# Theory of condensates in superfluids

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A gauge-symmetry-breaking parameter associated with the fluctuation of the zero-mode particles renders a source term to the continuity equation of interacting Bose systems. As a consequence, additional terms appear in the generalized Ward identities, which modify in an essential way the relation between the superfluid and condensate densities. An energy gap results as a function of the symmetry-breaking parameter and interatomic potential. The general results are worked out in the shielded potential approximation, and an equation for the condensate fraction $n_0(T)$ is obtained in closed form. At absolute zero the condensate fractions in dilute Bose systems and superfluid $^4$He are found to be $n_0=0.438$ and $n_0=0.126$, respectively.

## I. INTRODUCTION

Since London[1] proposed that the unique properties of superfluid $^4$He might be related to the phenomenon of Bose-Einstein condensation, the problem of demonstrating, theoretically and experimentally, the existence of condensate in liquid helium has been of continuing interest. In the present paper we use generalized Ward indentities to obtain an equation for the condensate fraction as a function of the superfluid density. Those identities have their origins in the field-theoretic formalism of Gavoret and Nozières,[2] which has extensively been used in the study of the excitations and response functions of condensed systems. Charged and neutral Bose gases at zero temperatures have been investigated by Ma and Woo, and Wong and Gould,[3] respectively. Talbot and Griffin[4] have studied the response functions at finite temperatures and Szépfalusy and Kondor[5] the dynamics of the phase transition. In this formalism the Dyson-Beliaev one-particle matrix propagator and the density-density correlation function are expressed in terms of irreducible and proper diagrams, the so-called regular contributions. Generalized Ward identities, which relate regular contributions of the one- and two-particle Green's functions, are direct consequences of the equation of continuity. However, this conservation law is no longer satisfied if one uses the Bogoliubov prescription[6] of treating the zero-mode amplitudes as $c$ numbers. In this regard the equation of continuity is usually imposed on the problem. Talbot and Griffin[4] have circumvented this difficulty by adopting another well-known procedure, also due to Bogoliubov,[7] of introducing into the Hamiltonian an infinitesimal symmetry-breaking term. As a result, the continuity equation is satisfied to within an infinitesimal term.[4]

In this paper we consider still another approach to the breaking of the gauge symmetry which consists in defining the zero-mode amplitude as[8]

$$b_0\equiv(1-\xi)^{1/2}a_0+(\xi N_0')^{1/2}\ , \tag{1.1}$$

together with the following condition on the ensemble averages,

$$\langle a_0\rangle=\langle a_0^\dagger\rangle=0\ , \tag{1.2}$$

so that

$$\langle b_0^\dagger b_0\rangle=(1-\xi)\langle a_0^\dagger a_0\rangle+\xi N_0'\ . \tag{1.3}$$

The annihilation and creation operators keep their usual commutation relation, $[a_0,a_0^\dagger]=1$, $N_0'$ is a finite fraction of the total number of particles, even in the thermodynamic limit, and $\xi$ a real parameter that can take on values in the interval [0,1]. The parameter $\xi$ may be interpreted as the probability of finding particles with negligible fluctuation in the $k=0$ state.[8]

In Sec. II the generalized Ward identities are derived for the case of a source term in the continuity equation. General expressions for the superfluid density and the energy gap are determined from the Ward identities in Sec. III. In Sec. IV we illustrate the results of the preceding sections by considering a dilute Bose system. We discuss our results in Sec. V.

## II. GENERALIZED WARD IDENTITIES

Consider a system of bosons of mass $m$ enclosed in a volume $V=L^d$, $d$ being the space dimension, and interacting via a two-body potential $U(x)$. The continuity equation of the system where (1.1) is taken into account has already been derived for a hard-sphere potential.[8] The generalization to an arbitrary potential is trivial and we obtain

$$\partial_t\hat n(x,t;\xi)+\nabla\cdot\hat J(x,t;\xi)=i\hat S(xt;\xi)\ , \tag{2.1}$$

$$\hat S(x,t;\xi)=\frac{\xi}{V}\int d^dx'd^dx''U(x'-x'')[\psi^\dagger(x,t;\xi)\,|\,\psi(x'',t;\xi)\,|\,^2\psi(x',t;\xi)-\psi^\dagger(x',t;\xi)\,|\,\psi(x'',t;\xi)\,|\,^2\psi(x,t;\xi)]\ , \tag{2.2}$$

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where $\psi$ is the boson field, $\hat{n}$ and $\hat{J}$ are the number-density and current operators, respectively, and the Hermitian operator $i\hat{S}$ is the source term that depends on $\xi$ and the interaction (we set $\hbar=1$ throughout this paper). It is convenient to Fourier transform Eq. (2.1) in terms of the imaginary time $\tau=it$,

$$\partial_\tau \rho_k + kJ_k = S_k \ , \tag{2.3}$$

where the density, longitudinal-current, and source operators are given, respectively, by

$$\rho_k=\sum_p a_p^\dagger a_{p+k} = b_0^\dagger a_k + a_{-k}^\dagger b_0 + \sum_{p(\neq 0)(-k)} a_p^\dagger a_{p+k} \ , \tag{2.4}$$

$$J_k = \frac{\hat{k}}{m}\sum_p (p+\tfrac{1}{2}k) a_p^\dagger a_{p+k} \ , \tag{2.5}$$

$$S_k = \frac{\xi}{V}\sum_{p,q} U_p (a_{-k}^\dagger a_{p+q}^\dagger a_p a_q - a_p^\dagger a_q^\dagger a_{p+q} a_k) \ , \tag{2.6}$$

with the convention that a zero-mode amplitude implies $b_0$ or $b_0^\dagger$, as illustrated in Eq. (2.4).

From (2.4)–(2.6) one can define several correlation functions that have the general structure,

$$\chi_{AB}(k,\tau) \equiv -\frac{1}{V}\langle T_\tau A_k(\tau) B_k^\dagger \rangle \ , \tag{2.7}$$

$$C_\mu^A(k,\tau) \equiv -\frac{1}{\sqrt{V}}\langle T_\tau A_k(\tau) a_{k\mu}^\dagger \rangle \ , \tag{2.8}$$

where $T_\tau$ is the $\tau$-ordering operator, $A_k$ and/or $B_k$ stand for the operators $\rho_k$, $mJ_k$, and $S_k$. The Bose amplitude in (2.8) obeys the standard convention

$$a_{k\mu} = \begin{cases} a_k, & \mu=+ \\ a_{-k}^\dagger, & \mu=- \ . \end{cases}$$

The usual procedure to obtain equations that relate the various correlation functions among themselves consists of three steps: (i) take $\tau$ derivatives of (2.7) and (2.8) so that inside the ensemble average only the density operator is involved in the differentiation, (ii) substitute $\partial\rho_k/\partial\tau$ according to the continuity equation, and (iii) expand the result in finite-temperature Fourier series.[3,4] The distinction of the present procedure resides solely in the fact that we do not neglect the source term, however small. It is straightforward to obtain the following equations,

$$\omega\chi_{nn}(k,\omega) = \frac{k}{m}\chi_{Jn}(k,\omega) + \frac{1}{V}\langle[\rho_k,\rho_k^\dagger]\rangle - \chi_{Sn}(k,\omega) \ , \tag{2.9}$$

$$\omega\chi_{Jn}(k,\omega) = \frac{k}{m}\chi_{JJ}(k,\omega) + \frac{m}{V}\langle[J_k,\rho_k^\dagger]\rangle - \chi_{SJ}(k,\omega) \ , \tag{2.10}$$

$$\omega\chi_{Sn}(k,\omega) = \frac{k}{m}\chi_{SJ}(k,\omega) + \frac{1}{V}\langle[S_k,\rho_k^\dagger]\rangle - \chi_{SS}(k,\omega) \ , \tag{2.11}$$

$$\omega C_\mu^n(k,\omega) = \frac{k}{m}C_\mu^J(k,\omega) + (\xi n_0')^{1/2}\beta_\mu - C_\mu^S(k,\omega) \ , \tag{2.12}$$

where $\beta_\mu \equiv \mathrm{sgn}\mu$ and $n_0' \equiv N_0'/V$. (To keep the notation similar to that of Ref. 4 we have denoted by $n$, instead of $\rho$, the sub- or superscripts related to the density. On the other hand, we have dropped the $l$ superscript associated with the longitudinal current.) The frequency $\omega$ is the analytic continuation of the Bose frequency $2\pi i n/\beta$, where $\beta \equiv 1/k_B T$ is the inverse temperature.

Clearly, Eq. (2.11) is absent from previous works,[3,4] and Eqs. (2.9), (2.10), and (2.12) differ from those by the presence of the parameter $\xi$ and correlations involving the source term. In addition, the commutators now read

$$[\rho_k,\rho_k^\dagger] = \xi(a_k^\dagger a_k - a_{-k}^\dagger a_{-k}) \ , \tag{2.13}$$

$$[J_k,\rho_k^\dagger] = \frac{k}{m}\left[ b_0^\dagger b_0 + \frac{\xi}{2}(a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) + \sum_{p(\neq 0)} a_p^\dagger a_p \right] . \tag{2.14}$$

Therefore, the commutator (2.13) does not vanish, as it usually does, and (2.14) is such that the total particle density, $n \equiv N/V$, is no longer equal to $n = m\langle[J_k,\rho_k^\dagger]\rangle/kV$. Only in the normal phase ($\xi=0$) the values of (2.13) and (2.14) become identical to those obtained previously.[3,4]

We proceed now to determine the Ward identities. An analysis similar to the dielectric formulation for the density and longitudinal-current response functions[3,4] shows that the new response functions, the ones that depend on the source term, may also be expressed in terms of irreducible contributions denoted with overbars,

$$\chi_{Sn}(k,\omega) = \bar{\chi}_{Sn}(k,\omega)/\epsilon(k,\omega) \ , \tag{2.15}$$

$$\chi_{SJ}(k,\omega) = \bar{\chi}_{SJ}(k,\omega) + \bar{\chi}_{Sn}(k,\omega)\frac{U_k}{\epsilon(k,\omega)}\bar{\chi}_{nS}(k,\omega) \ , \tag{2.16}$$

$$\chi_{SS}(k,\omega) = \bar{\chi}_{SS}(k,\omega) + \bar{\chi}_{Sn}(k,\omega)\frac{U_k}{\epsilon(k,\omega)}\bar{\chi}_{nS}(k,\omega) \ , \tag{2.17}$$

$$C_\mu^S(k,\omega) = \bar{C}_\mu^S(k,\omega) + \bar{\chi}_{Sn}(k,\omega)\frac{U_k}{\epsilon(k,\omega)}\bar{C}_\mu^n(k,\omega) \ , \tag{2.18}$$

where the dielectric function is defined by

$$\epsilon(k,\omega) \equiv 1 - U_k\bar{\chi}_{nn}(k,\omega) \ . \tag{2.19}$$

Equations (2.9)–(2.12) remain valid when the correlation functions $\chi_{AB}$ and $C_\mu^A$ are replaced by their corresponding irreducible parts $\bar{\chi}_{AB}$ and $\bar{C}_\mu^A$. In order to express these equations in terms of regular (irreducible and proper) contributions only, one must introduce density, longitudinal-current, and source vertex functions, $\Lambda_\nu^A$, defined by

$$\bar{C}_\mu^A(k,\omega) = \Lambda_\nu^A(k,\omega)\bar{G}_{\nu\mu}(k,\omega), \quad A_k = \rho_k, mJ_k, S_k \ , \tag{2.20}$$

where $\bar{G}_{\nu\mu}$ are the irreducible Beliaev-type Green's functions and the summation convention over repeated indices ($\mu,\nu=+,-$) is assumed. Equations (2.20) allows us to split the irreducible $\bar{\chi}_{AB}$'s into regular (proper), $\bar{\chi}_{AB}^R$, and improper contributions, viz.,

$$\bar{\chi}_{AB} = \Lambda_\nu^A \bar{G}_{\nu\mu}\Lambda_\mu^B + \bar{\chi}_{AB}^R, \quad A_k, B_k = \rho_k, mJ_k, S_k \ . \tag{2.21}$$

Substituting Eqs. (2.21) into the irreducible version of Eqs. (2.9)–(2.12), we finally obtain

$$\omega\bar{\chi}^{R}_{nn}=\frac{k}{m}\bar{\chi}^{R}_{Jn}-(\xi n_0')^{1/2}\beta_\mu\Lambda^n_\mu+\frac{1}{V}\langle[\rho_k,\rho_k^\dagger]\rangle-\bar{\chi}^{R}_{Sn}\ , \tag{2.22}$$

$$\omega\bar{\chi}^{R}_{Jn}=\frac{k}{m}\bar{\chi}^{R}_{JJ}-(\xi n_0')^{1/2}\beta_\mu\Lambda^J_\mu+\frac{m}{V}\langle[J_k,\rho_k^\dagger]\rangle-\bar{\chi}^{R}_{SJ}\ , \tag{2.23}$$

$$\omega\bar{\chi}^{R}_{Sn}=\frac{k}{m}\bar{\chi}^{R}_{SJ}-(\xi n_0')^{1/2}\beta_\mu\Lambda^S_\mu+\frac{1}{V}\langle[S_k,\rho_k^\dagger]\rangle-\bar{\chi}^{R}_{SS}\ , \tag{2.24}$$

$$\omega\Lambda^n_\mu=\frac{k}{m}\Lambda^J_\mu+(\xi n_0')^{1/2}\beta_\nu\bar{G}^{\,-1}_{\nu\mu}-\Lambda^S_\mu\ . \tag{2.25}$$

Equations (2.22)–(2.25) are the generalized Ward identities which follow from the continuity equation (2.3). They become identical to those obtained previously[3,4] when $\xi=1$ and $S_k=0$.

### III. ENERGY GAP AND SUPERFLUID DENSITY

The analysis of a Bose system at temperature $T=0$ K carried out by Gavoret and Nozières displayed several infrared divergences, which they removed by introducing a fictitious energy gap $\Delta$ at $(k,\omega)=0$. Talbot and Griffin[4] have shown that this gap may be accounted for the infinitesimal positive energy of the symmetry-breaking term introduced into the Hamiltonian. We consider next the energy gap within the context of the $\xi$ formalism. For the Bogoliubov propagator the energy gap is defined by[4]

$$\Delta\equiv\lim_{k\to 0}[\bar{\Sigma}_{++}(k,0)-\bar{\Sigma}_{+-}(k,0)-\mu]\ , \tag{3.1}$$

where $\mu$ is the chemical potential (not to be confused with the indices) and $\bar{\Sigma}_{\mu\nu}(k,\omega)$ are the irreducible self-energies defined through Dyson's equation

$$G^{-1}_{\mu\nu}=(G^0_{\mu\nu})^{-1}-\Sigma_{\mu\nu}\ . \tag{3.2}$$

$G^0_{\mu\nu}$ is the unperturbed Green's function given by

$$G^0_{\mu\nu}=\delta_{\mu\nu}[(\mathrm{sgn}\mu)\omega-\epsilon_k+\mu]^{-1}\ , \tag{3.3}$$

where $\epsilon_k=k^2/2m$ is the free-particle energy. From Eqs. (3.1)–(3.3) the gap reads

$$\Delta=\lim_{k\to 0}-\tfrac{1}{2}\beta_\mu\bar{G}^{\,-1}_{\mu\nu}(k,0)\beta_\nu\ . \tag{3.4}$$

By comparing the zero-frequency and long-wavelength limit of the Ward identity (2.25) with (3.4), we obtain

$$\Delta=\lim_{k\to 0}\tfrac{1}{2}(\xi n_0')^{-1/2}\beta_\mu\left[\frac{k}{m}\Lambda^J_\mu(k,0)-\Lambda^S_\mu(k,0)\right]\ . \tag{3.5}$$

Up to the one-loop approximation the first term in (3.5) vanishes[3,4] so that the source vertex function is solely responsible for the energy gap in the excitation spectrum of the Bogoliubov propagators.

We now use the Ward identity (2.23) to derive an exact expression for the superfluid density. The starting point is the following relation in the thermodynamic limit

$$n=n_0+\tilde{n}=n_S+n_N,\quad N,V\to\infty;\ N/V=n\ , \tag{3.6}$$

where $n_S$ and $n_N$ are the superfluid and normal fluid densities, respectively, while the condensate density, $n_0$, and the noncondensate density, $\tilde{n}$, are defined as follows,

$$n_0=\langle b_0^\dagger b_0\rangle/V,\quad \tilde{n}=\sum_{p(\neq 0)}\langle a_p^\dagger a_p\rangle/V,\quad V\to\infty\ . \tag{3.7}$$

Talbot and Griffin[4] have shown that the most general definition of the normal fluid density is given by the zero-frequency and long-wavelength limit of the regular part of the longitudinal-current correlation function, i.e.,

$$n_N(T)\equiv-\lim_{k\to 0}m^{-1}\bar{\chi}^{R}_{JJ}(k,0)\ . \tag{3.8}$$

Also notice that Eqs. (1.3), (2.14), and (3.6) yield

$$\lim_{k\to 0}m\langle[J_i,\rho_k^\dagger]\rangle/kV=n+\xi\langle a_0^\dagger a_0\rangle/V\ . \tag{3.9}$$

Combining Eqs. (3.6)–(3.9) with the hydrodynamic limit ($\omega=0$, $k\to 0$) of the Ward identity (2.23), we finally have

$$n_S=\lim_{k\to 0}\{[(\xi n_0')^{1/2}\beta_\mu\Lambda^J_\mu(k,0)+\bar{\chi}^{R}_{SJ}(k,0)]/k\}-\xi\langle a_0^\dagger a_0\rangle/V\ , \tag{3.10}$$

where the thermodynamic limit is implied. The first term on the right-hand side of (3.10) equals the one obtained by Talbot and Griffin[4] for $\xi=1$, while the other two are due to the source in the continuity equation.

### IV. WEAKLY INTERACTING BOSE SYSTEMS

To illustrate the general results of the preceding section we consider a dilute hard-sphere Bose system[9] with interaction constant $U_0=4\pi a/m$ and diluteness parameter $(na^3)^{1/2}\ll 1$, $a$ being the $s$-wave scattering length. For such a system the shielded potential approximation (SPA) is well founded.[5,9] Basically, it amounts to evaluating all regular functions in the noninteracting Bose gas approximation. The Dyson-Beliaev matrix propagator is that of (3.3) and from previous SPA results,[3,4,9]

$$\bar{\Sigma}_{\mu\nu}=0,\quad \Lambda^n_\mu=(\xi n_0')^{1/2},\quad \Lambda^J_\mu=\tfrac{1}{2}k(\xi n_0')^{1/2}\beta_\mu\ . \tag{4.1}$$

In the SPA either Eqs. (3.1) and (4.1), or Eqs. (3.3) and (3.4), show that $\Delta=-\mu$. Therefore, Eq. (3.5) provides a consistent way of determining the chemical potential within the $\xi$ formalism. Since the longitudinal-current vertex function does not contribute to $\mu$, we proceed to evaluate the source vertex function. Let $A_k$ in (2.8) be the source operator (2.6), namely,

$$C^S_\mu(k,\tau)=\frac{\xi U_0}{V^{3/2}}\sum_{p,q}[\langle T_\tau a_p^\dagger(\tau)a_q^\dagger(\tau)a_{p+q}(\tau)a_k(\tau)a_{k\mu}^\dagger\rangle-\langle T_\tau a_{-k}^\dagger(\tau)a_{p+q}^\dagger(\tau)a_p(\tau)a_q(\tau)a_{k\mu}^\dagger\rangle]\ . \tag{4.2}$$

The ensemble average of a $\tau$-ordered product in the SPA factorizes in pairs of unperturbed Green's functions. As (4.2) has an odd number of amplitudes, an odd number of these must necessarily correspond to the $k=0$ mode ($b_0$ or $b_0^\dagger$). By singling out these modes in the double summation of (4.2) it is straightforward to show that the (irreducible) Fourier transform becomes

$$\bar{C}^S_\mu(k,\omega) = -\xi U_0(\xi n_0')^{1/2}\frac{1}{V}[\xi N_0 + 2(1-\xi)\langle a_0^\dagger a_0\rangle + 2\bar{N}] \times [G^0_{+\mu}(k,\omega) - G^0_{-\mu}(k,\omega)] \ . \quad (4.3)$$

By comparison with the vertex function definition (2.20), Eq. (4.3) yields immediately

$$\Lambda^S_\mu = -\beta_\mu \xi U_0(\xi n_0')^{1/2}(2n - \xi n_0') \ , \quad (4.4)$$

where use has been made of Eqs. (3.6). Substituting (4.4) in (3.5), we finally have

$$\mu = \xi U_0(\xi n_0' - 2n) \ . \quad (4.5)$$

For $\xi > 0$, one has $\mu < 0$, and the noninteracting distribution $\langle a_0^\dagger a_0\rangle$ does not diverge. Consequently, from (1.3) the condensate density is

$$n_0 = \lim_{V\to\infty} \langle b_0^\dagger b_0\rangle / V = \xi n_0', \ \ \xi > 0 \quad (4.6)$$

and the chemical potential becomes

$$\mu = \xi U_0(n_0 - 2n), \ \ \xi > 0 \ . \quad (4.7)$$

In the Bogoliubov limit ($\xi \to 1$) the first term on the right-hand side of Eq. (4.7) coincides with the value of the chemical potential furnished by the usual field-theoretical treatment of dilute Bose systems. The second term is new and on account of it $\mu$ is negative. On general grounds one cannot determine the sign of the chemical potential of interacting bosons below $T_\lambda$. We do know however that $\mu \le 0$ in noninteracting systems and also that superfluid helium has negative $\mu$ since it coexists with its vapor which is a Boltzmann gas.[10]

We now turn to the superfluid density (3.10) in the SPA. After factorization of the $\tau$-ordered product, the source-longitudinal-current correlation function defined through Eqs. (2.5)—(2.7) reduces to a simple expression, i.e.,

$$\chi_{SJ}(k,\tau) \equiv -m\langle T_\tau S_k(\tau) J_k^\dagger\rangle / V = \frac{\xi U_0 k}{V^2}[\langle T_\tau a_k(\tau) a_k^\dagger\rangle\langle T_\tau b_0^\dagger(\tau) b_0\rangle + \langle T_\tau a^\dagger_{-k}(\tau) a_{-k}\rangle\langle T_\tau b_0(\tau) b_0^\dagger\rangle]\sum_p \langle a_p^\dagger a_p\rangle \ , \quad (4.8)$$

where the $k=0$ propagator is given by[8]

$$\langle T_\tau b_0(\tau) b_0^\dagger\rangle = (1-\xi)[1 - e^{(1-\xi)\beta\mu}]^{-1} e^{(1-\xi)\mu\tau} \ . \quad (4.9)$$

Hence, Eq. (4.8) corresponds to a proper and irreducible diagram whose Fourier transform reads

$$\bar{\chi}^R_{SJ}(k,\omega) = \xi U_0 n k (V\beta)^{-1} \times \sum_{\omega'} [G^0_{++}(k,\omega'+\omega) G^0_{++}(0,\omega') + G^0_{++}(k,\omega') G^0_{++}(0,\omega'+\omega)] \ , \quad (4.10)$$

where, from (4.9),

$$G^0_{++}(0,\omega) = (1-\xi)[\omega + (1-\xi)\mu]^{-1} \ . \quad (4.11)$$

Performing the frequency summations (actually, $\omega \to 2\pi i n/\beta$) by standard techniques,[10] and carrying out the analytic continuation ($2\pi i n/\beta \to \omega$), we find

$$\bar{\chi}^R_{SJ}(k,\omega) = (1-\xi)\xi U_0 n k V^{-1}[N^0(\epsilon_k) - N^0(0)] \times \left[\frac{1}{\omega - \epsilon_k + \xi\mu} - \frac{1}{\omega + \epsilon_k - \xi\mu}\right] , \quad (4.12)$$

with the Bose distribution functions given by

$$N^0(\epsilon_k) = [e^{\beta(\epsilon_k - \mu)} - 1]^{-1} = \langle a_k^\dagger a_k\rangle \ , \quad (4.13)$$

$$N^0(0) = [e^{-(1-\xi)\beta\mu} - 1]^{-1} = (1-\xi)^{-1}\langle b_0^\dagger b_0\rangle \ . \quad (4.14)$$

Clearly, the second equality in both (4.13) and (4.14) is true only in the SPA. Considering now Eq. (1.3), one can easily show that

$$\lim_{k\to 0}[N^0(\epsilon_k) - N^0(0)] = -\xi(1-\xi)^{-1} N_0' \ , \quad (4.15)$$

which, combined with (4.6) and (4.12), gives

$$\lim_{k\to 0} k^{-1}\bar{\chi}^R_{SJ}(k,0) = -2U_0 n n_0 \mu^{-1} \ . \quad (4.16)$$

On account of the chemical potential (4.7) the last term in (3.10) proportional to the noninteracting distribution $\langle a_0^\dagger a_0\rangle$ vanishes in the thermodynamic limit. Substituting the vertex functions (4.1) and (4.16) in Eq. (3.10) and considering Eq. (3.6), we finally have

$$n_S = n_0(1 - 2U_0 n\mu^{-1}) \ . \quad (4.17)$$

Notice that (4.17) requires a negative-definite $\mu$ in order

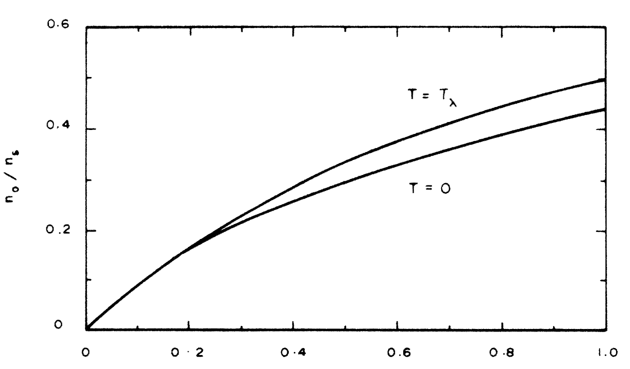


FIG. 1. The ratio $n_0/n_S$, Eq. (4.18), as a function of the parameter $\xi$, for $T=0$ (condensate fraction, $n_S = n$), and $T = T_\lambda$ ($n_S, n_0 \to 0$).

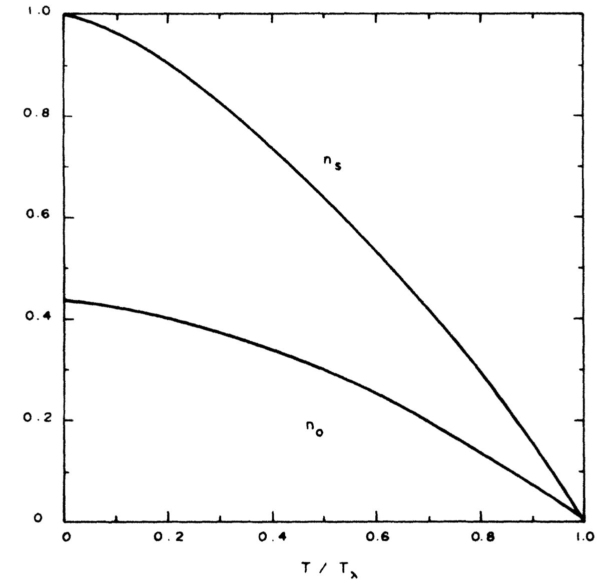


FIG. 2. Superfluid fraction $n_S$, Eq. (4.21), and condensate fraction $n_0$, Eq. (4.19), for $\xi=1$ and $n=1$, as a function of the reduced temperature $T/T_\lambda$.

that $n_S \geq n_0$, and this is precisely what we have found in (4.7). Upon this replacement, Eq. (4.17) reads

$$n_s = n_0[1+2(2-n_0/n)^{-1}\xi^{-1}] \ . \tag{4.18}$$

As $T \to T_\lambda^-$, both $n_S$ and $n_0$ approach zero and Eq. (4.18) gives $n_0/n_S = \xi/(1+\xi)$, as shown in Fig. 1. To determine the condensate fraction we solve Eq. (4.18) for $n_0/n$ and the allowed root of the quadratic equation is

$$n_0 = (2\xi)^{-1}(2+(2+n_S)\xi - \{[2+(2+n_S)\xi]^2 - 8\xi^2 n_S\}^{1/2}) \ . \tag{4.19}$$

This central result expresses the condensate fraction $n_0$ (hereafter we set $n=1$) in terms of the superfluid fraction $n_S$ and $\xi$. In Fig. 1 we plot $n_0$ at $T=0$ K ($n_S=1$) as a function of $\xi$. In particular, this value in the Bogoliubov limit turns out to be

$$n_0 = 0.438, \quad T=0, \quad \xi=1 \ , \tag{4.20}$$

corresponding to the largest condensate fraction. This is to be compared with $n_0 \cong 1$, that results from field-theoretic perturbation treatments of dilute Bose systems.[11]

In order to determine $n_0(T)$ in the whole range $0<T<T_\lambda$, we approximate the superfluid density by the noninteracting Bose gas formula,

$$n_S(T) = 1-(T/T_\lambda)^{3/2}, \quad n=1 \tag{4.21}$$

which is quite reasonable an approximation for dilute, weakly interacting systems.[12] In Fig. 2 we compare $n_0$ and $n_S$ given by Eqs. (4.19) and (4.21) in the Bogoliubov limit.

## V. DISCUSSION

If the Bogoliubov prescription ($\xi=1$) is an adequate description of weakly interacting Bose-condensed systems, and as long as $n_S/n=1$ at $T=0$ K, then the condensate fraction (4.20) should hold for each of these systems. A test on the theory might possibly be carried out with a low-density $^4$He system[13] and, hopefully, with spin-polarized $H_\downarrow$.

In spite of the fact that the results of Sec. IV were obtained in the SPA, we shall see below that Eq. (4.19) is a convenient expression to fit high-density superfluid $^4$He data. We present first some theoretical arguments that may justify that otherwise surprising result.

The next step past the SPA consists in calculating the vertex functions and regular parts of the correlation functions by means of one-loop diagrams and Bogoliubov propagators.[2–4] Although this latter approach is more realistic, particularly in the case of strongly interacting systems, Talbot and Griffin[4] have pointed out that the SPA contains some of the same physics as the one-loop approximation. In fact, Payne and Griffin[9] have shown that for a dilute Bose gas the one-loop approximation gives only small corrections to the SPA.

We now attempt to estimate the effect of the source contribution to the continuity equation in both dilute and dense cases. Accordingly, let us introduce the dimensionless quantity

$$\kappa(\xi) \equiv k_B T/\xi U_0 n = [(a\lambda_T^{-1})/(na^3)^{1/2}]^2/2\xi \ , \tag{5.1}$$

where $\lambda_T$ is the thermal wavelength. The numerator and the denominator inside the square bracket are the usual parameters that specify the dense and dilute states of the system. The quantity $\kappa(\xi)$ is a convenient way of estimating the contribution of the source constant, $\xi U_0$, at a given temperature. For definiteness we consider the system at the transition temperature. Dense and dilute Bose systems are characterized by $k_B T_\lambda/U_0 n \ll 1$ and $k_B T_\lambda/U_0 n \gg 1$, respectively. In dilute systems, where the Bogoliubov prescription holds, one has $\kappa(\xi=1) \gg 1$. It is then natural to extend this condition to dense systems, in order to keep small the contribution of the source term. Hence, we assume that (5.1) must satisfy in general the condition,

$$\kappa(\xi) \gg 1, \quad T=T_\lambda \ . \tag{5.2}$$

Therefore, in dense systems one has

$$\xi \ll k_B T_\lambda/U_0 n \ll 1 \ . \tag{5.3}$$

Let us estimate $\xi$ in (5.3). The dilute parameters of the weakly interacting superfluid $^4$He are $a\lambda_T^{-1}=0.029$ and $(na^3)^{1/2}=0.013$, $T=T_\lambda$ (Refs. 13 and 14) and Eq. (5.1) gives

$$\kappa(\xi=1)=2.5, \quad T_\lambda = 10 \text{ mK} \ , \tag{5.4}$$

while for the usual bulk superfluid $^4$He, $a\lambda_T^{-1}=0.43$ and $(na^3)^{1/2}=0.61$,[15] so that

$$\kappa(\xi)=0.25/\xi, \quad T_\lambda = 2.17 \text{ K} \ . \tag{5.5}$$

From Eqs. (5.4) and (5.5) we obtain the $\xi$ value that makes the source contribution in the dense case equivalent to that of the dilute system, namely,

$$\xi = 0.10 \ . \tag{5.6}$$

Hence, if $\xi \cong 0.1$, the source contribution in strongly interacting systems may be regarded as a small perturbation.

We now turn to the analysis of the experimental data. In recent years condensate fractions have been extracted from the momentum distribution in superfluid $^4$He through high-momentum neutron scattering at standard vapor pressure.[16–19] These measurements, shown in Fig. 3, were analyzed by Sears[20] in terms of the following heuristic expression

$$n_0(T) = n_0(0)[1-(T/T_\lambda)^\alpha] \ . \tag{5.7}$$

A least-square fit to the data gives $n_0(0) = 0.133 \pm 0.012$ and $\alpha = 4.7 \pm 1.2$, and is represented by the dashed curve in Fig. 3. Sears argues that according to Josephson the condensate fraction is expected to have the form[21]

$$n_0(T) \cong (T_\lambda - T)^{2\beta} \ , \tag{5.8}$$

as $T \to T_\lambda^-$. A least-square fit of (5.8) for $T \geq 1.8$ K yields $2\beta = 0.5 \pm 0.2$ and corresponds to the dotted line in Fig. 3.[22]

Equation (5.7) does not depend on $n_S(T)$ and, as pointed out by Sears,[20] has no actual justification for superfluid $^4$He. It should be regarded as a convenient way of parametrizing the data and of estimating the condensate fraction ($n_0 = 0.133$) at absolute zero. On the other hand, if we use in Eq. (4.19) the real values of $n_S(T)$ derived from normal fluid-density measurements at the saturated vapor pressure,[23] and in addition determine the parameter $\xi$ by a least-square fit to the data,[16–19] we obtain $\xi = 0.154$ and the corresponding smooth curve represented in Fig. 3. This curve has three interesting features: (i) it depends on the superfluid density $n_S(T)$, (ii) it resembles that of Eq. (5.8) as $T \to T_\lambda$ (the dotted curve in Fig. 3), and (iii) it furnishes a lower condensate fraction at $T = 0$ K, namely $n_0 = 0.126$, as expected from theoretical estimates. In fact, the first calculation of the condensate fraction at absolute zero is due to Penrose and Onsager[24] who found $n_0 = 0.08$ by assuming a plausible ground-state wave function. More recent calculations have yielded values of the condensate fraction that vary from[25] 0.09 to[26] 0.119, depending on the interatomic-force and calculation method.[27]

Recently, Griffin[28] revised the usual approach of extracting $n_0(T)$ from neutron scattering experiments. By reanalyzing some experimental data,[17,29] Griffin found

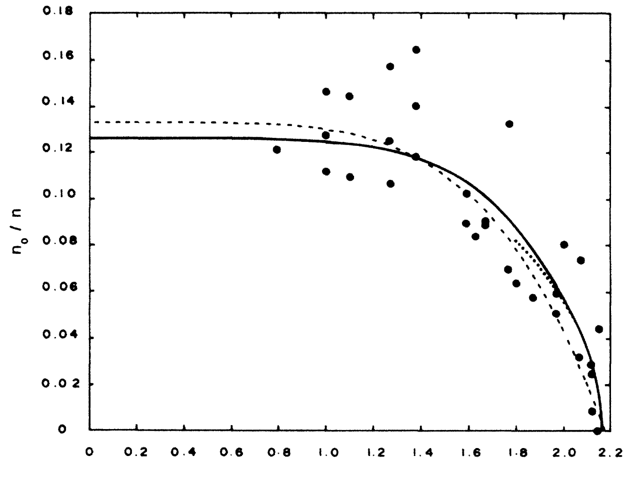


FIG. 3. Condensate fraction versus temperature. Solid curve is Eq. (4.19) for $\xi = 0.154$; dashed curve, Eq. (5.7); and dotted curve, Eq. (5.8). Solid circles are experimentally determined values in superfluid $^4$He from Refs. 16—19.

condensate fraction values that are approximately half the original ones. A least-square fit of Eq. (4.19) to these new values,[17,29] together with actual liquid-helium superfluid density $n_S(T)$,[23] gives $\xi = 0.059$ and, correspondingly, $n_0 = 0.054$ at $T = 0$ K.

Finally, we remark that the $\xi$ values found in both least-square fits above are consistent with (5.6). This calls to mind an argument given previously[8] that Bose systems might have two ordered phases: with ($\xi = 1$) and without ($0 < \xi \ll 1$) Bose-Einstein condensation, the former typical of weakly interacting systems and the latter of strongly interacting ones. The condition $0 < \xi \ll 1$ was meant to imply that $\xi$ ought to be infinitesimal based on the requirement that the source constant $\xi U_0$ should be small for the continuity equation to hold to within an infinitesimal term. The result (5.3) does in fact confirm that $0 < \xi \ll 1$ in strongly interacting systems. However, as shown explicitly for superfluid $^4$He, Eq. (5.6), the parameter $\xi$ need not necessarily be infinitesimal. Therefore, in the case of dense systems that display the phenomenon of Bose-Einstein condensation, the condition $0 < \xi \ll 1$ must be extended to allow for small, yet noninfinitesimal, values of the parameter $\xi$.


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