Relativistic dynamics of sine-Gordon solitons trapped in confining potentials

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A collective-coordinate method is used to study theoretically and numerically the stability and the dynamics of a sine-Gordon soliton trapped in a confining potential. The example of a harmonic well is emphasized. A remarkably simple approximated solution is found and checked by numerical simulations. The perturbed soliton is stable up to high (relativistic) energies and its profile has the following kinklike dependence on space and time:

$$U(x,t)=4\tan^{-1}\left\{\exp\left[\left[\frac{1+\frac{1}{4}V(y)}{[1-(y')^2]}\right]^{1/2}[x-y(t)]\right]\right\},$$

where $V(y)$ is the potential energy of the particlelike kink at $x=y(t)$. When an external driving force is present, resonances are pointed out and their nonlinear nature is stressed.

I. INTRODUCTION

The study of the dynamics of a sine-Gordon (SG) soliton under the influence of an external driving force has been developed within the last decade.[1-7] Using varied theoretical tools, authors first claimed that the SG soliton behaves as a rigid Newtonian particle, i.e., its acceleration is proportional to the amplitude of the applied field.[2] Then a controversy, now widely settled, developed about the particular case of a uniform external driving field:[3-6]

$$U_{tt}-U_{xx}+\sin U=\chi,\ \ \chi\ll 1\ . \tag{1}$$

It has been shown that, on a short time scale [of order of the characteristic phonon period: here, in the dimensionless units used in (1), this period is 2π], the soliton does not react upon the field χ as a particle, but is dressed with the linear (phonon) waves excited in the original vacuum state $U(x,0)\equiv 0$ by the field χ.[4,5] As a consequence, a complex process of interactions between the soliton's particlelike and wavelike aspects occurs, leading to anomalous (with respect to the reference Newtonian trajectory) soliton dynamics *within this short time scale.*

Actually, this anomalous effect is a transient process happening mainly because the initial soliton profile,

$$U(x,0)=4\tan^{-1}\exp(-\sigma x) \tag{2}$$

($\sigma=-1$ for a kink; $\sigma=1$ for an antikink), is not a solution of Eq. (1). As noted in Ref. 6, in order to avoid this transient effect, one may add the so-called vacuum state $U_{\rm vac}=\sin^{-1}\chi$ to the kink (2). The resulting solitary wave $4\tan^{-1}\exp(-\sigma x)+\sin^{-1}\chi$ connects two states of minimum effective energy, and therefore almost no phonon waves are excited. The dynamics of this "renormalized" kink wave is fairly Newtonian.

Another way to avoid the presence of phonon waves is to change the expression of the driving force in (1). Assuming a potential $V(x)$, whose characteristic length L satisfies the usual pointlike assumption in classical mechanics when applied to the SG kink (2),

$$L\gg 1\ , \tag{3}$$

where 1 is the soliton dimensionless width, we obtain, in first approximation with respect to $1/L$, the Newtonian equation of motion of the soliton position $y(t)$:[7]

$$8y''=-\left.\frac{\partial V}{\partial x}\right|_{x=y}, \tag{4}$$

provided the potential $V(x)$ enters the perturbed SG equation in the following way:

$$U_{tt}-U_{xx}+[1+\tfrac{1}{4}V(x)]\sin U=0\ . \tag{5}$$

Note that the soliton rest mass equals 8 in dimensionless units.

The advantage of Eq. (5) in comparison with Eq. (1) lies in the fact that the term $\frac{1}{4}V(x)\sin U$ vanishes on the soliton wings $U\sim 0$ [mod(2π)], so that the main source of phonon waves disappears.

A simple case of (5) is the harmonic problem:

$$V(x)=\kappa x^2,\ \ \kappa\ll 1\ . \tag{6}$$

A good test for studying the resulting soliton dynamics is the Fourier spectrum of the soliton position variable $y(t)$, when the range of t allows a great number of oscillations of the kink inside the potential (6). We find out that this Fourier spectrum is sharply peaked at a frequency Ω_S very close (to a few percent) to the Newtonian frequency

$$\Omega_N=\tfrac{1}{2}\sqrt{\kappa} \tag{7}$$

[see (4)], as long as the maximum amplitude $y_{\max}$ of the soliton oscillations remains small enough to allow perturbative regimes (see Fig. 1):

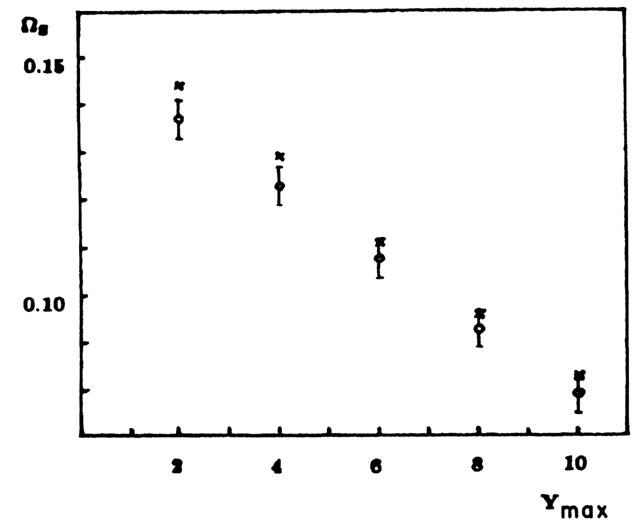

FIG. 1. Kink oscillations frequency Ω_S vs y_{max}. The parameters are $\kappa=0.09$, $\alpha=0$, $\epsilon=0$. The numerical points are represented by circles (○) with a $\Delta\Omega_S=\pm 4\times 10^{-3}$. The theoretical points given by (23d) and (23e) are represented by crosses (×). $\Omega_N=0.15$.

$$\tfrac{1}{4}\kappa y_{max}^2 \ll 1 \ . \tag{8}$$

Since the Hamiltonian corresponding to (5),

$$\begin{aligned} H &= \int_{-\infty}^{+\infty} h\, dx \\ &= \int_{-\infty}^{+\infty} dx \{ \tfrac{1}{2}U_t^2 + \tfrac{1}{2}U_x^2 + [1+\tfrac{1}{4}V(x)](1-\cos U)\} \\ &= \text{const} \equiv E \ , \end{aligned} \tag{9}$$

is an invariant for the evolution problem (5), the condition (8) simply means that the soliton dynamics is nonrelativistic:

$$\Delta E = E - 8 \simeq 4(y')^2 + \kappa y^2 = \kappa y_{max}^2 \ll 8 \ . \tag{10}$$

Therefore, as long as the problem is conservative, one may reasonably claim that the oscillations of a SG kink *at the bottom* of a harmonic potential well described by Eqs. (5) and (6) are Newtonian, in good agreement with the present state of the art of the literature.

Things drastically change when an (even very weak) external driving field is added on the right-hand side (rhs) of (5):

$$U_{tt} - U_{xx} + (1+\tfrac{1}{4}\kappa x^2)\sin U = \epsilon \cos(\Omega t), \quad \epsilon \ll 1 \ , \tag{11}$$

and resonant regimes are investigated: $\Omega \sim \Omega_N$.

Then, due to the resonance, inequalities (8) and (10) become invalid with time. The effect of the confining potential (6) onto the soliton may no longer be considered as a perturbative one, and two basic questions arise.

(i) Is the soliton still stable when it enters strong perturbation regimes in which its potential energy κy_{max}^2 is not small compared to its rest mass?

(ii) If so, what are its new dynamics?

The present paper answers these two questions by using a simple two-degrees-of-freedom collective-coordinate method already suggested by Rice through the following ansatz:[8]

$$U(x,t) = 4\tan^{-1}\exp\{-\sigma k(t)[x-y(t)]\} \ . \tag{12}$$

The agreement with direct numerical simulation of the original partial differential equation (11) is quite amazing.

The answer to question (i) is definitely positive. More precisely, due to the separation of the respective length scale of the soliton and of its confining potential [cf. (3)], we show that the adiabatically modulated profile deduced from (12),

$$U(x,t) = 4\tan^{-1}\exp\left[-\sigma\left[\frac{1+\frac{1}{4}\kappa y^2}{1-\dot{y}^2}\right]^{1/2}[x-y(t)]\right] , \tag{13}$$

is a stable kinklike wave which approximates fairly well the solution of the inhomogeneous problem [Eqs. (5) and (6)]: see Fig. 2.

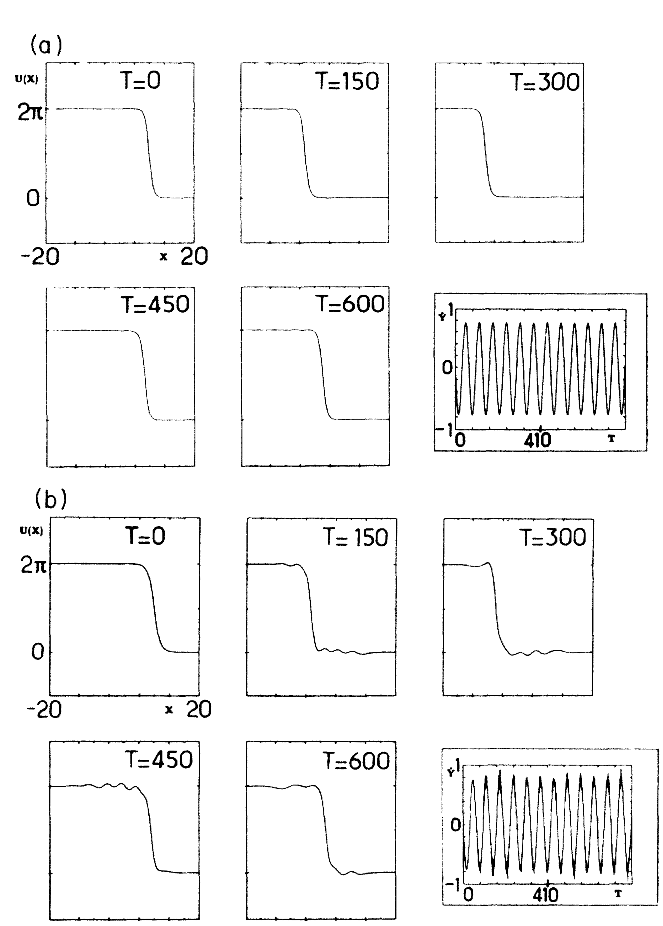

FIG. 2. A time sequence shows the stability of the adiabatic-kink-like profile (13) for $\kappa=0.09$, $\epsilon=0$, $\alpha=0$, $y(0)=8$, $y'(0)=0$, $k'(0)=0$ and, (a) $k(0)=[1+\kappa y^2(0)/4]^{1/2}\simeq\bar{k}$, (b) $k(0)=1$. The presence of stationary linear waves on Fig. 2(b) is due to the strong departure of $k(0)$ from $\bar{k}$ [see (23e)] and cannot be accounted for in the present collective-coordinate method. They are described by the formalism sketched in Sec. IV. The sixth panel of each sequence displays the kink dynamics y' vs t.

Concerning question (ii), we show that the dynamics of the kinklike profile (13), undergoing the external driving force $\epsilon\cos(\Omega t)$, is equivalent to that of a forced oscillator of variable mass. Actually, the "instantaneous" Lorentz contraction

$$d\tau = k^{-1}(t)dt \ , \quad (14)$$

maps the dynamical problem (11), onto the following forced-harmonic-oscillator dynamics:

$$y_{\tau\tau} + \tfrac{1}{4}\kappa y = \frac{\pi}{4}\sigma\epsilon k(t(\tau))\cos[\Omega t(\tau)] \ . \quad (15)$$

As a consequence, the (relativistic) resonance frequency Ω_R may significantly differ from the Newtonian value (7). Choosing typical values $\epsilon = 10^{-2}$, $\kappa = 0.09$, we obtain $\Omega_R \sim 0.127$ instead of $\Omega_N = 0.15$. System of Eqs. (14) and (15) is closed by [cf. (13)]:

$$k(t) = k(y(t)) = \left[\frac{1 + \frac{1}{4}\kappa y^2}{1 - (y')^2} \right]^{1/2} . \quad (16)$$

Formulas (15) and (16) clearly display the nonlinear character of the frequency shift $\Omega_N - \Omega_R$. Hence, adding a small damping term αU_t on the left-hand side (lhs) of (11) affects this shift through its dependence on the amplitude of the kink oscillations. Indeed, it leads to the additive term αy_τ on the lhs of (15), and the answer to question (ii) in the dissipative case is an oscillation whose resonance frequency is given by

$$\Omega_R = \Omega_R(\epsilon, \alpha, \kappa) \ . \quad (17)$$

We illustrate such a relativistic nonlinear resonant dynamics of the kinklike wave (13) in Sec. II.

Inasmuch as the forced dissipative partial differential equation (PDE)

$$U_{tt} - U_{xx} + [1 + \tfrac{1}{4}V(x)]\sin U = \epsilon\cos(\Omega t) - \alpha U_t \quad (18)$$

is an appropriate model for the propagation of the phase difference U between the two macroscopic wave functions of each superconductor layer of a long inhomogeneous Josephson junction,[9] the result (17)—where κ measures an average spatial curvature of the maximum Josephson current density dip—gives some hope for experimentally checking the present theory. Indeed, there are two parameters which may easily be varied: the amplitude of the ultrahigh frequency (Ω is in the GHz range) ac bias ϵ and the amplitude of the damping α through its strong dependence over the temperature of the junction [α increases by an order of magnitude when T varies from 2 to 4 K (Ref. 9)]. The main technical problem is the buildup of a junction with a Josephson current dip satisfying (3).

The intent of this experiment is to elucidate the nonlinear resonances of value (17) in the fluxon dynamics and it is currently being worked on in collaboration with other experimentalists.[10] The underlying idea is the construction of a low-noise uhf Josephson fluxon oscillator, which could be of interest for an uhf detection device in millimeter radioastronomy.[11]

II. TOWARD A DISCRETE TWO-DEGREES-OF-FREEDOM DYNAMICAL SYSTEM

The present theory is restricted to the case (6) for the sake of simplicity. In the Appendix we generalize it to a wider class of confining potentials $V(x)$, satisfying (3).

By direct substitution of the ansatz (12) into (18) we obtain:

$$\left[1 + \tfrac{1}{4}\kappa y^2 + k^2[(y')^2 - 1] + 2\sigma\xi\left[y'k' - \frac{\kappa y}{4k} \right] + \frac{\xi^2}{k^2}\left[\frac{\kappa}{4} + (k')^2 \right] \right] U_S''$$

$$+ \left[\sigma k y'' + \sigma(2k' + \alpha k)y' + \frac{\xi}{k}(k'' + \alpha k') \right] U_S' = \epsilon\cos(\Omega t) \ , \quad (19a)$$

where

$$U_S(\xi) = 4\tan^{-1}\exp\xi \ , \quad (19b)$$

$$\xi = -\sigma k(t)[x - y(t)] \ , \quad (19c)$$

$$\sigma = -1 \text{ for a kink; } \sigma = +1 \text{ for an antikink} \ , \quad (19d)$$

and the prime stands for a derivation with respect to the argument. Projecting[2] both members of (19a) onto the soliton translation mode $f_b(\xi) = (-\sigma/2\sqrt{2})U_S'(\xi)$ leads to the equation of motion of the degree of freedom $y(t)$:

$$k y'' + k'y' + \alpha k y' + \frac{\kappa y}{4k} = \frac{\pi}{4}\sigma\epsilon\cos(\Omega t) \ , \quad (20a)$$

while the behavior of the remaining degree of freedom $k(t)$ is determined by the projection of (19a) onto any mode orthogonal to $f_b(\xi)$: for instance $f_b'(\xi)$. We obtain

$$\frac{2\pi^2}{3k}(k'' + \alpha k') - \frac{\pi^2}{k^2}\left[\frac{\kappa}{4} + (k')^2 \right] - 4\left[1 + \frac{\kappa}{4}y^2 + k^2[(y')^2 - 1] \right] = 0 \ . \quad (20b)$$

The system (20) can be recovered in the Hamiltonian case ($\alpha=0$) by substituting the ansatz (12) into the Hamiltonian density defined by (9)—with the additional term $[-U\epsilon\cos(\Omega t)]$—, then by integrating over space:

$$H=\int_{-\infty}^{+\infty} h(U(y,y',k,k'),t)dx$$
$$=\frac{\pi^2}{3k^3}\left[\frac{\kappa}{4}+(k')^2\right]+\frac{4+\kappa y^2}{k}+4k[1+(y')^2]$$
$$-\epsilon\cos\Omega t\int_{-\infty}^{+\infty} U(y,y',k,k')dx \ , \qquad (21a)$$

and by writing the Hamiltonian equations for the canonical momenta

$$P=8ky'; \ \ Q=\frac{2\pi^2 k'}{3k^3} \ . \qquad (21b)$$

Indeed, $P'=-\partial H/\partial y$ gives

$$P'=-\frac{2\kappa y}{k}+2\pi\sigma\epsilon\cos(\Omega t) \ , \qquad (22a)$$

while $Q'=-\partial H/\partial k$ gives

$$Q'=\frac{1}{16k^2}P^2-\frac{9k^2}{4\pi^2}Q^2$$
$$+4\left[\frac{1}{k^2}-1\right]+\frac{\kappa}{k^2}\left[y^2+\frac{\pi^2}{4k^2}\right] \ . \qquad (22b)$$

We check that the system (22) is equivalent to (20) in the case where $\alpha=0$. This system is the reduction of the problem (18) with an infinite number of degrees of freedom to a two-degrees-of-freedom dynamical system. Its (obvious) numerical simulation shows a perfect agreement with the direct numerical simulation of the original partial differential equations (6) and (18): see Fig. 3. In Fig. 4, we display the resonance effect in a typical nondissipative case.

III. THE FINAL REDUCTION TO A ONE-DEGREE-OF-FREEDOM DYNAMICAL SYSTEM

In the conservative case, Eq. (20a) admits the invariant:

$$(y')^2+\frac{\kappa}{4k^2}y^2=\text{const} \ , \qquad (23a)$$

and then a particular solution of system (20) is

$$y(t)=y_0\cos(\Omega_S t) \qquad (23b)$$

$$k(t)=\bar{k}=\text{const} \ , \qquad (23c)$$

with

$$\Omega_S=\frac{\sqrt{\kappa}}{2\bar{k}} \ , \qquad (23d)$$

and

$$\bar{k}=\bar{k}(y_0)$$
$$=\frac{1}{\sqrt{2}}\omega_P(y_0)\left[1+\left[1+\frac{\kappa\pi^2}{4\omega_P^4(y_0)}\right]^{1/2}\right]^{1/2} \ , \qquad (23e)$$

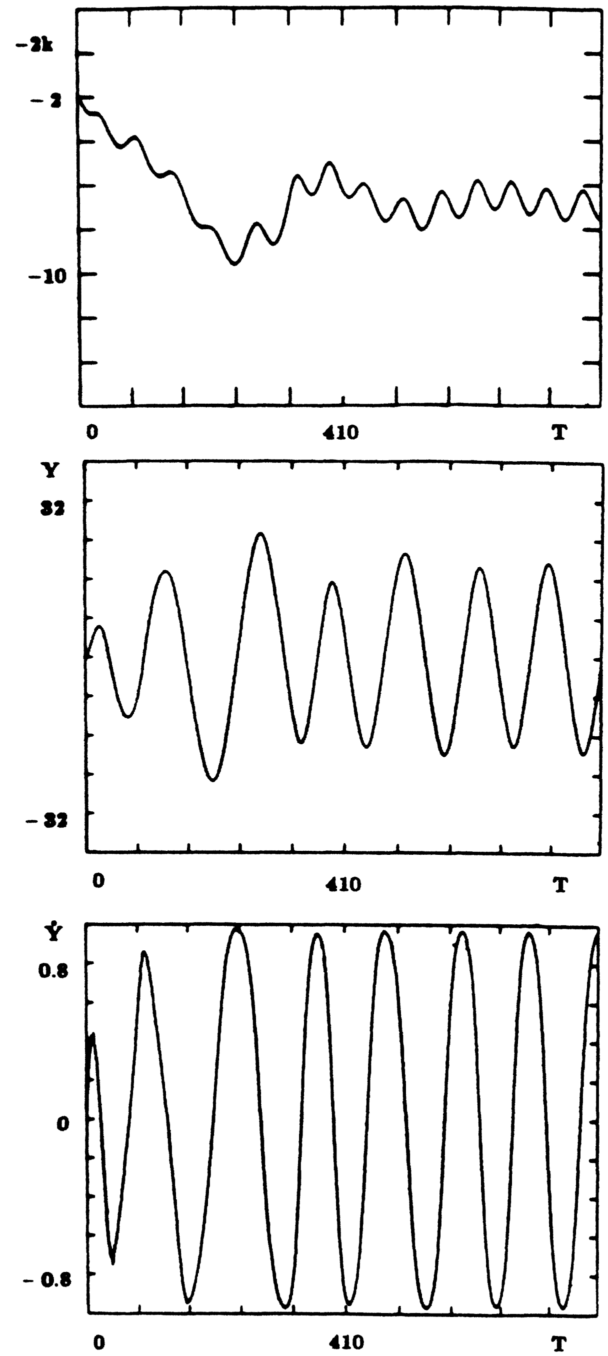

FIG. 3. A typical resonance obtained either by direct numerical simulation of the original PDE (18) with: $\kappa=0.09$, $\Omega=0.055$, $\alpha=0.01$, $\epsilon=0.1$, or by numerical solution of the corresponding two-degrees-of-freedom system (20). The initial condition is given by (2) with $\sigma=+1$. Top panel: the value of the slope $\partial U/\partial x$ at $U=\pi$ versus time. In terms of the ansatz (12), this slope is equal to $-2k(t)$. Note the huge increase of (relativistic) energy; indeed the "average" Lorentz factor k increase by a factor of the order of 3 for $t>400$. Middle panel: the kink position versus time defined at $U=\pi$. In terms of the ansatz (12), this position is given by $y(t)$. Bottom panel: the kink velocity obtained from the ratio $-[\partial U/\partial t]/[\partial U/\partial x]$ at $U=\pi$ versus time. In terms of the ansatz (12), this velocity is $y'(t)$. We insist on the fact that both numerical solutions of the PDE (18) and of the system (20) give *exactly* the same plots within the scale precision.

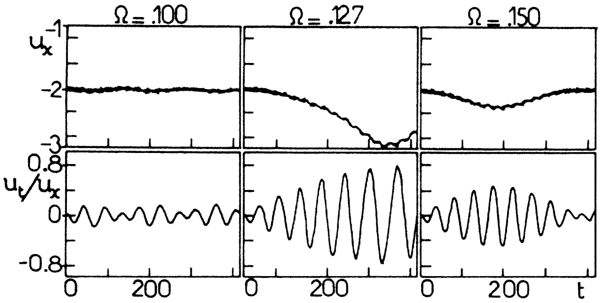

FIG. 4. Resonance effect about $\Omega=0.127$ for $\kappa=0.09$, $\alpha=0$, $\epsilon=0.01$, obtained by numerical simulation of either the PDE (18) or the ordinary differential system (20).

where

$$\omega_P(y)=(1+\tfrac{1}{4}\kappa y^2)^{1/2} . \tag{23f}$$

An initially static adapted kink [i.e., the initial conditions are $y(0)=y_0$, $y'(0)=0$, $k(0)=\bar{k}$, $k'(0)=0$] oscillates with a frequency Ω_S [see (23d)] equal to $\Omega_N/\bar{k}$: see Figs. 1 and 2.

Besides the above kink oscillations (23b), an initial static kink (2) also sees its width k^{-1} oscillating. Indeed, in this case, an expansion of (20b) in terms of $(1-\bar{k})/\bar{k}$ gives

$$k=\bar{k}\left[1+\left[\frac{1-\bar{k}}{\bar{k}}\right]\cos(\omega t)+O\left[\frac{1-\bar{k}}{\bar{k}}\right]^2\right] , \tag{24a}$$

with

$$\omega^2=\frac{12}{\pi^2}\bar{k}^2+\frac{3\kappa}{4\bar{k}^2}\gg\Omega_S^2 \quad [\text{cf.}(23\text{d})] . \tag{24b}$$

We note the order of magnitude separating the period of the high-frequency (HF) oscillations of the degree of freedom $k(t)$ [cf. (24a)] from that of the low-frequency (LF) oscillations of the position $y(t)$ [cf. (23b)]. Accordingly, one may average over the rapidly oscillating k variable in (20b) and extract the following relation:

$$k^2[(y')^2-1]+1+\tfrac{1}{4}\kappa y^2+\frac{\kappa\pi^2}{16k^2}=0 , \tag{25a}$$

which gives the only acceptable solution:

$$k(t)=\bar{k}(t)=\frac{1}{\sqrt{2}}\gamma(t)\omega_P(y)\left[1+\left[1+\frac{\kappa\pi^2}{4\gamma^2\omega_P^4}\right]^{1/2}\right]^{1/2} , \tag{25b}$$

where

$$\gamma(t)=\frac{1}{[1-(y')^2]^{1/2}} . \tag{25c}$$

The physical meaning of (25) is transparent. The factor $\gamma\omega_P$ is the adiabatic or instantaneous—in the LF time scale—adaptation of the kinklike profile (12) to

(i) its velocity $y'(t)$ through a Lorentz contraction $k\propto[1-(y')^2]^{-1/2}$ [cf. (25c)].

(ii) its local plasma frequency $[1+\frac{1}{4}\kappa y^2(t)]^{1/2}$ through the dependence $\kappa\propto(1+\frac{1}{4}\kappa y^2)^{1/2}$ [cf. (23f)]. This is immediately seen from (2) by transforming the lhs of (5) into the lhs of (1)—the dimensionless SG equation—through the local change of variable: $\tilde{t}=[1+\frac{1}{4}V(x)]t$; $d\tilde{x}=[1+\frac{1}{4}V(x)]dx$, then by using the separation of length scales (3).

The departure of k from unity [cf. (25b)] means an overall contraction of a static kink at the bottom of the potential well (6) due to the increase of energy $\Delta H=\kappa\pi^2/12$ [when $k\sim1$:cf. (21a)]. In the highly energetic case, i.e., $y'\lesssim1$ and $\kappa y^2\gg1$, it may be neglected (see Fig. 2) and one is left with the adiabatic solution (13). The trajectory $y(t)$ is checked by the numerical simulations of the system (20a) and (25b). The agreement with the previous results obtained in Sec. II is perfectly within the scale precision: in the case corresponding to Fig. 3, we recover exactly the kink dynamics displayed by Fig. 3: middle and bottom.

IV. BEYOND THE ADIABATIC CASE: OPEN PROBLEMS

When the external frequency Ω is high compared to Ω_N, i.e.,

$$\Omega\gtrsim\omega>1 \quad [\text{cf.}(24\text{b})] , \tag{26}$$

an adiabatic response like (12) is no longer possible; the external field oscillates too rapidly to allow the kink to instantaneously adapt its profile according to the local plasma frequency and to its velocity. As shown in Ref. 12, the exhaustive description of the kink confining potential system needs an infinite discrete series of degrees of freedom. Indeed, the external field excites stationary eigenmodes of the Schrödinger type inside the potential and the soliton appears, in this situation, as a perturbation of such an eigenproblem. By numerical integration and theoretical work (linearization) on Eq. (18), the authors have established that the discrete eigenfrequency spectrum of the solution $U(x,t)$ splits into two independent eigenfrequency spectra. One spectrum describes the internal vibrations of the kink and is referred to as the odd (or vibrational) spectrum (i.e., the eigenmodes are odd functions of the space variable x and describe the internal vibrations of the kink). The second spectrum, called the even (or dynamical) spectrum, determines the motion of the kink under the action of both the potential V and the external driving $\epsilon\cos(\Omega t)$. Resonances are possible only when the external frequency Ω is tuned on even eigenfrequencies.

Tuning the external frequency Ω on the fundamental eigenfrequency $n=0$ (which belongs to the even spectrum) corresponds to the LF resonance. The next even eigenfrequency $n=2$ is greater than 1 and gives rise to the so-called high-frequency (HF) resonance.

In this HF resonance, a quite different behavior of the system is observed, namely, strong HF internal vibrations develop [i.e., the first $(n=1)$ and second $(n=3)$ odd eigenmodes are strongly excited] together with oscillations of the soliton position described by the $n=0$ and the $n=2$ (original driving) eigenmodes. This might be translated in terms of usual spectroscopy as follows: when "pumping" on the $n=2$ level, one observes both the (stronger) "Stokes" satellite feature $n=1$ and the (weaker) "anti-Stokes" satellite $n=3$, together with the (LF) beating frequency $n=0$ which describes the motion of the kink. As a matter of fact, the observed mechanism can be compared to a stimulated Brillouin effect in an excited SG kink, but more work is needed before going further into this analogy.

APPENDIX

Let us consider a wider class of confining potentials $V(x)$ with the only restriction that $V(x)$ increase sufficiently slowly at both infinities [in fact $V(x)$ should increase less rapidly than $\exp(2|x|)$ at infinity] and that $V(x)$ be proportional to a small parameter κ. Assuming the conservative case, if we are interested in recovering a quasi-adiabatic expression for k, let us first see what becomes of Eqs. (22a) and (22b):

$$P'=\frac{\partial}{\partial y}\int_{-\infty}^{+\infty}dx\frac{V(x)}{2\cosh^2[k(x-y)]}\ , \tag{A1}$$

$$Q'=\frac{P^2}{16k^2}-\frac{9k^2}{4\pi^2}Q^2+4\left[\frac{1}{k^2}-1\right]-\frac{\partial}{\partial k}\int_{-\infty}^{+\infty}dx\frac{V(x)}{2\cosh^2[k(x-y)]}\ . \tag{A2}$$

If $V(x)$ is Taylor expandable everywhere we may write

$$\int_{-\infty}^{+\infty}dx\frac{V(x)}{2\cosh^2[k(x-y)]}=\frac{V(y)}{k}+\sum_{p=1}^{\infty}\frac{V^{(2p)}(y)}{k^{2p+1}}\frac{1}{(2p)!}\int_{-\infty}^{+\infty}dz\frac{z^{2p}}{2\cosh^2 z}$$
$$=\frac{V(y)}{k}+\sum_{p=1}^{\infty}\frac{V^{(2p)}(y)}{k^{2p+1}}\frac{1}{(2p)!}\frac{(2^{2p}-2)\pi^{2p}}{2^{2p}}|B_{2p}|\ , \tag{A3}$$

where B_{2p}'s are the Bernouilli numbers.

Let us provide some insight into the oscillatory behavior of k. The equilibrium solution k_0 of Eq. (A2) at $y=0$ ($y'=k'=0$) is given by

$$4(1-k_0^2)+V(0)+\sum_{p=1}^{\infty}\frac{V^{(2p)}(0)}{k_0^{2p}}A_p=0\ , \tag{A4a}$$

where

$$A_p=\frac{2p+1}{(2p)!}\frac{(2^{2p}-2)}{2^{2p}}\pi^{2p}|B_{2p}|\ . \tag{A4b}$$

Linearizing (A2) about $y=0$ and $k=k_0$ yields

$$k''=\frac{6}{\pi^2}(k-k_0)\left[1-3k_0^2+\frac{V(0)}{4}-\frac{1}{4}\sum_{p=1}^{\infty}\frac{V^{(2p)}(0)}{k_0^{2p}}(2p-1)A_p\right]\ , \tag{A5a}$$

where

$$k_0^2=1+\frac{V(0)}{4}+\frac{1}{4}\sum_{p=1}^{\infty}\frac{V^{(2p)}(0)}{k_0^{2p}}A_p\ . \tag{A5b}$$

Hence we get a frequency ω such that

$$\omega^2=\frac{6}{\pi^2}\left[2+\frac{V(0)}{2}+\sum_{p=1}^{\infty}\frac{V^{(2p)}(0)}{4k_0^{2p}}(2p+2)A_p\right]\ . \tag{A6}$$

Obviously, for a wide class of potentials (typically even potentials), ω is still much greater than Ω_S [${\Omega_S}^2$ being proportional to $V^{(2)}(0)$ which is assumed to be small compared to unity]. Therefore we may still separate the time scales of y and k as long as y still exhibits low-frequency phenomena compared with the rapid oscillations of k and Eq. (25a) is generalized to

$$k^2[(y')^2-1]+1+\frac{1}{4}\left[V(y)+\sum_{p=1}^{\infty}\frac{V^{(2p)}(y)}{k^{2p}}\frac{2p+1}{(2p)!}\frac{(2^{2p}-2)}{2^{2p}}\pi^{2p}|B_{2p}|\right]=0\ . \tag{A7}$$

For small $[1-(y')^2]$, one deduces from the above formula

$$k=\left[\frac{1+V(y)/4}{1-(y')^2}\right]^{1/2}+O\{[1-(y')^2]^{1/2}\}\ .$$

As previously mentioned, the adiabatic modulation of the kinklike profile is even more effective as the soliton reaches relativistic velocities.

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