# Phase diagrams and correlation exponents for quantum spin chains of arbitrary spin quantum number

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The low-temperature properties of the spin-$S$ quantum spin chain are studied representing a spin-$S$ operator as the sum of $2S$ spin-$\frac{1}{2}$ operators. The resulting system of $2S$-coupled spin-$\frac{1}{2}$ systems is studied in the weak-coupling limit, using a continuum representation. It is shown that under scaling, the coupling becomes strong. Under the additional hypothesis (which is shown to be true for $S=1$) that strong coupling represents correctly the properties of the spin-$S$ system, the following results are obtained: (i) there are, in general, two types of planar (massless) phases $XY1$ and $XY2$, separated by an Ising-like transition; (ii) for half-odd-integer $S$ the exponent $\eta$ governing the asymptotic power-law decay of transverse spin correlations takes the universal value $\eta=1$ at the boundary between $XY1$ and the adjacent uniaxial antiferromagnetic phase; (iii) for integer $S$ there is an additional singlet phase between $XY1$ and the antiferromagnetic state, with $\eta=\frac{1}{4}$ at the limit of $XY1$; (iv) a spin-Peierls instability only exists for half-odd-integer $S$; (v) a magnetic field along the $z$ direction may lead to a transition from the singlet or antiferromagnetic state to a planar phase. Universal scaling relations between exponents for transverse and longitudinal correlations in the phases $XY1$ and $XY2$ and an explicit asymptotic expression for correlation functions are derived. Finally, implications of the present results for some generalizations of the spin chain problem are briefly discussed. The above points (ii) and (iii) confirm predictions by Haldane, which were derived using quite different methods.

## I. INTRODUCTION

A theoretical understanding of the properties of quantum spin chains is important for a number of reasons: on the one hand, there is a large number of quasi-one-dimensional magnetic systems[1,2] like $CsNiF_3$. Their properties are often one-dimensional over a considerable range of temperatures. On the other hand, spin-chain models pose interesting theoretical problems. For example, Haldane[3] has argued that the physical properties are vastly different depending on whether the spin quantum number $S$ is an integer or a half-odd integer. Moreover, a number of theoretical problems can be described in terms of spin-chain models. An interesting recent example is the relation between spin chains and their SU($n$) generalizations and the theory of the integer quantum Hall effect.[4]

In the present paper I consider the anisotropic Heisenberg spin chain, described by the Hamiltonian

$$H=-\sum_{i=1}^{N}[(S_i^xS_{i+1}^x+S_i^yS_{i+1}^y)+J_zS_i^zS_{i+1}^z-D(S_i^z)^2]\ , \tag{1.1}$$

where $\mathbf{S}_i^2=S(S+1)$, and the index $i$ labels consecutive lattice sites. $D=0$, $J_z=1$ is the isotropic ferromagnet, $D=0$, $J_z=-1$ is the isotropic antiferromagnet (after a unitary transformation $S^x\rightarrow -S^x, S^y\rightarrow -S^y$ on every second site), and in general there is both exchange anisotropy ($|J_z|\neq 1$) and single-ion anisotropy ($D\neq 0$), but the model is isotropic in the $xy$ plane. For $S=\frac{1}{2}$ the single-ion term is a constant, and the model is exactly solvable.[5-7] For $S>\frac{1}{2}$ no exact solution exists. According to Haldane[3,8] the properties of the model depend drastically on the spin quantum number: for half-odd-integer $S$ the excitation spectrum at the isotropic antiferromagnetic point is gapless ("massless"), and spin-spin correlations decay algebraically at large distances, with a universal exponent $\eta=1$ [cf. Eq. (2.11) below]; on the other hand, for integer $S$ there is a gap in the excitation spectrum, and spin-spin correlations decay exponentially. Though initially controversial, there is by now a considerable amount of numerical evidence that this prediction is indeed correct, at least for small $S$ ($S=\frac{1}{2},1$).[9-11] Very recent experiments also seem to confirm Haldane's prediction for $S=1$.[12]

For general spin quantum number $S$ no exact solution of the model (1.1) exists. However, for $S=1$, considerable progress in understanding the properties of the model (1.1) has been made representing the spin-1 chain as two coupled spin-$\frac{1}{2}$ chains and analyzing the resulting field-theoretical problem.[13-15] Here I generalize this method to arbitrary spin quantum number. However, before doing so, in Sec. II I shall describe and extend the method used by Timonen and Luther (TL).[15] This will help to clarify the subsequent derivation for $S>1$. Moreover, correcting some errors in TL, it is actually possible to obtain considerably more information about the phase diagram, correlation functions, etc., than obtained previously. In Sec. III the general $S$ case is treated. It will be seen that the present results largely agree with those of Haldane, though the method used here is quite different, and considerably more information can be obtained. In Sec. IV, expressions for the asymptotic behavior of correlation functions in different massless phases are derived. Section V is devoted to a discussion of the results.

## II. THE CASE $S=1$

To represent the spin-1 operators, one writes $\mathbf{S}_i=\sigma_i+\tau_i$, where $\sigma$ and $\tau$ are spin-$\frac{1}{2}$ operators. The ground-state properties of the original spin-1 model are the same as those of the resulting system of two coupled spin-$\frac{1}{2}$ chains.[13] One now uses a Jordan-Wigner transformation[16] from spin-$\frac{1}{2}$ to fermion operators $a_i^\dagger,a_i$,

$$\sigma_i^z=a_z^\dagger a_i-\tfrac{1}{2}\ ,$$
$$\sigma_i^+=\sigma_i^x+i\sigma_i^y=a_i^\dagger \exp\left[i\pi\sum_{n=1}^{i-1}a_n^\dagger a_n\right] \tag{2.1}$$

and analogously for the $\tau$'s with fermion operators $b_i^\dagger,b_i$. TL then go to the continuum limit by using the boson representation of fermion operators. Doing this carefully, I find

$$H=H_b+H'\ ,$$

where $H_b$ is bilinear in fermion density operators,

$$H_b=\frac{\pi}{L}\sum_{k(\neq 0)}\sum_{s=1,2}\{v_s[\rho_{+,s}(k)\rho_{+,s}(-k)+\rho_{-,s}(k)\rho_{-,s}(-k)]+2\lambda_s\rho_{+,s}(k)\rho_{-,s}(-k)\}\ . \tag{2.2}$$

Here $\rho_{\pm,1}=(\rho_{\pm,a}+\rho_{\pm b})/\sqrt{2},\rho_{\pm,2}=(\rho_{\pm,a}-\rho_{\pm,b})/\sqrt{2}$, and $\rho_{\pm,a},\rho_{\pm,b}$ are the standard density operators for right- $(+)$ and left-moving $(-)$ fermions of types $a$ and $b$, respectively. The coefficients in (2.2) are

$$v_1=1-(3J_z-D)/\pi\ ,$$
$$v_2=1-(J_z+D)/\pi\ , \tag{2.3}$$
$$\lambda_s=v_s-1\ .$$

TL calculate slightly different coefficients which in my notation are $v_1=1-(J_z-D)/\pi$, $v_2=1+(J_z-D)/\pi$, $\lambda_1=(D-3J_z)/\pi,\lambda_2=-\lambda_1$. The difference is due to TL's neglect of Hartree-Fock-like corrections to the Fermi velocity[17] ($g_4$ processes in standard "g-ology" notation[18,19]) and an additional sign error in their calculation of $\lambda_2$. The $\rho$ operators obey bosonlike commutation relations.[18–20] It is now convenient to introduce the boson phase fields

$$\psi_s(x)=-\frac{i\pi}{L}\sum_{p(\neq 0)}\frac{1}{p}[\rho_{+,s}(p)+\rho_{-,s}(p)]e^{-\alpha|p|/2-ipx}\ , \tag{2.4}$$

$$X_s(x)=\frac{i\pi}{L}\sum_{p(\neq 0)}\frac{1}{p}[\rho_{+,s}(p)-\rho_{-,s}(p)]e^{-\alpha|p|/2-ipx}\ , \tag{2.5}$$

where $\alpha$ is a short-range cutoff which may be identified with a lattice constant. $X_s$ is related to the momentum density conjugate to $\psi_s$ via $\partial X_s/\partial x=\pi \chi_s(x)$. The canonical transformation $\rho_{+,s}\to-\rho_{+,s}$ leads to the interchange $\psi_s\leftrightarrow X_s$. Using the continuum representation

$$i\pi\sum_{n=1}^{i-1}a_n^\dagger a_n=i\pi x/2-i[\psi_1(x)+\psi_2(x)]/\sqrt{2}\ , \tag{2.6}$$

an analogous relation for the $b$ fermions, and the boson representation of single-fermion operators,[21–24] $H$ decomposes into separate and commuting parts for $\psi_1$ and $\psi_2$

$$H=H_1+H_2\ ,$$

with

$$H_1=\int dx\left[\frac{\pi}{2}\chi_1^2(x)+\frac{1}{2\pi}[1+2(D-3J_z)/\pi]\left[\frac{\partial\psi_1}{\partial x}\right]^2\right]+\frac{1}{(\pi\alpha)^2}\int dx\,\mu_1\cos[\sqrt{8}\psi_1(x)]\ , \tag{2.7}$$

$$H_2=\int dx\left[\frac{\pi}{2}\chi_2^2(x)+\frac{1}{2\pi}[1-2(D+J_z)/\pi]\left[\frac{\partial\psi_2}{\partial x}\right]^2\right]+\frac{1}{(\pi\alpha)^2}\int dx\{\mu_2\cos[\sqrt{8}\psi_2(x)]+\mu_3\cos[\sqrt{2}X_2(x)]\}\ , \tag{2.8}$$

with $\mu_1=\mu_2=D+J_z$, $\mu_3=-1$. In the fermion description the $\mu_1$ and $\mu_2$ operators represent umklapp and backward scattering between $a$ and $b$ fermions ($g_3$ and $g_{1\perp}$ processes), which come from the $\sigma^z\tau^z$ coupling between the spin-$\frac{1}{2}$ systems.[25] The $\mu_3$ operator comes from the $\sigma^+\tau^-$ coupling and does not have a simple representation in terms of local fermion operators. In going to the continuum limit, lattice-renormalization effects on the coefficients in $H_1$ and $H_2$ are neglected, so that the coefficients given in Eqs. (2.7) and (2.8) are strictly valid only to lowest order in $D$ and $J_z$. Moreover, only the most relevant operators [in a renormalization group sense, see the scaling equations (3.13)–(3.19) below] have been kept.

In a way similar to Eqs. (2.7) and (2.8) the spin operators are found as

$$S^+(x)=\frac{1}{\pi\alpha}e^{-iX_1(x)/\sqrt{2}}\times\{\cos[X_2(x)/\sqrt{2}]+e^{i\pi x}\cos[\sqrt{2}\psi_2(x)+X_2(x)/\sqrt{2}]\times e^{-i\sqrt{2}\psi_1(x)}\}\ , \tag{2.9}$$

$$S^z(x)=-\frac{\sqrt{2}}{\pi}\frac{\partial\psi_1}{\partial x}+\frac{2}{\pi\alpha}e^{i\pi x}\cos[\sqrt{2}\psi_2(x)]\cos[\sqrt{2}\psi_1(x)]\ . \tag{2.10}$$

We are now in a position to discuss the phase diagram of the model described by Eqs. (2.7) and (2.8). First consider $H_2$. Apart from the Gaussian (bilinear) part, there are two operators with coefficients $\mu_2$ and $\mu_3$. Their scaling dimensions $x_2,x_3$ can be obtained from correlation functions, as briefly outlined in Sec. IV. One finds $x_2=2[1-2(D+J_z)/\pi]^{-1/2}$, $x_3=1/x_2$, so that there is always at least one relevant operator ($x_i<2$). As argued by den Nijs,[14] $H_2$ can be considered as the transfer matrix of the two-dimensional $XY$ model with a twofold anisotropy field $\mu_2$ and order parameter $\exp[i\sqrt{2}\psi_2(x)]$. That model is expected to have properties similar to those of the two-dimensional Ising model. Consequently, there are two possible phases for $H_2$.

(1) A disordered phase, where correlation functions containing $\exp(i\sqrt{2}\psi_2)$ decay exponentially. On the other hand, $\cos(\sqrt{2}X_2)$ is a disorder (or vortex) operator,[26,27] and consequently correlation functions involving $X_2$ go to a nonzero constant at large distances. In the disordered state $\cos(\sqrt{2}X_2)$ is the dominant (i.e., most relevant) operator, and therefore this state is realized for not too negative $D+J_z$ (where $x_3<x_2$).

(2) An ordered phase with long-range order in the $\psi_2$ correlations and exponentially decaying $X_2$ correlations. In this state $\cos(\sqrt{8}\psi_2)$ is most relevant, and it exists for sufficiently negative $D+J_z$ (where $x_2<x_3$). From the scaling equations (3.13)–(3.19) below, taken for $S=1$, the boundary between the two phases is expected to be $D+J_z\approx-3\pi/2$ for small $\mu_2,\mu_3$. This is line 4 in Fig. 1.

We now turn to $H_1$. For $3J_z-D>\pi/2$ (line 1 in Fig. 1) the coefficients of $(\partial\psi_1/\partial x)^2$ is negative, indicating a breakdown of the continuum limit. For spin-$\frac{1}{2}$ this corresponds to the onset of ferromagnetic long-range order in the lattice model,[5,6,17] and it can reasonably be assumed that this also occurs for $S=1$. For $3J_z-D<\pi/2$ the only potentially relevant operator in $H_1$ is $\cos(\sqrt{8}\psi_1)$, with scaling dimension $x_1=2[1+2(D-3J_z)/\pi]^{-1/2}$. This operator is relevant for $x_1<2$, i.e., $D-3J_z>0$ (line 2 in Fig. 1). To the left of line 2 there is a mass gap in the excitation spectrum of $H_1$, corresponding to a long-range-ordered $\psi_1$ field and exponentially decaying $X_1$ correlations. On the other hand, for $x_1>2$ ($D-3J_z<0$) $\cos(\sqrt{8}\psi_1)$ is an irrelevant operator, there is no mass gap, and both $\psi_1$ and $X_1$ correlations decay algebraically at large distances. In addition, there is a special line

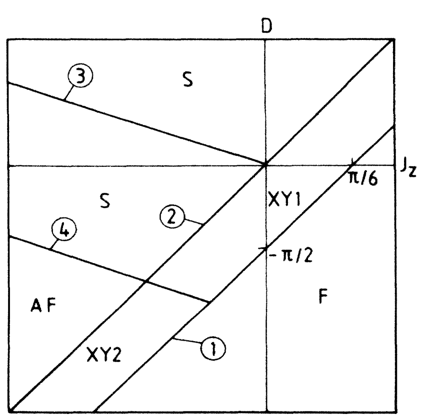


FIG. 1. Phase diagram for $S=1$, with ferromagnetic (F), antiferromagnetic (AF), planar ($XY1$,$XY2$), and singlet (S) phases. The different lines are explained in the text. Line 3 is shifted upwards for convenience.

$D+J_z=0$ (line 3 in Fig. 1) where the $\cos(\sqrt{8}\psi_1)$ operator does not exist ($\mu_1=0$). Here again $H_1$ is massless.

Apart from the ferromagnetic region there are four different phases in Fig. 1. In order to discuss their physical properties, I consider the following three spin-spin correlation functions:

$$G_\perp(x,t)=\langle S^+(x,t)S^-(0,0)\rangle =C_\perp|x^2-v^2t^2|^{-\eta/2} +D_\perp e^{i\pi x}(x^2+v^2t^2)|x^2-v^2t^2|^{-1-\eta'/2}, \quad (2.11)$$

$$G_{\perp 2}(x,t)=\langle[S^+(x,t)]^2[S^-(0,0)]^2\rangle =C_{\perp 2}|x^2-v^2t^2|^{-\eta_2/2} +D_{\perp 2}e^{i\pi x}(x^2+v^2t^2)|x^2-v^2t^2|^{-1-\eta_2'/2}, \quad (2.12)$$

$$G_z(x,t)=\langle S_z(x,t)S_z(0,0)\rangle =C_z\frac{x^2+v^2t^2}{(x^2-v^2t^2)^2}+D_ze^{i\pi x}|x^2-v^2t^2|^{-\eta_z/2} . \quad (2.13)$$

Here the alternating ($\propto e^{i\pi x}$) and slowly varying pieces come from the corresponding pieces in the operator representation, Eqs. (2.9) and (2.10). Equations (2.11), (2.12), and (2.13) define the exponents governing the asymptotic behavior in the massless phases, $v$ is the spin-wave velocity (determined by $H_1$),

$$v=[1+2(D-3J_z)/\pi]^{1/2} ,$$

and in $G_{\perp 2}$ only the leading asymptotic behavior is retained.

In the region $XY1$ $X_2$ is long-range ordered and therefore contributes a constant to the prefactors $C_\perp,C_{\perp 2}$, etc. A standard calculation[15,21,18] (see also Sec. IV, this type of result will be used repeatedly) then results in

$$\eta=1/2x_1,\quad \eta_2=4\eta . \quad (2.14)$$

From the condition $x_1>2$ one finds $0<\eta<\frac{1}{4}$, with $\eta=\frac{1}{4}$ at the boundary to the region $S$. On the other hand, $\psi_2$ correlations, and therefore the alternating parts of $G_\perp$ and $G_z$, decay exponentially (formally $\eta'=\eta_z=\infty$). In summary, there are power-law in-plane correlations, whereas the alternating part of the out-of-plane correlations is short ranged. $XY1$ and its boundary towards $S$ are the only parts of the phase diagram considered by TL, and, of course, their result for $\eta$ agrees with the one given here.

In the $XY2$ region the roles of $\psi_2$ and $X_2$ are interchanged. $X_2$ correlations and therefore $G_\perp$ decay exponentially ($\eta=\eta'=\infty$). However, in $(S^+)^2$ there is a contribution not containing $X_2$ at all, and consequently $G_{\perp 2}$ obeys power laws,

$$\eta_2=2/x_1,\eta_2'=\eta_2+1/\eta_2 . \quad (2.15)$$

One has $0<\eta_2<1$, with $\eta_2=1$ at the boundary to the AF region. Because $\psi_2$ is long-range ordered, there are now also power-law contributions to the alternating part of $G_z$,

$$\eta_z=1/\eta_2 . \quad (2.16)$$

The $XY2$ phase is thus quite different from $XY1$; no power-laws in $G_\perp$, but additional power-law contributions in the alternating parts of both $G_{\perp 2}$ and $G_z$. The situation is reminiscent of the spin-$\frac{1}{2}$ case, where one also has both in-plane and out-of-plane power laws. The analogy is easily understood from the original Hamiltonian, Eq. (1.1); for large negative $D$, the dominating states are those with $S^z = \pm 1$, whereas $S^z = 0$ has a very high energy and therefore is nearly frozen out. Working with basis states containing only $S^z = \pm 1$, it is possible to derive an effective spin-$\frac{1}{2}$ problem.[28] To pass from $S^z = -1$ to $S^z = 1$, two applications of $S^+$ are needed, leading to power laws in $G_{\perp 2}$ instead of $G_\perp$. From the known exponents for $S = \frac{1}{2}$ (Refs. 29 and 30) one then finds the scaling relations (2.15) and (2.16). Nevertheless, it is quite rewarding to obtain these results directly from the continuum representation. From the above discussion of the properties of $H_2$ the transition between $XY1$ and $XY2$ is expected to be Ising-like. This is analogous to the transition between two different planar phases found in a generalized two-dimensional $XY$ model.[31]

To the left of line 2 there is a mass gap in $H_1$, and $\psi_1$ is long-range ordered. In the region marked AF, $\psi_2$ is also long-range ordered, and from Eq. (2.10) one then sees that there is long-range antiferromagnetic order along the $z$ axis, i.e., a uniaxial Néel State. On the other hand, $X_i$ correlations and therefore $G_\perp$ and $G_{\perp 2}$ decay exponentially. Finally, crossing line 3 (via an Ising transition) one finds long-range order in $\psi_1$ and $X_2$, whereas $\psi_2$ and $X_1$ correlations decay exponentially. Inspection of Eqs. (2.9) and (2.10) then shows that all of the correlation functions discussed above decay exponentially, i.e., the system is in a nonmagnetic singlet state. Within this phase there is line 3, along which the mass-generating operator vanishes. Here correlations are similar to the $XY1$ phase ($\eta_2 = 4\eta$, $\eta_z = \eta' = \eta_2' = \infty$), however, now $\eta > \frac{1}{4}$. A classical two-dimensional analogue of this phase is the low-temperature critical line of an $XY$ model with sixfold anisotropy.[32] The whole line 2 is characterized by a single operator ($\propto \mu_1$) becoming relevant, and therefore is of the Kosterlitz-Thouless type.

The above perturbative analysis is valid for small coefficients $\mu_i$ of the nonlinear operators. On the other hand, in the continuum representation of the original model, Eqs. (2.7) and (2.8), these parameters are quite large, especially $\mu_3$. If our discussion is to apply to the spin-chain model, Eq. (1.1), there must be a continuous renormalization-group connection between the small parameter region discussed here and the parameters appropriate for the spin-1 chain. Such a connection exists in many related problems. For example, the strong-repulsion behavior of the one-dimensional Hubbard model is continuously connected to the weakly interacting continuum model.[18,33] In the present case, numerical work shows that the topology of the phase diagram of the spin-1 chain is the same as that of Fig. 1.[9,11] Of course, the precise shape of the phase boundaries is different, probably due both to the large values of the parameters $\mu_i$ and to the neglect of all lattice renormalization effects in Fig. 1. For small $D$, $J_z$ ($\mu_1,\mu_2$ small) even these differences are small; for $D=0$ the present model predicts the $S$-$XY1$ boundary at $J_z = 0$, whereas the numerical result is $J_z \approx -0.1$. Moreover, the scaling relations $\eta_2 = 4\eta$ ($XY1$) and $\eta_z = 1/\eta_2$ ($XY2$) have been verified numerically.[11] It thus seems fairly clear that the weak-coupling continuum theory indeed describes correctly the qualitative features of the spin-1 chain model. The main discrepancy is that in the spin-1 chain lines 1 and 2 (and similarly 3 and 4) coalesce into a single line at some finite but quite large value of $D$ and $J_z$. At such large coupling, however, it is quite likely that more than just the operators kept in Eqs. (2.7) and (2.8) are necessary for a correct description. In this context it is interesting to remark that preliminary numerical results[34] indicate that the multicritical point where lines 3 and 4 meet is in the same universality class as a special, exactly solvable spin-1 model.[35,36] Approaching the multicritical point along line 4 $\eta$ tends towards $\frac{1}{2}$ ($\eta_2 \to 2$, $\eta_z = \eta' = \infty$), whereas along line 3 one has the Ising value $\eta_z = \frac{1}{4}$. Precisely at the multicritical point one has $\eta = \eta_z = \frac{3}{4}$ (Refs. 4 and 37), i.e., the exponents jump discontinuously. It may be noticed that this last value ($\frac{3}{4}$) is just the sum of the two limits along lines 3 and 4 ($\frac{1}{4} + \frac{1}{2}$). Finally, we remark that it seems unlikely that lines 2 and 3 cross without any effect on each other (decoupled Ising and Kosterlitz-Thouless behavior). It is rather likely that some higher-order operator coupling $\psi_1$ and $\psi_2$ changes this situation, even though at the present stage no precise prediction can be made.

## III. GENERAL SPIN QUANTUM NUMBER

Encouraged by the correct qualitative results for $S=1$ I now use an analogous method for arbitrary $S$: spin-$S$ operators are represented as a sum of $2S$ distinct spin-$\frac{1}{2}$ operators $\boldsymbol{\sigma}_n$

$$\mathbf{S}_i = \sum_{n=1}^{2S} \boldsymbol{\sigma}_{n,i} \tag{3.1}$$

($i$ is the site index) and this is inserted into Eq. (1.1). By arguments similar to those of Luther and Scalapino[13] the ground-state properties of the new model are expected to be the same as those of the original one. Via a Jordan-Wigner transformation $2S$ fermion fields are introduced, one for each $\boldsymbol{\sigma}_n$. I then go to the continuum limit and use the boson representation of fermion operators. Introducing the phase fields

$$\phi_n(x) = -\frac{i\pi}{L} \sum_{p(\neq 0)} \frac{1}{p} [\rho_{+,n}(p) + \rho_{-,n}(p)] e^{-\alpha|p|/2 - ipx} , \tag{3.2}$$

$$\theta_n(x) = \frac{i\pi}{L} \sum_{p(\neq 0)} \frac{1}{p} [\rho_{+,n}(p) - \rho_{-,n}(p)] e^{-\alpha|p|/2 - ipx} , \tag{3.3}$$

[cf. Eqs. (2.4) and (2.5)] the Hamiltonian becomes [with the same approximations as in Eqs. (2.7) and (2.8)]

$$H=\int dx\left[\frac{\pi}{2}\Pi^2(x)+\frac{1}{2\pi}[1-2(D+J_z)/\pi]\left[\frac{\partial\boldsymbol{\phi}}{\partial x}\right]^2+\frac{1}{\pi^2}(D-J_z)\frac{\partial\boldsymbol{\phi}^T}{\partial x}\tilde{M}\frac{\partial\boldsymbol{\phi}}{\partial x}\right]$$
$$+\frac{1}{(\pi\alpha)^2}\sum_{\substack{i,j\\ i<j}}\int dx(\mu_1\cos\{2[\phi_i(x)+\phi_j(x)]\}+\mu_2\cos\{2[\phi_i(x)-\phi_j(x)]\}+\mu_3\cos[\theta_i(x)-\theta_j(x)])\ , \tag{3.4}$$

where, as before, $\mu_1=\mu_2=D+J_z$, $\mu_3=-1$, and the operators with coefficients $\mu_1,\mu_2,\mu_3$ are of the same origin as in Sec. II. A vector notation has been introduced: $\boldsymbol{\phi}=(\phi_1,\ldots,\phi_{2S})$, $\tilde{M}$ is a matrix with all elements equal to unity, and $\Pi$ is the momentum density conjugate to $\boldsymbol{\phi}$. The spin operators are

$$S^+(x)=\frac{1}{2\pi\alpha}\sum_{n=1}^{2S}\exp[-i\theta_n(x)][1+e^{i\pi x}e^{-2i\phi_n(x)}]\ , \tag{3.5}$$

$$S^z(x)=-\frac{1}{\pi}\sum_{n=1}^{2S}\frac{\partial\phi_n}{\partial x}+\frac{1}{\pi\alpha}\sum_{n=1}^{2S}e^{i\pi x}\cos[2\phi_n(x)]\ . \tag{3.6}$$

The bilinear part of (3.4) is diagonalized by a unitary transformation

$$\boldsymbol{\phi}=\tilde{U}\boldsymbol{\psi},\ \ \Pi=\tilde{U}\boldsymbol{\chi},\ \ \boldsymbol{\theta}=\tilde{U}\mathbf{X}\ , \tag{3.7}$$

which defines the transformed fields $\psi,\chi$. Clearly, the scalar products in (3.4) are unaffected by the transformation. The matrix $\tilde{M}$ has one eigenvalue $2S$ and $2S-1$ zero eigenvalues. Due to this degeneracy, there is some freedom in the determination of $\tilde{U}$. In all cases the first column is

$$U_{m1}=1/\sqrt{2S}\ \ (1\le m\le 2S)\ . \tag{3.8}$$

A possible (but not unique) choice for $n\ge 2$ is

$$\begin{aligned}&U_{mn}=1/\sqrt{(n-1)n}\ \ (n>m)\ ,\\ &U_{nn}=-\sqrt{(n-1)/n}\ ,\\ &U_{mn}=0\ \ (n<m)\ .\end{aligned} \tag{3.9}$$

From Eq. (3.8) one has

$$\psi_1(x)=\frac{1}{\sqrt{2S}}\sum_{n=1}^{2S}\phi_n(x)\ , \tag{3.10}$$

i.e., $\psi_1$ is the "average" value of the $\phi_n$. On the other hand, for $n\ge 2$ the $\psi_n$ are invariant under the global transformation $\phi_n(x)\to\phi_n(x)+\Phi(x)$, with $\Phi$ independent of $n$, i.e., the $\psi_n$ with $n\ge 2$ depend only on relative displacements of the $\phi_n$ with respect to each other.

After the unitary transformation the Hamiltonian becomes

$$H=\int dx\left[\frac{\pi}{2}\boldsymbol{\chi}^2(x)+\frac{1}{2\pi}[1-2(D+J_z)/\pi]\left[\frac{\partial\boldsymbol{\psi}}{\partial x}\right]^2+\frac{2S}{\pi^2}(D-J_z)\left[\frac{\partial\psi_1}{\partial x}\right]^2\right]$$
$$+\frac{1}{(\pi\alpha)^2}\sum_{\substack{i,j\\ i<j}}\int dx\{\mu_1\cos[2(U_{ik}+U_{jk})\psi_k(x)]+\mu_2\cos[2(U_{ik}-U_{jk})\psi_k(x)]+\mu_3\cos[(U_{ik}-U_{jk})X_k(x)]\}\ , \tag{3.11}$$

where summation over repeated indices is implied. Because $U_{m1}$ is independent of $m$, the terms in $\mu_2,\mu_3$ involve only the "relative" fields with $k\ge 2$. However, the $\mu_1$ operator couples $\psi_1$ to $\psi_k$'s with $k\ge 2$. Only for $S=1$ this term involves $\psi_1$ only, and then (3.11) is identical to (2.7), (2.8).

I now proceed as in Sec. II and first discuss the region of small parameters $\mu_1,\mu_2,\mu_3$. The corresponding scaling dimensions are

$$\begin{aligned}&x_1=2K_1/S+2(1-1/S)K_2\ ,\\ &x_2=2K_2\ ,\\ &x_3=1/x_2\ ,\end{aligned} \tag{3.12}$$

with

$$K_1=\{1-2[2J_z-(2S-1)(D-J_z)]/\pi\}^{-1/2}\ ,$$
$$K_2=[1-2(D+J_z)/\pi]^{-1/2}\ .$$

The constants $K_i$ determine the long-distance behavior of the average ($K_1$) and relative ($K_2$) fields. From the lowest-order corrections in $\mu_i$ to the $K_i$ I obtain in a way similar to José *et al.*[32] the scaling equations under a change of length scale $\alpha \rightarrow \alpha e^l$,

$$\frac{dK_1}{dl} = -K_1^2 \alpha_{+1} \left[\frac{v_1}{v_2}\right] \left[\frac{\mu_1}{\pi v_1}\right]^2 , \tag{3.13}$$

$$\frac{dK_2}{dl} = -(S-1)K_2^2 \alpha_{+2} \left[\frac{v_1}{v_2}\right] \left[\frac{\mu_1}{\pi v_2}\right]^2 - SK_2^2 \left[\frac{\mu_2}{\pi v_2}\right]^2 + S\left[\frac{\mu_3}{2\pi v_2}\right]^2 , \tag{3.14}$$

$$\frac{d\mu_1}{dl} = \{2-2[K_1+(S-1)K_2]/S\}\mu_1 , \tag{3.15}$$

$$\frac{d\mu_2}{dl} = [2-2K_2]\mu_2 , \tag{3.16}$$

$$\frac{d\mu_3}{dl} = [2-1/(2K_2)]\mu_3 , \tag{3.17}$$

$$\frac{dv_1}{dl} = K_1 \alpha_{-1} \left[\frac{v_1}{v_2}\right] \left[\frac{\mu_1}{\pi v_1}\right]^2 , \tag{3.18}$$

$$\frac{dv_2}{dl} = (S-1)K_2 \alpha_{-2} \left[\frac{v_1}{v_2}\right] \left[\frac{\mu_1}{\pi v_2}\right]^2 , \tag{3.19}$$

where the initial values of the velocities of the average ($v_1$) and relative ($v_2$) excitations are

$$v_1 = \{1-2[2J_z-(2S-1)(D-J_z)]/\pi\}^{1/2} ,$$

$$v_2 = [1-2(D+J_z)/\pi]^{1/2} ,$$

and I have introduced the abbreviations

$$\alpha_{\pm 1}(x) = \frac{x}{2\pi} \int_0^{2\pi} d\theta [1-(1\mp x^2)\sin^2\theta] \times [1-(1-x^2)\sin^2\theta]^{-2K_1/s} ,$$

$$\alpha_{\pm 2}(x) = \frac{1}{2\pi} \int_0^{2\pi} d\theta [1-(1\mp 1)\sin^2\theta] \times [1-(1-x^2)\sin^2\theta]^{-2K_1/S} .$$

Equations (3.18) and (3.19) for the renormalization of $v_1$ and $v_2$ arise because in general $v_1$ and $v_2$ are different, but the corresponding excitations are coupled by the $\mu_1$ term. For the following discussion these two equations are unimportant, and they are included here only for completeness.

To discuss the physical properties in different parts of the $D-J_z$ plane, I consider, as in Sec. II, the correlation functions

$$G_\perp(x,t) = \langle S^+(x,t)S^-(0,0)\rangle = C_\perp |x^2-v^2t^2|^{-\eta/2} + D_\perp e^{i\pi x}(x^2+v^2t^2)|x^2-v^2t^2|^{-1-\eta'/2} , \tag{3.20}$$

$$G_{\perp n}(x,t) = \langle [S^+(x,t)]^n [S^-(0,0)]^n \rangle = C_{\perp n} |x^2-v^2t^2|^{-\eta_n/2} + D_{\perp n} e^{i\pi x}(x^2+v^2t^2)|x^2-v^2t^2|^{-1-\eta_n'/2} , \tag{3.21}$$

$$G_z(x,t) = \langle S_z(x,t)S_z(0,0)\rangle = C_z \frac{x^2+v^2t^2}{(x^2-v^2t^2)^2} + D_z e^{i\pi x}|x^2-v^2t^2|^{-\eta_z/2} . \tag{3.22}$$

I first treat the case of sufficiently small $K_2$ ($D+J_z$ sufficiently negative), so that $\mu_2$ increases more quickly than $\mu_3$. Then, from Eq. (3.14) $K_2$ will further decrease, and so on, i.e., $\mu_2$ is the most relevant operator in this case. In analogy to the discussion in Sec. II one has long-range order in the $\psi_k$ fields ($k \geq 2$), whereas $X_k$ correlations decay exponentially. Due to the long-range order of the relative $\psi$'s, only $\psi_1$ contributes to the renormalization of $\mu_1$, and instead of Eq. (3.15) one has

$$\frac{d\mu_1}{dl} = (2-2K_1/S)\mu_1 . \tag{3.23}$$

There are now two regions, separated by line 2 in Figs. 2 and 3. For $\mu_1 \rightarrow 0$ it is given by $K_1 = S$. On the left of this line $\mu_1$ is a relevant operator, and thus in addition to the relative $\psi$'s also $\psi_1$ is long-range ordered. From Eqs. (3.6) and (3.22) one sees that this corresponds to a uniaxially ordered antiferromagnetic phase. All $X_k$ correlations and therefore all correlation functions containing $S^+$ or $S^-$ decay exponentially at large distances.

On the right of line 2 (Figs. 2 and 3) $\mu_1$ renormalizes to zero [cf. Eq. (3.23)], and therefore the $\psi_1$ field remains massless. The fluctuations of the relative $\psi$'s are frozen out, so that from Eq. (3.6) one finds

$$S^z(x) \approx -\frac{\sqrt{2S}}{\pi}\frac{\partial \psi_1}{\partial x} + \frac{2S}{\pi\alpha} e^{i\pi x} \cos[\sqrt{2/S}\,\psi_1(x)] . \tag{3.24}$$

The exponent in $G_z$ follows as

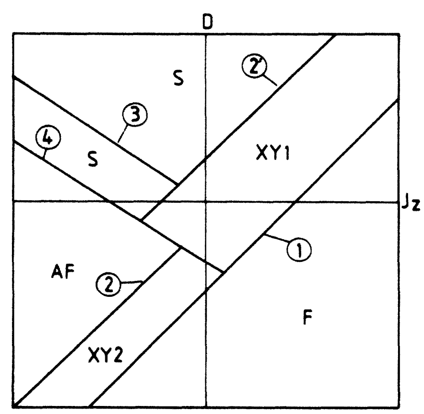


FIG. 2. Schematic phase diagram for integer spin $S$, with the same phases as in Fig. 1, obtained for small parameters $\mu_i$. The axes are in arbitrary units.

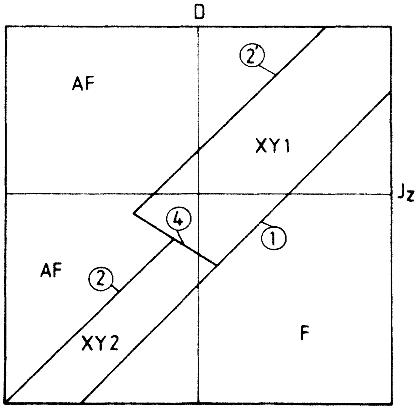


FIG. 3. Same schematic phase diagram as Fig. 2, but for half-odd-integer spin $S$.

$$\eta_z = K_1/S, \quad \eta_z \geq 1 , \tag{3.25}$$

with $\eta_z = 1$ on the boundary to the AF region. One should note that here, as well as in similar formulas below, $K_1$ contains the renormalization effects from the $\mu_i$'s, i.e., $K_1(\infty)$ from Eqs. (3.13)–(3.19) is used. The operator $S^+$ always contains some contribution from relative $X$'s, and therefore $G_\perp$ (and $G_{\perp n}$ for $n < 2S$) decays exponentially. On the other hand, in $(S^+)^{2S}$ there is a contribution containing only $X_1$,

$$[S^+(x)]^{2S} \approx \frac{2S}{(2\pi\alpha)^{2S}} \exp[-i\sqrt{2S}X_1(x)] \times (1 + e^{i\pi x} e^{-i\sqrt{2/S}\,\psi_1(x)}) . \tag{3.26}$$

This leads to

$$\eta_{2S} = 1/\eta_z = S/K_1, \quad 0 \leq \eta_{2S} \leq 1 ,$$
$$\eta'_{2S} = \eta_{2S} + 1/\eta_{2S} . \tag{3.27}$$

This result is analogous to the $XY2$ phase for $S=1$ [see Eqs. (2.15) and (2.16)], and can be explained in the same way; for large negative $D$ the ground state is built essentially out of states with $S_i^z = \pm S$, and one has an effective spin-$\frac{1}{2}$ problem. To pass from $S^z = S$ to $S^z = -S$ $2S$ applications of $S^+$ are needed, and therefore the massless excitations show up in $G_{\perp 2S}$. From the spin-$\frac{1}{2}$ analogy one does indeed obtain the same scaling relations as Eq. (3.27), and it is quite satisfying to observe that the small-$\mu_i$ approach used here does indeed reproduce correctly, for all $S$, these relations.

To its right, the $XY2$ region (Figs. 2 and 3) is limited by line 1, which is determined by $v_1 = 0$. As in the case $S=1$ this is identified as the transition to the fully ordered ferromagnetic state.

Now consider the opposite case: $K_2$ sufficiently large, so that the $\mu_3$ operator is the most relevant one [i.e., increases most quickly, see Eqs. (3.16) and (3.17)]. Equation (3.14) then leads to a further increase of $K_2$, and so on. The $\mu_3$ operator tends to order the relative $X$ fields, and therefore this region is characterized by long-range order in the $X_k$'s with $k \geq 2$, whereas $\psi_k$ correlations decay exponentially ($k \geq 2$). This is the region above line 4 in Figs. 2 and 3. The only potentially massless modes are those associated with the $\psi_1$ field. The only cos operator containing $\psi_1$ in the Hamiltonian (3.11) is the one with coefficient $\mu_1$. However, apart from the special case $S=1$ this operator always contains also some of the relative $\psi$'s, which lead to an exponential decay of the correlation functions appearing in a perturbation expansion [cf. Eq. (3.29)]. One would thus be tempted to conclude that the $\mu_1$ operator gives only finite corrections and is therefore always irrelevant. The $\psi_1$ excitations should remain massless in the whole region above line 4. This conclusion is however somewhat hasty; consider the correlation function

$$D_n(x,\tau) = \langle T_\tau (e^{-in\psi_1(x,\tau)} e^{in\psi_1(0,0)}) \rangle , \tag{3.28}$$

where $\tau$ is the imaginary time and $T_\tau$ is the time-ordering symbol. A typical $n$th order contribution from $\mu_1$ to $D_n$ is proportional to

$$\mu_1^n \langle T_\tau (e^{-in\psi_1(x,\tau)} e^{in\psi_1(0,0)} \cos\{2[\phi_{i_1}(\overline{x_1,\tau_1}) + \phi_{j_1}(\overline{x_1,\tau_1})]\} \times \cos\{2[\phi_{i_2}(\overline{x_2,\tau_2}) + \phi_{j_2}(\overline{x_2,\tau_2})]\} \cdots \cos\{2[\phi_{i_n}(\overline{x_n,\tau_n}) + \phi_{j_n}(\overline{x_n,\tau_n})]\})\rangle . \tag{3.29}$$

Here a bar indicates integration over the corresponding variable, and for convenience we have used the untransformed $\phi$ fields instead of the $\psi$'s. In most cases, the correlation function in Eq. (3.29) decays exponentially as a function of the variable differences $x_m - x_n$, $\tau_m - \tau_n$, and consequently (3.29) gives a finite and irrelevant correction to $D_n$. However, there are some very important exceptions. As an example, consider the fourth-order contribution for $S=2$ with $i_1 = i_3 = 1$, $i_2 = i_4 = 3$, $j_1 = j_3 = 2$, $j_2 = j_4 = 4$. Due to the exponential decay, only the regions $(x_1,\tau_1) \approx (x_2,\tau_2)$, $(x_3,\tau_3) \approx (x_4,\tau_4)$ and $(x_1,\tau_1) \approx (x_4,\tau_4)$, $(x_3,\tau_3) \approx (x_2,\tau_2)$ are important. Up to higher-order corrections, one then can set $(x_1,\tau_1) = (x_2,\tau_2)$, $(x_3,\tau_3) = (x_4,\tau_4)$, and obtains from (3.29)

$$\langle T_\tau (e^{-in\psi_1(x,\tau)} e^{in\psi_1(0,0)} \cos\{2[\phi_1(\overline{x_1,\tau_\perp}) + \phi_2(\overline{x_1,\tau_1})]\} \cos\{2[\phi_3(\overline{x_1,\tau_1}) + \phi_4(\overline{x_1,\tau_1})]\} \times \cos\{2[\phi_1(\overline{x_2,\tau_2}) + \phi_2(\overline{x_2,\tau_2})]\} \cos\{2[\phi_3(\overline{x_2,\tau_2}) + \phi_4(\overline{x_2,\tau_2})]\})\rangle$$

and after transforming products of cosines at the same point into a sum of cosines

$$\langle T_\tau(e^{-in\psi_1(x,\tau)}e^{in\psi_1(0,0)}\cos[4\psi_1(\overline{x_1,\tau_1})]\cos[4\psi_1(\overline{x_2,\tau_2})])\rangle\ . \tag{3.30}$$

Here all additional terms containing relative $\psi$'s have been neglected. Expression (3.30) now contains only $\psi_1$, and therefore has a power-law dependence on its variables. It looks exactly like a second-order term in a perturbation expansion with perturbation proportional to

$$\cos(4\psi_1)\ ,$$

which has scaling dimension $4K_1$ and therefore drives the $\psi_1$ excitations massive for $K_1 < \frac{1}{2}$.

Obviously, the above argument can be applied to arbitrary values of the spin quantum number $S$; one takes a product of $\cos[2(\phi_i+\phi_j)]$ operators, so as to obtain one cosine term which contains only $\psi_1$ as an argument. In this process, an important difference between integer and half-odd-integer values of $S$ arises. Consider $S=\frac{3}{2}$. To fourth order in $\mu_1$ one finds the term

$$\cos[2(\phi_1+\phi_2)]\cos[2(\phi_2+\phi_3)]$$

and cyclic permutations of the indices. Each of these terms contains a relative $\psi$ and therefore is always irrelevant. However, at the sixth order one obtains

$$\cos[2(\phi_1+\phi_2)]\cos[2(\phi_2+\phi_3)]\cos[2(\phi_2+\phi_3)] \approx \cos(4\sqrt{3}\psi_1)\ ,$$

which has scaling dimension $12K_1$ and thus generates a mass for $K_1 < \frac{1}{6}$. It should now be obvious how to generalize this construction to arbitrary $S$. For integer $S$ a product of $S\cos[2(\phi_i+\phi_j)]$ operators at (nearly) identical points is needed to produce

$$\cos[2\sqrt{2S}\,\psi_1]\ , \tag{3.31}$$

which has scaling dimension $2SK_1$ and therefore generates a mass for the $\psi_1$ excitations for $K_1 < 1/S$. This is line 2′ in Fig. 2.

On the other hand, for half-odd-integer $S$ a product of $2S\cos[2(\phi_i+\phi_j)]$ operators is needed and results in

$$\cos[4\sqrt{2S}\,\psi_1] \tag{3.32}$$

which has scaling dimension $8SK_1$ and therefore generates a mass for the $\psi_1$ excitations for $K_1 < 1/4S$. This is the line 2′ in Fig. 3.

I now discuss the physical properties of the different phases above line 4 (Figs. 2 and 3) in terms of the correlation functions defined in Eqs. (3.20)–(3.22). First consider the case where the $\psi_1$ excitations are massless ($XY1$ in Figs. 2 and 3). All the relative $X$'s are long-range ordered, and thus from Eq. (3.5) the effective (nonalternating) $S^+$ operator is

$$S^+(x) \approx \frac{1}{\pi\alpha} S e^{-iX_1(x)/\sqrt{2S}}\ . \tag{3.33}$$

From this expression I obtain the exponents

$$\eta = 1/(4SK_1),\quad \eta_n = n^2\eta\ . \tag{3.34}$$

From the above discussion one has for integer $S$ $K_1 \geq 1/S$ and therefore

$$0 < \eta \leq \tfrac{1}{4}\ , \tag{3.35}$$

with the universal, $S$ independent value $\eta=\frac{1}{4}$ at the transition to the massive region. In contrast, for half-odd-integer $S$ $K_1 \geq 1/4S$, and therefore

$$0 < \eta \leq 1\ , \tag{3.36}$$

with another universal value $\eta=1$ at the transition (line 2′).

The nonalternating part of $S^z$ contains only $\psi_1$, and this gives rise to the first term in Eq. (3.22). The alternating part of $S^z$ always contains some of the relative $\psi$'s, and one thus naively would expect exponential decay. However, an argument as given above for the $\mu_1$ operator shows that this is not always true. An $n$th order (in $\mu_1$) term in the perturbation expansion for the alternating part of $G_z$ is

$$\langle T_\tau(\cos[2\phi_1(x,\tau)]\cos[2\phi_1(0,0)]\cos\{2[\phi_{i_1}(\overline{x_1,\tau_1})+\phi_{j_1}(\overline{x_1,\tau_1})]\} \times \cos\{2[\phi_{i_2}(\overline{x_2,\tau_2})+\phi_{j_2}(\overline{x_2,\tau_2})]\}\cdots\cos\{2[\phi_{i_n}(\overline{x_n,\tau_n})+\phi_{j_n}(\overline{x_n,\tau_n})]\})\rangle\ . \tag{3.37}$$

Via the same mechanism as discussed above, this can give rise to a function containing only $\psi_1$. Consider a second order term for $S=\frac{3}{2}$. Due to the exponential decay of the correlation function, only the regions $(x_1,\tau_1)\approx(x,\tau),(x_2,\tau_2)\approx(0,0)$ and $(x_2,\tau_2)\approx(x,\tau),(x_1,\tau_1)\approx(0,0)$ give appreciable contributions to the integrals, and, taking for example $i_1=i_2=2, j_1=j_2=3$, (3.37) transforms into

$$\langle T_\tau(\cos[2\phi_1(x,\tau)]\cos[2\phi_1(0,0)]\cos\{2[\phi_2(x,\tau)+\phi_3(x,\tau)]\}\cos\{2[\phi_2(0,0)+\phi_3(0,0)]\})\rangle \approx \langle T_\tau\{\cos[2\sqrt{3}\psi_1(x,\tau)]\cos[2\sqrt{3}\psi_1(0,0)]\}\rangle\ . \tag{3.38}$$

Even though this is a perturbative "correction," it decays as a power law at large distances, contrary to the zeroth-order term which decays exponentially. Consequently, the term (3.38) dominates the asymptotic properties of $G_z$. This reasoning is straightforwardly generalized to all half-odd-integer $S$, where one has to go to order $(S-\frac{1}{2})$ to produce a term like (3.38). One obtains for the effective (alternating) $S^z$ operator, governing the long-range behavior of $G^z$,

$$S^z(x) \approx e^{i\pi x}\cos[2\sqrt{2S}\,\psi_1(x)] , \qquad (3.39)$$

and this leads to the exponent

$$\eta_z = 4SK_1 = 1/\eta, \quad \eta_z \geq 1 , \qquad (3.40)$$

with $\eta_z = 1$ at the transition to the AF region. A similar argument can be given for the alternating part of $G_\perp$ and leads to

$$\eta' = \eta + \eta_z . \qquad (3.41)$$

The relations (3.36), (3.40), and (3.41) are the same as those of the well-known case $S=\frac{1}{2}$ (Refs. 29 and 30) and one concludes that spin correlations are generally equal for all half-odd-integer $S$ in our description. Similar relations hold also in the $XY2$ phase, see Eq. (3.27). However, one has to note that in $XY1$ all transverse correlation functions have power-law decay, whereas for $XY2$ there is a power law only in $G_{2S}$. Consequently, there has to be phase transition between $XY1$ and $XY2$.

In contrast to the half-odd-integer case, for integer $S$ one sees that it is not possible to obtain a $\cos(\psi_1)$ operator from contractions of integration variables in (3.37); $\psi_1$ is the sum of an even number of $\phi$'s, whereas from (3.37) one can only obtain a sum of an odd number of $\phi$'s. Thus one is always left with some relative $\psi$'s, and the alternating part of $G^z$ decays exponentially ($\eta_z = \infty$) in the region $XY1$ for integer spin. This is a generalization of the $XY1$ phase discussed in Sec. II for $S=1$.

Finally, in the region on the left of line 2′ in Figs. 2 and 3 the operators (3.31) and (3.32) are relevant and lead to long-range order of the $\psi_1$ field. Consequently, all correlation functions containing $X_1$ such as $G_\perp$ and $G_{\perp n}$ decay exponentially at large distances ($\eta = \eta_n = \infty$). For integer $S$, due to the presence of some relative $\psi$'s in $G_z$, this function decays exponentially, too. Thus, all the magnetic correlations are short ranged, one has a nonmagnetic singlet ground state. In contrast to this situation, for half-odd-integer $S$ the effective $S^z$ operator (3.39) contains $\psi_1$ only, and one therefore has antiferromagnetic long-range order.

Within the region where $\psi_1$ is massive there is the special line 3 (Fig. 2) along which the mass-generating term vanishes; $\mu_1 = D + J_z = 0$. For integer $S$ one then has the same type of correlations as in the $XY1$ phase, but with $\eta > \frac{1}{4}$. On the other hand, for half-odd-integer $S$ this line can only exist for $\eta > 1$, and at these values other operators, e.g., the umklapp operator of the individual $S=\frac{1}{2}$ systems,[26,30,38] become relevant and are likely to generate a mass for the $\psi_1$ excitations. Thus the massless line 3 probably only exists for integer $S$.

The phase transition along the lines 2 and 2′ is characterized by a single operator becoming relevant. This is typical for transitions of the Kosterlitz-Thouless type,[32,39] and therefore the transitions along 2 and 2′ are of that type. The transition along the line 4 is governed by the competition between the $\mu_2$ and $\mu_3$ operators and is more complicated to analyze. In the case $S=\frac{3}{2}$ these two terms become

$$\mu_2\{\cos[\sqrt{8}\psi_2]+\cos[\sqrt{2}(\psi_2+\sqrt{3}\psi_3)]+\cos[\sqrt{2}(\psi_2-\sqrt{3}\psi_3)]\} + \mu_3\{\cos[\sqrt{2}X_2]+\cos[(X_2+\sqrt{3}X_3)/\sqrt{2}]+\cos[(X_2-\sqrt{3}X_3)/\sqrt{2}]\} . \qquad (3.42)$$

If the quadratic part of the Hamiltonian (3.11) is considered as the transfer matrix of a two-dimensional elastic solid with displacement components $\psi_2$ and $\psi_3$, the $\mu_2$ term represents an underlying periodic potential with centered rectangular symmetry. Extending arguments given in Refs. (14) and (40) the $\mu_3$ operators represent thermally excited dislocations with Burgers vectors of length $\sqrt{2}\pi$. Consequently the state with long-range ordered $(\psi_2,\psi_3)$ is twofold degenerate: either $(\psi_2,\psi_3)\approx(0,0)$ or $(\psi_2,\psi_3) \approx(\pi,\pi/\sqrt{3})$. The Hamiltonian can then be considered as the continuum transfer matrix of an antiferromagnetic Ising model on a centered rectangular lattice, and the transition along line 4 thus should be of the two-dimensional Ising type. This type of argument can be applied to all values of $S$, and one always finds a twofold-degenerate ordered state and therefore an Ising transition along line 4. Taking this reasoning literally, for half-odd-integer $S$ this transition would also occur between two antiferromagnetic phases. It seems more likely that in this region the transition either disappears completely, due to some operator neglected here (e.g., the umklapp operator mentioned above for line 3), or becomes of first order. A first-order transition between two antiferromagnetic phases with different amplitudes of the order parameter has actually been proposed for $S=\frac{3}{2}$.[28] We finally remark that, as already discussed in Sec. II, our analysis most likely becomes inapplicable in the close vicinity of the points of contact between lines 2, 2′, and 4. Especially, it seems rather unlikely that there should be, for integer $S$, a direct transition between the $XY1$ and the antiferromagnetic phase.

Up to here our analysis has been perturbative, assuming small parameters $\mu_i$. In order to be applicable to the original spin-chain model, Eq. (1.1), a continuous connection to the region of relatively large couplings has to exist. I defer the discussion of this possibility and its consequences to Sec. V and rather extend the present treatment by a brief study of the effects of some symmetry-breaking terms.

First, consider the effect of an external magnetic field $h$ applied along the $z$ direction. Using Eq. (3.6) this leads to a new term in the Hamiltonian,

$$H_h = -\frac{\sqrt{2S}}{\pi} h \int dx \frac{\partial\psi_1}{\partial x} . \qquad (3.43)$$

Away from the critical line 4 (Figs. 2 and 3), apart from the terms bilinear in $\psi_1$, there are only the cos terms (3.31) or (3.32). The effects of the additional gradient term (3.43) has been studied extensively in the context of commensurate-incommensurate transitions.[41–44] Using this analogy, the following conclusions can be drawn.

(1) In the massless phases the field leads to a finite magnetization (a nonzero gradient of $\psi_1$) for arbitrary small fields, i.e., there is a finite susceptibility. In addition the correlation exponents depend continuously on $h$.[43,44]

(2) In the massive regions S and AF a field smaller than some critical value $h_c$ (which is proportional to the gap in the $\psi_1$ excitations) does not change the ground-state properties at all. This corresponds to the commensurate phase. When $|h|$ exceeds $h_c$, there is a transition into a massless region, which corresponds to the incommensurate state. The correlation exponents obey the same relations as in the $h=0$ massless phases, however, one now has $\eta > \frac{1}{4}$ ($XY1$, $S$ integer), $\eta > 1$ ($XY1$, $S$ half-odd-integer), or $\eta_{2S} > 1$ ($XY2$). In general, the exponents are continuous functions of the parameters $D$, $J_z$, and $h$. However, for $h$ close to $h_c$, following Ref. (43), one finds $\eta = \frac{1}{2}$ ($XY1$, $S$ integer), $\eta=2$ ($XY1$, $S$ half-odd-integer), or $\eta_{2S}=2$ ($XY2$). For the special case $S=\frac{1}{2}$, where only the $XY1$ and AF phases exist, this is indeed the observed behavior.[30]

A second interesting question is the effect of an additional staggered interaction, i.e., introducing a factor

$$1+\beta(-1)^i$$

in Eq. (1.1). The most important effect is the alternation of the $S^+S^-$ interaction,[26,45] which in the continuum form gives an additional term in the Hamiltonian,

$$H_{\rm st}=\beta \int dx \cos(\pi x) S^+(x) S^-(x) \ . \tag{3.44}$$

Inserting (3.5), the most relevant terms are those with $n=n'$ in the double sum, giving

$$H_{\rm st}=\frac{\beta}{\pi\alpha}\sum_{n=1}^{2S}\int dx \cos[2\phi_n(x)] \ . \tag{3.45}$$

From this form a number of conclusions can be drawn. It is obvious that in general two $\cos(2\phi_n)$ terms coming from $H_{\rm st}$ can replace one $\cos[2(\phi_m+\phi_n)]$ terms in Eq. (3.29). This leads to the replacement $\mu_1 \to \mu_1+\beta^2$. For integer $S$, and below line 3, where $\mu_1$ is negative, this reduces the amplitude of the mass-generating term, and consequently I expect the $XY1$ region to be increased at the expense of region S. Especially, for finite $\beta$ the massless region should extend further along the line $D=0$. Above line 3 $\mu_1$ is positive, so that the massless region shrinks.

For half-odd-integer $S$ the situation is quite different; it is now possible to form terms like (for $S=\frac{3}{2}$)

$$\cos[2(\phi_1+\phi_2)]\cos(2\phi_3) \approx \cos(2\sqrt{3}\psi_1) \ ,$$

and more generally a term like (3.31), which for $\beta=0$ could only exist for integer $S$. This term thus will generate a mass for $K_1 < 1/S$, e.g., for $\eta \geq \frac{1}{4}$ [cf. Eq. (3.34)]. A staggered interaction thus largely reduces the region of the massless $XY1$ phase for half-odd-integer spin.

The discussion can be extended to the possibility of the spin-Peierls transition,[45,46] i.e., the spontaneous dimerization of a spin chain on an elastic lattice. In order to have a net energy gain the operator in Eq. (3.44) has to have dimension less or equal to unity.[45] This requires $\eta \geq \frac{1}{2}$, and therefore the spin-Peierls transition occurs for half-odd-integer $S$, but not for integer $S$ [cf. Eqs. (3.36) and (3.35)].

## IV. CORRELATION FUNCTIONS IN THE MASSLESS PHASES

In the planar phases $XY1$ and $XY2$ discussed in the preceding two chapters, the massless excitations are only associated with the $\psi_1$ field. The long-distance power laws of different correlation functions are entirely determined by the dynamics of $\psi_1$. As in the massless regions none of the cos operators in the Hamiltonian (3.11) are relevant, the leading asymptotic properties of correlation functions can be correctly obtained from the effective Hamiltonian

$$H=\int dx \left[\frac{\pi v_1 K_1}{2}\chi_1^2(x)+\frac{v_1}{2\pi K_1}\left[\frac{\partial\psi_1}{\partial x}\right]^2\right] , \tag{4.1}$$

where the coefficients $K_1, v_1$ include all renormalization effects from the irrelevant operators, and to lowest order are given by the solution of the scaling equations (3.13)—(3.19) for $l \to \infty$. A standard calculation[18,21,26] then gives the correlation functions at temperature $T$,

$$\langle \exp[i\epsilon\psi_1(x,t)]\exp[-i\epsilon\psi_1(0,0)]\rangle = F(x,t;\epsilon^2K_1/4,T) \tag{4.2}$$

$$\langle \exp[i\epsilon X_1(x,t)]\exp[-i\epsilon X_1(0,0)]\rangle = F(x,t;\epsilon^2/(4K_1),T) \ , \tag{4.3}$$

where $\epsilon$ is an arbitrary parameter, and

$$F(x,t;\xi,T)=\left[\frac{\pi\alpha T}{v_1}\right]^{2\xi}\left[\frac{1}{(\alpha+iv_1t)^2+x^2}\,\frac{x^2-v_1^2t^2}{\sinh[\pi T(x/v_1-t)]\sinh[\pi T(x/v_1+t)]}\right]^{\xi}$$

$$=\alpha^{2\xi}[(\alpha+iv_1t)^2+x^2]^{-\xi} \ \ (T=0) \ . \tag{4.4}$$

A short-distance cutoff $\alpha$ has been introduced as in Eqs. (2.4) and (2.5), and $\xi$ is the scaling dimension of the operators in Eqs. (4.2) and (4.3).[26,27]

These results can now be used to calculate the retarded correlation functions which determine, e.g., scattering cross sections. For integer $S$ in the $XY1$ phase only the transverse correlations are massless and determined by the operator

$$S^+(x)=c_\perp e^{-iX_1(x)/\sqrt{2S}}\ , \tag{4.5}$$

where $c_\perp$ is the proportionality factor omitted in Eq. (3.33), which is determined by the contribution from the relative $\psi$'s and cannot be obtained by the methods used here. The transverse correlation function now is

$$\begin{aligned}S_\perp(x,t)&=-i\theta(t)\langle[S^+(x,t),S^-(0,0)]\rangle\\&=2c_\perp^2\theta(t)\mathrm{Im}[F(x,t;\eta/2,T)]\ ,\end{aligned} \tag{4.6}$$

with $\eta\le\frac{1}{4}$ [cf. Eq. (3.35)]. I now make the approximation

$$F(x,t;\xi,T)\approx\left[\frac{\pi\alpha T}{v_1}\right]^{2\xi}\{\sinh[\pi T(x-v_1t+i\alpha)/v_1]\sinh[\pi T(x+v_1t-i\alpha)/v_1]\}^{-\xi}\ , \tag{4.7}$$

which introduces only errors of order $\alpha/x$, $\alpha/(v_1t)$, has the correct behavior both for large and small arguments, and moreover has the correct singularity structure at the origin. Then the Fourier transform of Eq. (4.6) can be calculated as

$$\begin{aligned}S_\perp(q,\omega)&=\int\int dx\,dt\,e^{i(\omega t-qx)}S_\perp(x,t)\\&=-c_\perp^2\sin(\pi\eta/2)\frac{\alpha^2}{v_1}\left[\frac{2\pi\alpha T}{v_1}\right]^{\eta-2}\\&\quad\times B(\eta/4-i(\omega+v_1q)/(4\pi T),1-\eta/2)B(\eta/4-i(\omega-v_1q)/(4\pi T),1-\eta/2)\ ,\end{aligned} \tag{4.8}$$

where $B(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$. To obtain this result, I have used the integral[47]

$$\int_0^\infty ds[\sinh(\pi Ts)]^{-\eta/2}e^{isz}=\frac{2^{\eta/2-1}}{\pi T}B(\eta/4-iz/(2\pi T),1-\eta/2)$$

and all corrections of order $\alpha q$, $\alpha\omega/v_1$, $\alpha T/v_1$ are neglected. An accurate calculation of these corrections is in any case beyond the scope of the present approach. However, Eq. (4.8) is uniformly valid irrespective of the ratios $\omega/T$, $v_{1q}/T$. For $\omega$, $v_1q\gg T$, and in particular for $T=0$, Eq. (4.8) becomes

$$S_\perp(q,\omega)=-\frac{c_\perp^2}{v_1}\sin(\pi\eta/2)\Gamma^2(1-\eta/2)\left[\frac{\alpha}{v_1}\right]^\eta[v_1^2q^2-(\omega+i\delta)^2]^{\eta/2-1}\ . \tag{4.9}$$

At any nonzero temperature the form (4.8) has a discrete series of poles in the complex $\omega$ plane. In contrast, the approximate form of $S_\perp(q,\omega)$ proposed by TL (Ref. 15) has a cut in the complex $\omega$ plane. In principle, Eq. (4.8) provides a more suitable form for fitting experimental results than the form proposed by TL, but in practice the differences may well be minor.

For the other massless phases a completely analogous calculation can be done. For half-odd-integer $S$ one finds in the $XY1$ phase exactly the same form for $S_\perp$ as Eq. (4.8), but now with the restriction $0<\eta\le1$. For the longitudinal correlation function a similar result holds; from Eqs. (3.39) and (4.3) one obtains

$$\begin{aligned}S_\parallel(q,\omega)&=-i\int\int dx\,dt\,e^{i(\omega t-qx)}\theta(t)\langle[S^z(x,t),S^z(0,0)]\rangle\\&=-c_\parallel^2\sin(\pi\eta_z/2)\frac{\alpha^2}{v_1}\left[\frac{2\pi\alpha T}{v_1}\right]^{\eta_z-2}B(\eta_z/4-i(\omega+v_1q')/(4\pi T),1-\eta_z/2)\\&\quad\times B(\eta_z/4-i(\omega-v_1q')/(4\pi T),1-\eta_z/2)+\frac{c_\parallel^2\alpha^2\pi}{v_1(1-\eta_z/2)}\ ,\end{aligned} \tag{4.10}$$

where $q'=q-\pi$, $\eta_z=1/\eta\ge1$, and the last term comes from a careful treatment of the short-range cutoff and is important for $\eta_z\approx2$ in order to cancel a spurious divergence. Note that only the alternating part of the correlation function has been taken into account. In general one has $\eta_z>\eta$, and therefore $S_\parallel$ is less divergent than $S_\perp$. Only on the transition line to the antiferromagnetic region both functions have equal singularities. One should also note that a real antiferromagnet is correctly described by the Hamiltonian (1.1) only after the unitary transformation $S_i^x\to-S_i^x,S_i^y\to-S_i^y$ on every second site. Singular behavior then appears in $S_\perp$ for $q\approx\pi$, i.e., one has to make the replacement $q\to q-\pi$ in Eq. (4.8). Analogous results can also be obtained for the $XY2$ phase.

The effective Hamiltonian (4.1) only describes correctly the asymptotic behavior of correlation functions. Corrections to the asymptotic power laws, which arise from the finite strength of various perturbations (i.e., $\mu_1,\mu_2,\mu_3$) at finite-length scale are not present. These corrections may be of some importance on the transition lines, where they are logarithmic.

## V. DISCUSSION AND CONCLUSION

In the present paper I have derived a description of the spin-$S$ quantum spin chain in terms of $2S$ appropriately coupled spin-$\frac{1}{2}$ chains, and have analyzed the properties of the resulting model for weak coupling between the individual spin-$\frac{1}{2}$ systems. Specifically, in Sec. II the spin-1 problem, previously treated by the same method by Timonen and Luther,[15] has been reanalyzed. I find good qualitative agreement with numerical results, especially concerning the correlation properties of different planar (massless) phases and the transitions to massive regimes. The analysis has subsequently been generalized to arbitrary $S$, where higher-order terms in the coupling are important to treat the transition between the planar phase $XY1$ and an adjacent massive phase. The most important feature of the model is that its properties do not depend dramatically on the value of the spin quantum number $S$, but only on whether $S$ is integer or half-odd-integer. For example, on the border of the $XY1$ phase one has $\eta=\frac{1}{4}$ for all integer $S$, whereas for half-odd-integer $S$ the result is $\eta=1$. This remarkable fact is in obvious agreement with Haldane's conclusions.[3,8] In Sec. IV explicit expressions for the spin-spin correlation functions in the different massless phases have been derived, which are valid uniformly for small energy, wave number, and temperature.

In the present treatment the low-energy properties of the model are essentially determined by the excitation of the average field $\psi_1$. In addition, however, there are also the usually massive relative excitations associated with $\psi_k$ fields with $k\geq 2$. Even though these excitations have not been studied here in detail, one can tentatively associate them with states where locally the $2S$ individual $\sigma_n$'s do not align to form a multiplet with maximum spin $S$, but rather form a state with lower spin. This interpretation has been proposed previously for $S=1$.[14]

Up to this point the analysis has been perturbative assuming small parameters $\mu_i$. If the analysis is to be valid for the finite values appropriate for the original spin-chain model, Eq. (1.1), the scaling trajectories of the renormalization equations (3.13)—(3.19) have to continue up to fairly strong couplings without going to some intermediate fixed point. Even though it is not easy to see which physical situation such a fixed point might represent, its existence cannot be *a priori* ruled out. In this context it is interesting to note that a Hamiltonian identical to (3.4) can be obtained coupling $2S$ spin-$\frac{1}{2}$ systems in a different way. Consider

$$H=\sum_{n=1}^{2S} H_n - W\sum_{i=1}^{N} \mathbf{S}_i^2 \, , \tag{5.1}$$

where $H_n$ is one of the individual spin-$\frac{1}{2}$ Hamiltonians and $\mathbf{s}_i$ is the total spin at site $i$ [cf. Eq. (3.1)],

$$\mathbf{S}_i=\sum_{n=1}^{2S} \sigma_{n,i} \; .$$

In the continuum limit one obtains the form (3.4), with $\mu_1=\mu_2=\mu_3=-W$. On the other hand, it is obvious that for large $W$ the low-energy properties of (5.1) are dominated by states with $\mathbf{S}_i^2=S(S+1)$, and therefore are identical to those of a spin-$S$ chain. Under the above hypothesis on the scaling trajectories one then is led again to the conclusion that the Hamiltonian (3.4) represents correctly the spin-$S$ chain.

Independent evidence that there is indeed a continuous connection between weak and strong coupling exists for $S=1$ (from a comparison with numerical results, see Sec. II) and for $S=\frac{1}{2}$ (Refs. 26, 38, and 30) where an exact solution is available.[5-7] If one assumes the existence of such a connection for all values of $S$ one finds radically different properties for the spin-chain according to whether the spin quantum number $S$ is integer or half-odd-integer.

(1) For integer $S$ one has a (massless) planar phase $XY1$ where only the transverse spin correlation functions show power-law decay, whereas the alternating longitudinal ($z$) correlations decay exponentially. Thus this phase can in no way include the isotropic antiferromagnetic point, where transverse and longitudinal correlations have to be identical. The massive phase adjacent to $XY1$ is nonmagnetic (singlet type). On the transition line to the singlet phase one has the universal value $\eta=\frac{1}{4}$ for all $S$.

(2) For half-odd-integer $S$ the $XY1$ phase has power-law decay both in the transverse and in the longitudinal correlations, with $\eta=\eta_z=1$ along the transition line to the adjacent antiferromagnetic state, and in general $\eta=1/\eta_z$. As at the isotropic antiferromagnetic point one has to have $\eta=\eta_z$, it seems almost certain that this point lies on the transition line (as is known to be true for $S=\frac{1}{2}$).

(3) In addition to the $XY1$ phase there is another planar phase $XY2$, both for integer and half-odd-integer $S$. The in-plane massless modes appear in correlation functions of the operator $(S^+)^{2S}$, but not for $S^+$ itself, and one has the scaling relation $\eta_z=1/\eta_{2S}$. Consequently, there has to be a phase transition where $S^+$ correlations change from algebraic to exponential.

(4) From the present analysis, both the singlet-antiferromagnetic ($S$ integer) and the $XY1$-$XY2$ transitions are of Ising type. This is quite natural for the AF-S transition, but may seem somewhat surprising for the $XY1$-$XY2$ transition. Nevertheless, a similar situation has recently been found in a two-dimensional classical $XY$ model.[31]

(5) For integer $S$ within the singlet phase there exists a special line along which the mass-generating operator vanishes, so that one has $XY1$-type correlations, but with $\eta>\frac{1}{4}$.

(6) The effect of a staggered interaction is quite different according to whether $S$ is integer or half-odd-integer. For half-odd-integer $S$ staggering generates an excitation gap in a large portion of the $XY1$ phase (for $\eta\geq\frac{1}{4}$), and thus greatly decreases the $XY1$ region. In addition, there is an instability against spontaneous dimerization for $\eta\geq\frac{1}{2}$ (spin-Peierls instability). On the other hand, for integer $S$ staggering increases the region of existence of the $XY1$ phase in certain directions, especially along the line $D=0$, and there is no spin-Peierls transition. These points are in agreement with semiclassical arguments.[4]

(7) A magnetic field along the $z$ direction leads to a transition from the massive (S or AF) to a massless phase at a nonzero critical field. The critical field is of the order of the excitation gap in the massive phase. In the massless phase correlations are similar to $XY1$, but now with $\eta > \frac{1}{4}$ ($S$ integer) or $\eta > 1$ ($S$ half-odd-integer). Because now $\eta > \frac{1}{4}$, a spin-Peierls instability (which requires $\eta \geq \frac{1}{2}$) is now possible also for integer $S$. This instability occurs, however, with a field-dependent, and in general incommensurate, wave number, due to the finite gradient of $\psi_1$.

(8) In the massless phases, the asymptotic form of spin-spin correlation functions for low temperature, frequency, and wave number is given by the formulas derived in Sec. IV, where fully renormalized parameters have to be used.

The above points 1 and 2 are in complete agreement with Haldane's predictions,[3,8] even though derived by a quite different method. Nevertheless, the underlying physical mechanism is identical; from Eq. (3.33), $X_1/\sqrt{2S}$ can be interpreted as the azimuthal angle in the $xy$ plane. Then, following arguments given in Refs. 14 and 40, the operator [cf. Eq. (3.31)] $\cos[2\sqrt{2S}\,\psi_1(x,t)]$ generates, for integer $S$, a "phase slip" of $\pm 2\pi$ at point $x$ and time $t$. In the S region this operator becomes strong, indicating a large number of phase slips, which leads to exponential decay of the transverse correlations. For half-odd-integer $S$, the analysis of Sec. II shows that only $\cos(4\sqrt{2S}\,\psi_1)$ exists, which generates $\pm 4\pi$ phase slips and is much less relevant. The crucial point is that the $\pm 2\pi$ phase slips do not appear for half-odd-integer $S$ because, as argued by Haldane,[3] the corresponding states have time-reversal symmetry different from the ground state and therefore the matrix element of the Hamiltonian (which is time-reversal invariant) between the two types of states is zero (see also Ref. 48).

The present method is perturbative, assuming small $\mu_1,\mu_2,\mu_3$, and therefore cannot make precise predictions on the detailed shape of the phase diagram of model (1.1) or on the dependence of correlation exponents on the parameters. These problems need numerical calculations.[9–11] On the other hand, if the above hypothesis on the scaling trajectories is correct, scaling relations between different exponents [(3.34) and (3.35) for $XY1$, $S$ integer; (3.40) and (3.41) for $XY1$, $S$ half-odd-integer; (3.27) for $XY2$] and the values of the exponents on the critical lines are given correctly. Similarly, the global topology of the phase diagrams (excluding possibly the vicinity of the crossing between lines 2 and 4) is expected to hold for the spin-chain model, as verified explicitly for $S=1$.

At first sight the present result that correlation exponents are qualitatively independent of the value of the spin quantum number $S$, and in particular that $\eta=1$ for the isotropic antiferromagnet for all half-odd-integer $S$, seems to be in contradiction with spin-wave theory,[49] which predicts $\eta \approx 1/S$. One should, however, notice that spin-wave theory runs into problems in the vicinity of the isotropic antiferromagnet. A possible scenario, strongly suggested by recent expansions up to order $1/S^2$,[50] is that in most of the massless region $\eta$ is indeed of order $1/S$. The critical values (1 or $\frac{1}{4}$) then are approached only in the close vicinity, of width decreasing with increasing $S$, of the critical lines. In addition, or alternatively, the length scale beyond which the asymptotic power laws are valid (the parameter $\alpha$ of Sec. IV) might diverge for $S\to\infty$, and at shorter distances the spin-wave results would be valid.

The present calculation predicts $\eta=1$ for all half-odd-integer-$S$ isotropic antiferromagnets. On the other hand, there is a class of SU(2)-symmetric Wess-Zumino field theories for which one finds the $S$-dependent value $\eta=3/(2S+2)$.[4] These field theories have been identified[37] as the continuum representation of the exactly solvable (massless) spin-$S$ antiferromagnet,[35,36] with a Hamiltonian which is a $2S$-degree polynomial of exchange operators and therefore cannot be represented by Eq. (1.1). Thus there seems to be two different universality classes for the isotropic antiferromagnet, one with $\eta=1$, the other with $\eta=3/(2S+2)$. A possible way out is suggested by the case $S=1$ (see the discussion at the end of Sec. II), where the exactly solvable model seems to represent a special multicritical point in the phase diagram and does not have the same properties as the standard quadratic-exchange antiferromagnet. Generalized to arbitrary $S$ this would imply that the quadratic-exchange antiferromagnet has (for half-odd-integer $S$) $\eta=1$, whereas the exactly solvable models [$\eta=3/(2S+2)$] represent special multicritical points. An alternative, more complicated scenario would be that the SU(2)-symmetric Wess-Zumino models represent multicritical points for integer $S$ only, whereas for half-odd-integer $S$ they would represent indeed the isotropic antiferromagnetic point of model (1.1). Then $\eta$ has to be discontinuous at $J_z=-1$, $D=0$. This possibility certainly cannot be ruled out by the present calculation; the planar anisotropic state $|J_z|<1$ has U(1) symmetry, and therefore the conformal anomaly has to be $c=1$.[51] This is the typical situation of a single free-boson field, $\psi_1$ in our case. On the other hand, for a SU(2)-symmetric point $c>1$, characteristic of the Wess-Zumino models, is possible,[37,51] and this would lead to a discontinuous exponent $\eta$ in the limit $J_z\to -1$, where the symmetry changes from U(1) to SU(2). The present result $\eta=1$ then would be the limiting value for $J_z=-1+0^+$, whereas at $J_z=-1$ one would have $\eta=3/(2+2S)$. This question certainly deserves more (numerical) investigation. One may further wonder whether different universality classes also exist for the SU($n$) generalization of spin chains. This possibility then might have consequences for the determination of critical properties of the integer quantum Hall effect.[4]

In conclusion, a new method for treating the properties of the spin-$S$ quantum spin chain has been developed, based on a perturbative treatment of $2S$ coupled spin-$\frac{1}{2}$ chains. Due to uncontrollable (by the present method) lattice-renormalization effects no precise predictions on detailed phase diagrams and similar properties can be made. However, under the additional plausible assumption that the strong-coupling regime reached under renormalization represents correctly the physical properties of the spin-$S$ system, a number of predictions concerning possible phases, the topology of phase diagrams, scaling relations between correlation exponents, and critical properties can be made. Moreover, the effect of some symmetry-breaking perturbations like a magnetic field or staggered interactions has been studied.

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