One-dimensional Ising model in a variety of random fields

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We study one-dimensional Ising models in the presence of various random-field (RF) distributions. The distribution which determines the average free energy and other thermodynamic properties is found to be a devil's staircase for discrete RF distributions and continuous for nondiscrete RF distributions. Thus, any experiment that can be done on these systems will not show this devil's-staircase behavior, due to the natural broadening of RF distributions in real physical systems. Experiments will show some reminiscence of the largest steps and this is associated with the broadening of the peaks of the RF distributions.

Quenched disorder in the form of impurities, dislocations, and gels in binary mixtures can affect drastically the behavior of condensed matter systems and has therefore recently been a subject of intense research efforts. Some of the most challenging systems with such a disorder are those described by the random-field Ising model (RFIM). On the theoretical side, since the pioneering work of Imry and Ma,[1] the value of the lower critical dimension d_l (below which long-range order does not exist) has been a matter of controversy. Some works[2] predicted $d_l=2$ (which we tend now to believe), whereas others[2] gave $d_l=3$. An additional complication in the RFIM is that metastable states play an important role and cannot be ignored.[3] The RFIM can be realized[4] experimentally by the use of a diluted antiferromagnet in a uniform magnetic field. Various experimental techniques, such as neutron scattering[5] and linear birefringence,[6] are currently used to probe RFIM systems. Although the RFIM in one dimension, $d=1$, is clearly below its lower critical dimension, we have studied it since it has very complex behavior which may also be present in higher dimensionalities.

Recently, it has been shown by Bruinsma and Aeppli[7] that for a RFIM with a binary random-field distribution $\pm h_0$, and for high enough temperatures, the integrated probability distribution for various observables is a devil's staircase (we recall that a devil's staircase is a stepwise function, such that between any two steps, there exists another step). They also suggested that this theoretical result could be tested experimentally with the use of nuclear magnetic resonance or Mössbauer spectroscopy. However, by performing a detailed numerical study of the $d=1$ RFIM in the presence of a *variety of random fields,* we conclude that the devil's staircase is a special characteristic of *discrete distributions*. For continuous distributions, we do not find these nonanalyticities. Therefore, we do not expect this devil's staircase to be observed experimentally as suggested in Ref. 7. This is because there is always a broadening of the random fields in real physical systems and their distribution is never discrete.

In order to make our results more comprehensive we will repeat some of the general formulation that had already been given by Fan and McCoy.[8] The Hamiltonian of the $d=1$ RFIM is

$$-\beta\mathscr{H}=J\sum_{i=1}^{N}S_iS_{i+1}+\sum_{i=1}^{N}h_iS_i\ , \tag{1}$$

where $S_i=\pm 1$ is the Ising spin, $\beta=1/k_BT$, J is the nearest-neighbor interaction, and h_i is the random field at site i. We assume that all the h_i's are independent random variables with the same probability distribution function $\rho(h)$. The aim, as always, is to calculate the partition function, $Z=\mathrm{Tr}_{\{S\}}e^{-\beta\mathscr{H}}$ in the thermodynamic limit $N\to\infty$. A convenient recursive way[8] is as follows.

Define Z_n^+ (Z_n^-) as the partition function of a RFIM chain of n spins with the last spin $S_n=+1$ ($S_n=-1$). Then

$$Z_n=Z_n^++Z_n^-\ , \tag{2}$$

where

$$Z_n^+=\mathrm{Tr}_{\{S\}}\exp\left[JS_{n-1}+h_n+\sum_{i=1}^{n-1}(JS_iS_{i+1}+h_iS_i)\right]\ , \tag{3a}$$

$$Z_n^-=\mathrm{Tr}_{\{S\}}\exp\left[-JS_{n-1}-h_n+\sum_{i=1}^{n-1}(JS_iS_{i+1}+h_iS_i)\right]\ , \tag{3b}$$

so that we have

$$Z_n^+=e^{J+h_n}Z_{n-1}^++e^{-J+h_n}Z_{n-1}^-\ , \tag{4a}$$

$$Z_n^-=e^{-J-h_n}Z_{n-1}^++e^{J-h_n}Z_{n-1}^-\ , \tag{4b}$$

with the initial condition $Z_1^+=e^{h_1}$, $Z_1^-=e^{-h_1}$. If z_n is defined as the ratio between Z_n^- and Z_n^+, one can obtain from Eqs. (4a) and (4b)

$$z_n=\frac{e^{-J-h_n}+z_{n-1}e^{J-h_n}}{e^{J+h_n}+z_{n-1}e^{-J+h_n}}\ . \tag{5}$$

Furstenberg[9] has shown that as $n\to\infty$, the random variable z_n will approach a limiting fixed integrated distribution $P(z)$, which is independent of the initial condition and is unique. This fixed distribution is the solution of the integral equation[8,9]

$$P(z)=\int_{-\infty}^{+\infty} dh\,\rho(h)\int_0^{\infty} dz'\frac{dP(z')}{dz'}\times\Theta\left[z-\frac{e^{-J-h}+z'e^{J-h}}{e^{J+h}+z'e^{-J+h}}\right], \quad (6)$$

where $\Theta(z)$ is the Heaviside step function. Integrating over z', Eq. (6) reduces to

$$P(z)=\int_{-\infty}^{+\infty} dh\,\rho(h)P(f(h,z)), \quad (7)$$

where

$$f(h,z)=e^{2h}\left[\frac{ze^{J}-e^{-J}}{e^{J}-ze^{-J}}\right] \quad (8)$$

is the inverse transform of Eq. (5), namely $z_{n-1}=f(h_n,z_n)$. Knowing $P(z)$ we can evaluate the free energy, the magnetization and other thermodynamic properties. The free energy per site of the RFIM is

$$\mathcal{F}=-\frac{1}{\beta}\lim_{N\to\infty}\frac{1}{N}\ln(Z_N^{+}+Z_N^{-})=-\frac{1}{\beta}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\ln\left[\frac{Z_{n+1}^{+}+Z_{n+1}^{-}}{Z_n^{+}+Z_n^{-}}\right]. \quad (9)$$

Using Eqs. (4), (5), and (9), the free energy $\mathcal{F}$ can be expressed as an average over z and h,

$$\mathcal{F}=-\frac{1}{\beta}\int_0^{\infty} dz\frac{dP(z)}{dz}\int_{-\infty}^{+\infty} dh\,\rho(h)\ln\left[\frac{2\cosh(J+h)+2z\cosh(J-h)}{1+z}\right]. \quad (10)$$

Note that although $\mathcal{F}$ is unique and well defined, the choice of the integrand in Eq. (10) is not. (For more details see Ref. 8.) Similarly, the local magnetization $m_i=\langle S_i\rangle$ is[8,10]

$$m_i=\frac{z_1z_2-e^{-2h_i}}{z_1z_2+e^{-2h_i}}, \quad (11)$$

where z_1,z_2 are two independent random variables with probability distribution $P(z_1)$ and $P(z_2)$, respectively. The probability density function of m_i, dQ/dm will be a convolution between $dP(z_1)/dz_1$, $dP(z_2)/dz_2$, and $\rho(h)$ subject to the constraint Eq. (11). The distribution dQ/dm is not always a δ function at $m=0$, but clearly $\langle m\rangle_{\text{av}}\equiv\int(m\,dQ/dm)dm=0$ for $T>0$, since there is no long-range order in $d=1$.

In this work, we concentrated on finding the solution $P(z)$ of Eq. (7) for general random-field distributions $\rho(h)$. This solution was previously found only for $T=0$,[11] or for very special $\rho(h)$.[7,12] For general $\rho(h)$, one has to rely on numerical solutions of Eq. (7). One way of finding it, is to iterate directly Eq. (7) from some initial guess $P_0(z)$ until convergence is achieved. Another way, which is simpler in our opinion, is to start with the recursion, Eq. (5), and to iterate it until very long chains are produced. (Most of our results are for chains with $N=10^5$ sites.) As one iterates along the chain, the various values of z_n are accumulated and used for the construction of $P(z)$. Numerical stability is checked by comparing results which are obtained with different chain lengths: N, $N/10$, and $2N$. The sequence from $n=1$ up to $n\approx N/10$ is always disregarded and not used in determining $P(z)$ (although this has only a minor effect). The integrated distribution $P(z)$ is plotted for various values of temperature and field strength. For more complicated $\rho(h)$, where the field strength is not a single parameter, results are obtained by varying only one combination of the field-strength parameters at a time. In the following, we will describe our results for several distributions, $\rho(h)$.

(i) As a check, we started with the binary $\pm h_0$ distribution, for which there are already known results.[7,10,11] In this case

$$\rho(h)=\tfrac{1}{2}[\delta(h-h_0)+\delta(h+h_0)]. \quad (12)$$

In Fig. 1, $P(z)$ is plotted from our data and as we lower

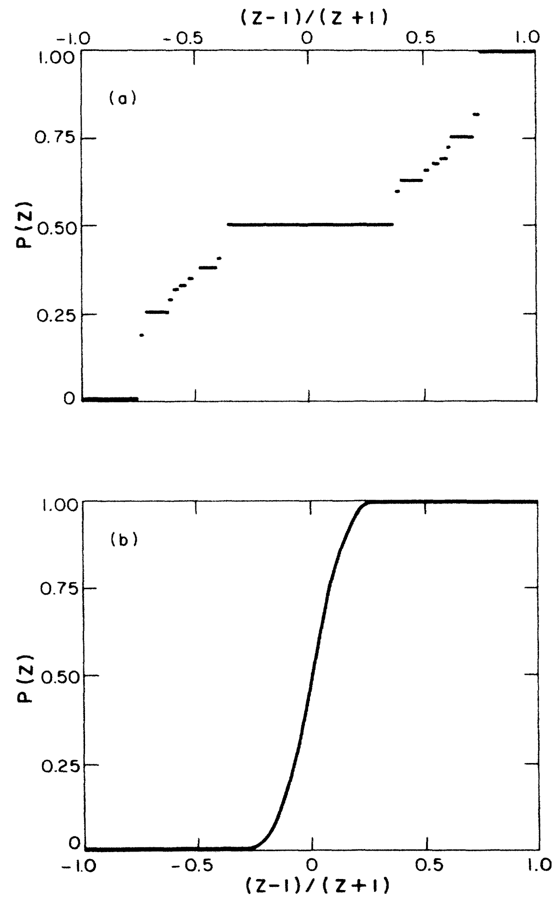

FIG. 1. The integrated distribution function $P(z)$ as a function of $(z-1)/(z+1)$ for a binary random field distribution, $\rho(h)$, from Eq. (12), with $J^{-1}=1.0$ and $h_0/J=1.5$ in (a) and $h_0/J=0.1$ in (b). The variable z is the $n\to\infty$ limit of z_n, Eq. (5).

the field strength h_0, keeping J constant, a transition from the devil's staircase to a continuous curve occurs. This transition is not a phase transition, but rather observables have an essential singularity on the transition line. For this special $\rho(h)$, the integration over the field h can be done in Eq. (7) and it reduces to the functional equation[7,10]

$$P(z)=\tfrac{1}{2}[P(f(h_0,z))+P(f(-h_0,z))]\ , \tag{13}$$

where $f(h,z)$ is given in Eq. (8). The solution of Eq. (13) can be obtained iteratively[7,13] when it is a devil's staircase [there is a crossover line in the (J,h_0) plane].

(ii) We also checked this devil's-staircase behavior for two others, more complicated discrete distributions,

$$\rho(h)=\frac{p}{2}[\delta(h-h_0)+\delta(h+h_0)]+(1-p)\delta(h) \tag{14a}$$

and

$$\rho(h)=\frac{p}{2}[\delta(h-h_1)+\delta(h+h_1)] + \frac{1-p}{2}[\delta(h-h_2)+\delta(h+h_2)]\ . \tag{14b}$$

Figures 2 and 3 correspond to the three-peak distribution, Eq. (14a) and to the four-peak distribution, Eq. (14b), respectively. In Fig. 2(b) the interval $[-0.1,0.1]$ is magnified 10 times in order to observe the self-similarity of the devil's staircase. Again, the transition from the nonanalytic to the continuous behavior occurs, as the field strength is reduced or as the temperature is reduced, similar to what is seen in Fig. 1. It is possible to generalize the solution that was given in Ref. 7 for Eq. (13) to these more complex distributions, Eqs. (14a) and (14b). The integration over h can be done and the resulting functional equation can be solved even though the enumeration of various steps is more tedious.

(iii) As examples of continuous distributions with a single central peak, we have chosen Gaussian and uniform distributions:

$$\rho(h)=\left[\frac{1}{2\pi h_0^2}\right]^{1/2} e^{-h^2/2h_0^2}\ , \tag{15a}$$

and

$$\rho(h)=\frac{1}{2h_0}[\Theta(h+h_0)-\Theta(h-h_0)]\ , \tag{15b}$$

where $\Theta(z)$ is the Heaviside step function. To our knowledge, it is not possible to find $P(z)$ analytically for these distributions. The numerical solution for typical values of J and h_0 is plotted in Fig. 4. From this figure and from additional scans that we did in the (J,h_0) plane, we conclude that $P(z)$ is always continuous.

(iv) In order to compare the binary distribution, Eq. (12), with more realistic distributions that have some width, we have also chosen a continuous distribution which has two very sharp Gaussian-type peaks centered at $\pm h_0$:

$$\rho(h)=\frac{1}{2}\left[\frac{1}{2\pi\sigma^2}\right]^{1/2}(e^{-(h-h_0)^2/2\sigma^2}+e^{-(h+h_0)^2/2\sigma^2})\ . \tag{16}$$

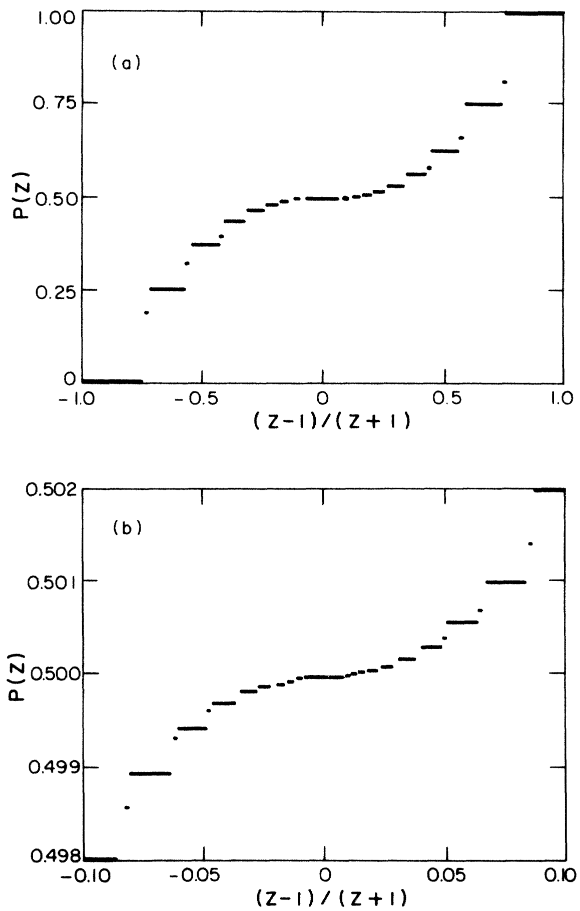

FIG. 2. Same as in Fig. 1 for a discrete three-peak distribution, Eq. (14a), with $J^{-1}=1.0$, $h_0/J=3.0$, and $p=0.5$. The horizontal scale of (a) is blown up 10 times in (b) and smaller steps are seen.

For $\sigma \ll h_0$, the two peaks are very narrow and in Fig. 5(a) [5(b)], we display $P(z)$ for $\sigma/h_0=0.1(0.02)$. Although $P(z)$ is smooth with no singularities, a reminiscence of the first three (five) steps can be seen in Fig. 5(a) [5(b)]. However for all values of T, σ, and h_0 that we

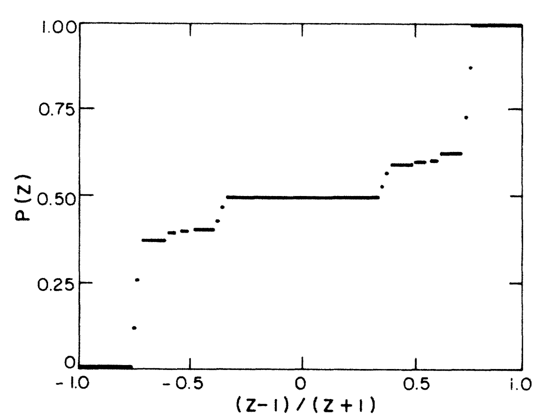

FIG. 3. Same as in Fig. 1 for a discrete four-peak distribution, Eq. (14b), with $J^{-1}=1.0$, $h_1/J=2h_2/J=3.0$, and $p=0.5$.

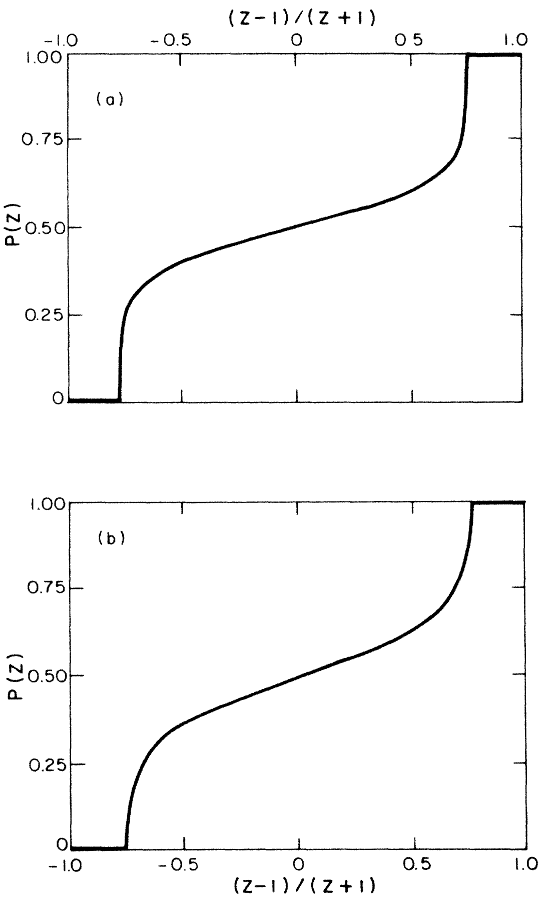

FIG. 4. Same as in Fig. 1 for (a) a Gaussian distribution, Eq. (15a) and (b) a uniform distribution, Eq. (15b). For both cases $J^{-1}=1.0$ and $h_0/J=3.0$.

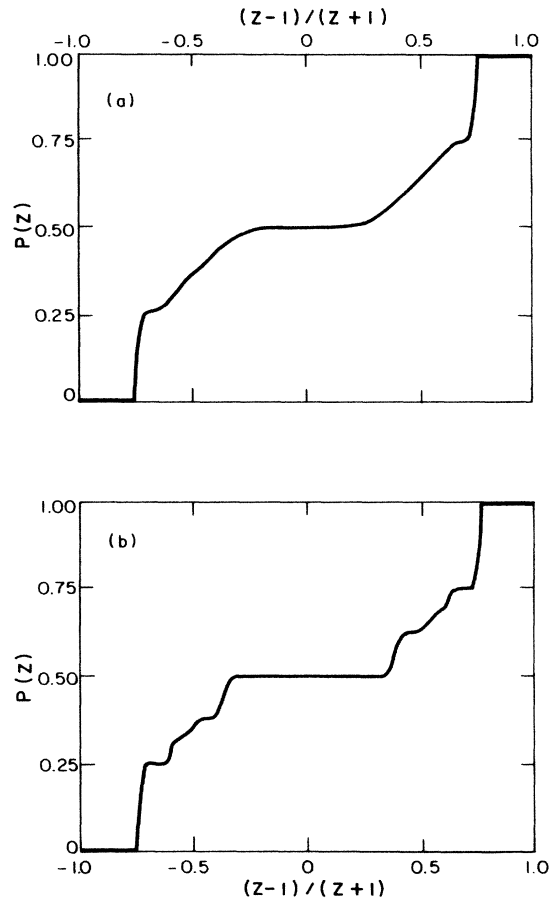

FIG. 5. Same as in Fig. 1 for a two-peak Gaussian distribution, Eq. (16) with $J^{-1}=1.0$, $h_0/J=1.5$, and (a) $\sigma/h_0=0.1$ and (b) $\sigma/h_0=0.02$.

have checked, we never found the full devil's-staircase behavior, namely, steps on all length scales. As we decrease even more the ratio between σ and h_0, a finite number of steps of smaller length scale start to appear.

To summarize, we have investigated the RFIM in the presence of various random-field distributions, Eqs. (12) and (14)–(16), and we found numerically the fixed distribution $P(z)$ from which the free energy and other thermodynamic properties can be determined. From our results it is apparent that the devil's-staircase behavior is a characteristic only of discrete distributions. If experiments are performed on RFIM chains (note that the Fishman-Aharony argument[4] does not hold in $d=1$), reminiscence of the first couple of steps could be seen, but not the full devil's staircase. It will be very interesting to compare our predictions with those of real experiments, where by measuring the various plateaus in the distribution of the local magnetization, one can estimate the broadening of the random fields.

Note added. The difference between a discrete random-field distribution and a continuous one was discussed within a linear response theory by G. Aeppli and R. Bruinsma, Phys. Lett. **97A**, 117 (1983). I thank G. Aeppli for bringing this paper to my attention.

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