interface fluctuates), the surface free energy should behave as

$$E_L^{\text{surf}}(h) \sim \beta h^2 L^{d-2} , \tag{3.3}$$

while the available bulk energy is of the order of (see above)

$$E_L^{\text{bulk}} \sim \epsilon L^{(d-1)/2} . \tag{3.4}$$

Note that the surface energy is proportional to h^2 rather than the naive h. This takes into account that to reach a height h by making random steps, the number of steps needed is, by the central-limit theorem, proportional to h^2.

To actually gain the bulk term, the interface must fluctuate and pay a price in surface energy. The maximal height can thus be estimated by equating both terms and solving for h. We find

$$h \sim \sqrt{\epsilon/\beta} L^{(3-d)/4} , \tag{3.5}$$

which of course in three dimensions really becomes

$$h \sim \sqrt{\epsilon/\beta}\sqrt{\ln L} . \tag{3.6}$$

At high temperatures, on the other hand, thermal excitations are abundant, and one would expect the surface free energy to be given by

$$E_L^{\rm surf}(h) \sim \beta \left[\frac{h}{L} \right]^2 L^{d-1} , \tag{3.7}$$

so that

$$h \sim \sqrt{\epsilon/\beta} L^{(5-d)/4} , \tag{3.8}$$

i.e., in three dimensions

$$h \sim \sqrt{\epsilon/\beta} L^{1/2} . \tag{3.9}$$

Notice that none of these estimates is correct in two dimensions. If they apply in three dimensions, a crossover from logarithmic to square-root fluctuations should take place. The more naive expectation that at high temperatures the disordered system should behave like the pure one, together with the estimate (3.6), on the other hand would suggest logarithmic fluctuations at all temperatures. A numerical investigation of this question would be highly desirable.

Our result for the critical dimension for fluctuations, $d_c = 3$, is at variance with that of the authors of Ref. 3, who find rigidity only in five and more dimensions. However, they are considering a continuum approximation to this model [corresponding to the surface-energy ansatz (3.7)], and this enhances fluctuations.

IV. NUMERICAL RESULTS

In this section we report on some numerical simulations for the two-dimensional model. In contrast to the calculations of Ref. 1, they were done at finite temperature. Thus, although (due to the more involved numerical process, resulting in smaller system sizes) we cannot match their accuracy with regard to the exponent α, we provide two pieces of complementary and not entirely uninteresting information.

TABLE I. $\sigma_n(t)$ for various n and t.

n \ t	30	40	50
1	7.87	9.38	10.82
2	7.78	9.28	10.69
3	7.67	9.17	10.56
4	7.59	9.06	10.44
5	7.50	8.97	10.33
6	7.42	8.88	10.23
7	7.34	8.79	10.13
8	7.27	8.71	10.35

(i) We have test the nature of the averaged distribution $[\rho_L(y_\perp)]_{\rm av}$ and found good agreement with a Gaussian ansatz.

(ii) We checked the dependence of α on the length scale for various strengths of the disorder. A transition from $\alpha = \frac{1}{2}$ to $\alpha = \frac{2}{3}$ is observed, at least qualitatively, and the disorder dependence of this crossover agrees (again qualitatively) with the predictions of perturbation theory.

Finite-temperature simulations have recently also been performed by Kardar *et al.*[4] For numerical reasons, we choose to simulate the distribution

$$P_{\{J\},\xi}(y,t) = \frac{\displaystyle\sum_{\omega:\, 0\to(y,t)} \prod_{\langle ij\rangle\in\omega} \xi^{J_{ij}}}{\displaystyle\sum_{y} \sum_{\omega:\, 0\to(y,t)} \prod_{\langle ij\rangle\in\omega} \xi^{J_{ij}}} , \tag{4.1}$$

with ω a SOS walk that furthermore is restricted to make steps of maximal height one in the y direction. The J_{ij} are taken to be one for all bonds $\langle ij\rangle$ oriented in the y direction, and for bonds $\langle ij\rangle$ oriented in the t direction,

$$J_{ij} = \begin{cases} \epsilon & \text{with probability } p , \\ 1 & \text{with probability } 1-p . \end{cases} \tag{4.2}$$

Our simulation is based on the following recursion relation:

$$P_{\{J\},\xi}(y,t) = \frac{\displaystyle\sum_{y':\,|y-y'|<1} \xi^{|y-y'|+J_{(y,t-1),(y,t)}} P_{\{J\},\xi}(y',t-1)}{\displaystyle\sum_{y=-t}^{t} \left[\sum_{y':\,|y-y'|\le 1} \xi^{|y-y'|+J_{(y,t-1),(y,t)}} P_{\{J\},\xi}(y',t-1) \right]} , \tag{4.3}$$

and starts with the initial condition

$$p_{\{J\},\xi}(y,0) = \delta(y) . \tag{4.4}$$

Note: Unfortunately this procedure becomes numerically unstable for large t. Empirically, we find for $\xi = \sqrt{2} - 1$, $\epsilon = 8$, $p = 0.5$, that the numerical values for expectations of $|y|^n$ become unreliable for $t \gtrsim 60$. We therefore cannot hope to reach the true asymptotic regime and measure accurately the value of α. It should be noted that for $\xi \to 0$, and a continuous distribution of the J values (as was the case in the simulations of Ref. 1) $P_{\{J\},0}$ remains a δ function at all t. In this situation it suffices to determine the largest term in the numerator of (4.3) at

TABLE II. The exponent α for various ϵ and t_0 as explained in the text.

ϵ \ t_0	1	4	8	12	16	20
2	0.5213 $\pm$0.0017	0.5252 $\pm$0.0016	0.5289 $\pm$0.0014	0.5320 $\pm$0.0017	0.5365 $\pm$0.008	0.5369 $\pm$0.009
4	0.5783 $\pm$0.0040	0.5883 $\pm$0.0031	0.5906 $\pm$0.0030	0.5970 $\pm$0.0036	0.6049 $\pm$0.0041	0.5954 $\pm$0.106
8	0.6047 $\pm$0.0031	0.6121 $\pm$0.0023	0.6174 $\pm$0.0021	0.6199 $\pm$0.0033	0.6261 $\pm$0.0059	0.6341 $\pm$0.0129

each step, which is of course much easier to do and does not become unstable. The authors of Ref. 1 were thus able to deal with systems up to $t=4000$ and obtained therefore a much more accurate value for α.

We calculated $P_{\{J\},\xi}(y,t)$ for $M=5\times10^4$ realizations of $\{J\}$, so that statistical errors with respect to the J distribution become irrelevant. We tested the Gaussian nature of the averaged distribution,

$$P_{\xi,p,\epsilon}(y,t)=\frac{1}{M}\sum_{i=1}^{M}P_{\{J\}_i,\xi}(y,t)\ , \tag{4.5}$$

by calculating the first eight moments:

$$E_n(t)=\int |y|^n dP_{\xi,p,\epsilon}(y,t)\ . \tag{4.6}$$

The function

$$\sigma_n(t)=\begin{cases}\sqrt{2}\left[\dfrac{E_n(t)}{(n-1)!!}\right]^{1/n}, & n \text{ even},\\[2ex] \sqrt{\pi}\left[\dfrac{E_n}{[(n-1)/2]!}\right]^{1/n}, & n \text{ odd}\ ,\end{cases} \tag{4.7}$$

would be independent of n for a Gaussian distribution. We list $\sigma_n(t)$ for $t=30, 40, 50$; $\epsilon=8$, $p=0.5$, $\xi=\sqrt{2}-1$ in Table I.

Evidently, our data are in good agreement with the assumption that σ_n is constant, and this agreement improves as t increases. We may thus conclude that

$$P_{\xi,p,\epsilon}(y,t)=\frac{1}{\sqrt{\pi}\sigma(\xi,p,\epsilon,t)}\exp\left[-\frac{y^2}{\sigma^2(\xi,p,\epsilon,t)}\right], \tag{4.8}$$

for large t.

For $\xi=0$, it was found in Ref. 1 that

$$\sigma(t)=t^\alpha,\ \text{as } t\to\infty\ ,$$

with $\alpha=\frac{2}{3}$, and from analytic results[3,5] this is expected for all $\epsilon>0$, ξ. However, for $\xi>0$, $\sigma(t)$ should, as we argued previously, show a crossover from

$$\sigma(t)\sim t^{1/2}\ \text{for } t<t_0(\xi,\epsilon,p)$$

to

$$\sigma(t)\sim t^{2/3}\ \text{for } t>t_0(\xi,\epsilon,p)\ .$$

We determined α from a least-squares fit to the relation

$$\ln E_2(t)=2\alpha\ln t+c\ , \tag{4.9}$$

fitting the parameters α and c. The results are presented in Table II. α is given as a function of t_0, in that all the data points t with $t_0\le t\le 50$ were included in the fit for $\alpha(t_0)$.

We observe a good qualitative agreement with our expectations. We also see that even with $\epsilon=8$, our simulations have not yet reached the asymptotic regime.

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