# Termination of hierarchy of fractional quantum Hall states: Scaling of impurity effect

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The hierarchy of fractional quantum Hall states of a two-dimensional electron gas in a high magnetic field is studied quantitatively. The effect of impurities is found to rescale, eventually terminating the hierarchy. The conditions on the impurities and the magnetic fields for the existence of the fractional quantum Hall effect at rational Landau-level fillings are derived.

## INTRODUCTION

The purpose of this paper is to report a quantitative study of the hierarchy of fractional quantum Hall states of the two-dimensional (2D) electron gas with no impurity as well as with randomly distributed static, charged impurities. We show that the effect of impurities rescales, eventually terminates the hierarchy. Thus only a finite number of fractional states are quantized in real samples. By assuming that the dominant interaction among the quasiparticles is a pairwise charged Coulomb interaction, the excitation energies at all rational Landau-level fillings $\nu$ of the pure system can be estimated in terms of those at $\nu=1/m$, with $m$ odd integers.

Soon after the observation of the integral quantum Hall effect in Si-MOSFETs (Ref. 1) and the other materials, the fractional quantum Hall effect (FQHE) was observed[2] in GaAs-$Ga_{1-x}Al_xAs$ heterostructures with high mobilities at low temperatures. Laughlin[3] proposed a variational ground state to explain the state at $\nu=1/m$, and argued that the elementary excitations of stable states are fractional-charged quasiparticles. Laughlin's theory has been extended to explain the other observed fractional Hall steps, which are the hierarchical high-order states of the $1/m$ state. Among the proposed hierarchical schemes are the descriptions of three kinds of statistics of the quasiparticles: Bose,[4,5] Fermi,[6] and fractional.[7,8] These schemes lead to the same hierarchy of rational quantization of the Hall effect. There are strong experimental evidences in which sample impurities play a crucial role in destroying the FQHE. The observation of FQHE at higher levels of the hierarchy requires higher mobility in samples. Yoshioka[9] has argued that the higher levels are more fragile to impurity effect because of the decreased Coulomb energy. Laughlin *et al.*[10] have recently proposed a scaling theory to explain the disorder effect. However, the theory is based on "guesswork." In this paper, we shall adopt Haldane's boson version of statistics for the quasiparticles[5] to study the hierarchy for both clean and dirty systems. We observe that the hierarchy scheme of Haldane's may be put in a more quantitative fashion. As a hierarchy develops, the impurity-quasiparticle interaction becomes more and more important, and the disorder of the sample in the quasiparticle point of view is rescaled. The increasing rescaling of the impurity effect eventually terminates the hierarchy by making the excitation energy of the incompressible liquid state vanish. The conditions on the impurities and the magnetic fields for FQHE are derived as a function of rational fillings.

Following Haldane,[5] we consider a 2D electron gas of $N$ particles on a spherical surface of radius $R$, in a radial magnetic field $B=\hbar cS/eR^2$, where $2S$ is the total magnetic flux through the surface in units of the flux quantum $h/e$, and $\nu=(N-1)/2S$. We assume that the magnetic field is sufficiently high so that only the spin polarized states in the lowest Landau level need to be considered. The total number of single-electron states is thus $2S+1$. We assume that the impurities are charged, static, and randomly distributed on the surface. The role of impurities in this model is to produce a spatial charge fluctuation of the background. In addition to the clean system's Hamiltonian $H_0$, we have the impurity-electron interaction

$$H'=\sum_{j=1}^{N}\int\frac{e\rho(\Omega)}{\epsilon\,|\mathbf{r}-\mathbf{r}_j|}d\Omega\ , \tag{1}$$

with $\rho(\Omega)$ being the inhomogeneous part of the background charge per solid angle at angle coordinate $\Omega$, $\epsilon$ the background dielectric constant. The impurity-impurity interaction in this model gives the Hamiltonian an overall constant and has no significance for physical quantities. In the thermodynamic limit, the statistical average of random impurity systems gives $\langle\rho(\Omega)\rangle=0$, and $\langle\rho(\Omega)\rho(\Omega')\rangle=W^2\delta(\Omega-\Omega')$, with $W$ the width of the charge fluctuation. For a system with point-charged impurities, $W^2=\sum(Z_ie)^2N_i$, with $Z_ie$ and $N_i$ being the charge and the number of the $i$th type of impurities, respectively.

## LAUGHLIN STATE AT $\nu=1/m$

The Hamiltonian (1) can be treated as a perturbation. The ground state of $H_0$ at $\nu=1/m$, according to Laughlin, is a quantum liquid state with a finite excitation energy (defined as the energy cost to produce a well-separated pair of quasiparticles with opposite charge). Since the magnetic length $l_0=(\hbar c/eB)^{1/2}$ is the only length scale in the problem, this energy can be written in a general form, $E(\nu)=(e^2/\epsilon l_0)f_F(m)$, where $f_F(m)$ is dimensionless. In the presence of impurities, the $(N+1)$-fold degeneracy of the excitation states[3,5] are broken and the excitation energy (defined as the minimum energy cost to produce a well-separated pair of quasiparticles) is reduced by a finite amount.[11] This reduction can be written in a general form based on dimensional analysis,

$$\Delta E(\nu)=(e^2/\epsilon l_0)g_F(\lambda_0,m) ,$$

where we have defined a disorder parameter $\lambda_0=W^2/2Se^2$. The function $g_F$ is dimensionless. For fixed $m$, $g_F$ is a monotonic increasing function of $\lambda_0$. The disorder parameter is related to the magnetic field $B$, $\lambda_0=w^2hc/eB$, with $w^2=W^2/4\pi R^2$. $\lambda_0$ is a measure of the charge fluctuation against the magnetic field. The quantum Hall state requires a gap between the ground state and the excitation state. The condition for the presence of FQHE at $\nu=1/m$ is given by $E(\nu)-\Delta E(\nu)>0$. If the sample is sufficiently dirty, i.e., $\lambda_0$ is large, the FQHE collapses. This corresponds to the lower mobility and lower magnetic field situation in experiments.

## HIERARCHY, EXCITATION ENERGY OF $H_0$

The hierarchy in the absence of the impurities has been studied in Ref. 5. Here we show that the excitation energy at any hierarchy level may be approximately related to the excitation energy for the $1/m$ state. According to Haldane, a Laughlin fluid state

$$\nu(m;\alpha_1,p_1;\alpha_2,p_2;\ldots;\alpha_n p_n)$$

of the excitation can be constructed from its parent state

$$\nu(m;\alpha_1,p_1;\alpha_2,p_2;\ldots;\alpha_{n-1},p_{n-1})$$

if $S_q=(N_q-1)p_n/2$, where $p_i$ are even integers (Bose statistics), $N_q$ and $2S_q+1$ are the number of the excitations and the degeneracy of the excitations of the parent state, respectively. The filling $\nu$ of state

$$\nu(m;\alpha_1,p_1;\alpha_2,p_2;\ldots;\alpha_n,p_n)$$

is given[5] by a finite continued fraction

$$[m;\alpha_1,p_1;\alpha_2,p_2;\ldots;\alpha_n p_n] ,$$

where $\alpha=\pm1$. The excitation energy of this new fluid state is equal to that of a system[12] consisting of $N_q$ particles with fractional charge $e_q$, obeying Bose statistics, on a spherical surface with radius $R$, in a radial magnetic field $B_q=\hbar c S_q/e_q R^2$. The magnetic length of the quasiparticle system is $l_q=(\hbar c/e_q B_q)^{1/2}$. If we assume that the interaction of the quasiparticles can be approximated by the pairwise Coulomb interaction between point particles of charge $e_q$, then the excitation energy of the quasiparticle system (hence that of the corresponding electron system) is

$$E(\nu)=(e_q^2/\epsilon l_q)f_B(p_n) , \tag{2}$$

where $f_B(p_n)$ is the value of the excitation energy of a Bose system at filling $\nu_q=1/p_n$, in units of $e_q^2/\epsilon l_q$. Neglecting the finite size of the quasiparticles, $f_B(p_n)$ does not depend on the details of the hierarchical construction, and on the hierarchy label $n$. The charge of the quasiparticle $e_q$ and $l_q$ can be easily found from the hierarchy equations,[5] and the results are $e_q=e/Q_{n-1}$ and $l_q=l_0(Q_n/p_n)^{1/2}$, with $Q_n$ and $Q_{n-1}$ being the denominators of the rational fillings of the state and its parent state, respectively. Furthermore, $f_B(p_n)$ can be related to $f_F(m)$, an excitation energy of the elementary Laughlin state for the electron system.

Using electron-hole symmetry, which is exact in the lowest Landau level, $E(1-1/m)=E(1/m)$, at fixed magnetic field, and that the $1-1/m$ state may be regarded as $[1;\ m-1]$ state, we derive

$$f_B(m-1)=[m/(m-1)]^{1/2}f_F(m) .$$

Thus the excitation energy at a hierarchy is

$$E(\nu)=\frac{e^2}{\epsilon l_0}\frac{1}{Q_{n-1}^2}\left[\frac{p_n+1}{Q_n}\right]^{1/2}f_F(p_n+1) . \tag{3}$$

Electron-hole symmetry is preserved in Eq. (3) for all rational fillings

The values of the excitation energy at various fillings are listed in Table I in units of the value at $\frac{1}{3}$. The latter has been studied extensively,[3,12,13] and $f_F(3)\simeq0.1$. In particular, we find $E(\nu=\frac{2}{5})=2$ K, while $E(\nu=\frac{1}{3})=23$ K at $B=20$ T and $\epsilon=12.8$.

## HIERARCHY, EXCITATION ENERGY REDUCTION BY IMPURITIES

In the presence of impurities, the excitation energy of a high-order fluid state is equal to that of a system with background charge fluctuation width $W$ in addition to the corresponding pure quasiparticle gas. Note that impurities themselves are not rescaled, while both the charge and the number of the quasiparticles are reduced as we consider higher and higher steps in the hierarchy. Thus, the dis-

TABLE I. Excitation energies $E(\nu)$ in the absence of impurities given by Eq. (3) and the critical disorder parameter $\lambda_c(\nu)$ in the presence of the random impurities given by Eq. (9). For point-charged $e$ impurities, the disorder parameter is $2\pi l_0^2 n_i$, with $n_i$ being the impurity concentration of the sample. $E(\frac{1}{3})\simeq0.1e^2/\epsilon l_0$, and $\lambda_c(\frac{1}{3})\sim4\times10^{-3}$. The upper bound of $E(\nu)$ is given in Eq. (13), which is about 1.8 times the values in the table for $\nu\neq\frac{1}{3}$.

| $\nu$ | $\frac{1}{3}$ | $\frac{2}{5}$ | $\frac{2}{7}$ | $\frac{3}{7}$ | $\frac{2}{9}$ | $\frac{2}{11}$ | $\frac{5}{13}$ | $\frac{4}{9}$ | $\frac{3}{11}$ |
|---|---|---|---|---|---|---|---|---|---|
| $E(\nu)/E(\frac{1}{3})$ | 1 | $\frac{1}{9}(\frac{3}{5})^{1/2}$ | $\frac{1}{9}(\frac{3}{7})^{1/2}$ | $\frac{1}{25}(\frac{3}{7})^{1/2}$ | $\frac{1}{25}(\frac{1}{3})^{1/2}$ | $\frac{1}{25}(\frac{3}{11})^{1/2}$ | $\frac{1}{25}(\frac{3}{13})^{1/2}$ | $\frac{1}{49}(\frac{1}{3})^{1/2}$ | $\frac{1}{49}(\frac{3}{11})^{1/2}$ |
| $\lambda_c(\nu)/\lambda_c(\frac{1}{3})$ | 1 | $\frac{1}{15}$ | $\frac{1}{21}$ | $\frac{3}{175}$ | $\frac{1}{75}$ | $\frac{3}{275}$ | $\frac{3}{325}$ | $\frac{1}{147}$ | $\frac{3}{539}$ |

order parameter of the quasiparticle system (or the effective disorder parameter of the electron system), $\tilde{\lambda}(\nu)=W^2/2S_q e_q^2$, rescales, and it is related to the original disorder parameter $\lambda_0$ by

$$\tilde{\lambda}(\nu)=\lambda_0 Q_{n-1}^2 Q_n/p_n \ . \tag{4}$$

As a hierarchy develops, $Q_{n-1}^2 Q_n$ increases rapidly. The same sample seems to be dirtier in a daughter state than in its parent state. When $\tilde{\lambda}(\nu)$ becomes sufficiently large, the quantum Hall state is destroyed, and the hierarchy is terminated.

By assuming that the impurity-quasiparticle interaction is Coulombic, and that the quasiparticle can be approximated by a point-charged particle, the excitation energy reduction at a filling $\nu$ can be written in the following form in analogy with that for the elementary Laughlin state we gave earlier:

$$\Delta E(\nu)=(e_q^2/\epsilon l_q) g_B[\tilde{\lambda}(\nu),p_n] \ . \tag{5}$$

Note that in Eq. (5) charge and magnetic length are those of quasiparticles, since we are dealing with the quasiparticle system. $g_B(\tilde{\lambda},p_n)$ is the value of the excitation energy reduction in units of $e_q^2/\epsilon l_q$ for a Bose system at filling $1/p_n$ with a disorder parameter $\tilde{\lambda}$. $\tilde{\lambda}$ is given in Eq. (4). The electron-hole symmetry enables us to relate the function $g_B$ for bosons to the function $g_F$ for fermions. Consider state [1; $m-1$] in a given magnetic field and with a disorder parameter $\lambda$. The excitation energy reduction due to disorder is, according to Eq. (5),

$$\Delta E(\nu=1-1/m)=\frac{e^2}{\epsilon l_0}\left[\frac{m-1}{m}\right]^{1/2} g_B(\tilde{\lambda},m-1) \ , \tag{6}$$

with $\tilde{\lambda}=\lambda m/(m-1)$. The state [1; $m-1$] on the other hand, is a conjugate state of the $\nu=1/m$ state. The excitation energy reductions due to impurities are exactly the same for these two states at the fixed magnetic field and fixed disorder parameter within the lowest Landau-level assumption, namely,

$$\Delta E(\nu=1-1/m)=\frac{e^2}{\epsilon l_0} g_F(\lambda,m) \ . \tag{7}$$

Using Eqs. (6) and (7), we obtain, for arbitrary $\tilde{\lambda}$,

$$g_B(\tilde{\lambda},p)=\left[\frac{p+1}{p}\right]^{1/2} g_F[\tilde{\lambda}p/(p+1),p+1] \ . \tag{8}$$

One finally finds that the excitation energy reduction at high-order state is related to that at $1/m$ state,

$$\Delta E(\nu)=\frac{e^2}{\epsilon l_0}\frac{1}{Q_{n-1}^2}\left[\frac{p_n+1}{Q_n}\right]^{1/2} g_F[\lambda(\nu),p_n+1] \ , \tag{9}$$

where

$$\lambda(\nu)=\lambda_0 Q_{n-1}^2 Q_n/(p_n+1) \ . \tag{10}$$

## CONDITIONS FOR FQHE

A quantum Hall state at zero temperature is destroyed by impurities if the excitation energy vanishes. Combining Eqs. (3), (9), and (10), the critical condition for FQHE at a rational $\nu$ is

$$f_F(p_n+1)-g_F[\lambda(\nu),p_n+1]=0 \ , \tag{11}$$

where $\lambda(\nu)$ is given by Eq. (10).

For each rational $\nu$, there is a value of $\lambda_0$, satisfying Eq. (11). This is the critical disorder parameter $\lambda_c(\nu)$. The quantum Hall state at $\nu$ may survive in the presence of impurities if $\lambda_0<\lambda_c(\nu)$. The critical disorder parameter of the high-order Laughlin state is related to that of the $1/m$ state by

$$\lambda_c(\nu)=\frac{P_n+1}{Q_{n-1}^2 Q_n}\lambda_c\left[\frac{1}{p_n+1}\right] \ . \tag{12}$$

Equation (12) is a necessary condition for the presence of FQHE. Only those Hall states, whose $\lambda_c(\nu)$ are larger than $\lambda_0$ in the experimental situation, are quantized. All the others are destroyed by impurities. By the hierarchy theory, the high-order states of the destroyed quantum Hall states cannot be constructed. We conclude that the hierarchy of FQHE must be terminated in the presence of impurities. Equation (12) explains why the FQHE is observed only at simple rational fillings. Values of $\lambda_c(\nu)$ for various fillings are listed in Table I in terms of $\lambda_c(\frac{1}{3})$. The fillings with larger $\lambda_c$ are most easily observed. From Eq. (12), at a similar magnetic field, the sample purity (i.e., mobility from the experimental view point) required for FQHE at filling $\frac{2}{5}$ is about one order higher than that at $\frac{1}{3}$. This is qualitatively consistent with the experimental fact: The first observation of FQHE was at $\frac{1}{3}$ and $\frac{2}{3}$ only (for sample mobility $\mu=80\,000-100\,000$ $\mathrm{cm}^2/\mathrm{v}$ sec, Ref. 2), and the higher-order steps require cleaner samples (mobility of at least $5\times10^5$ $\mathrm{cm}^2/\mathrm{v\,sec}$, Refs. 14 and 15, for example).

For the purpose of illustration, we have carried out numerical calculations for the three-electron system with point random impurities (charge $e$). In this case, $\lambda_0=2\pi l_0^2 n_i$. The impurity concentration $n_i$ is inversely proportional to the mobility of the sample. The very small system calculation suggests the value of $\lambda_c$ at $\nu=\frac{1}{3}$ to be order of $4\times10^{-3}$. This corresponds to the critical impurity concentration $n_i\sim10^9$ $\mathrm{cm}^{-2}$ at $B=10$ T.

## DISCUSSION

In the above analysis, we have treated the quasiparticles as if they were point-charged objects. The finite structure of the quasiparticles will certainly weaken the short-range component of the Coulomb interaction, and reduce the excitation energy of the incompressible states. However, we note that when we apply electron-hole symmetry to relate the excitation energy of a Bose system to that of a Fermi system, the bosons are quasiparticles with finite size. Therefore, what we did, in deriving Eqs. (3) and (9), is to approximate the structure of the quasiparticles of the parent state of $\nu$ by that of the parent state of [1; $p_n$]. Although the finite-size effects are different at different fillings, this replacement enables us to avoid dealing with a complicated structure problem directly. The reliability of the approximation may be discussed as follows.

The upper bound of the excitation energy may be estimated from Eq. (2) since the finite structure and nonpairwise interaction presumably tends to reduce the Coulomb interaction

$$E(\nu) < \frac{e^2}{\epsilon l_0} \frac{1}{Q_{n-1}^2} \left[ \frac{p_n}{Q_n} \right]^{1/2} f_B^{(0)}(p_n) , \tag{13}$$

where the superscript index on $f_B$ indicates the physical point-charged boson system. The ratio of this upper bound and the value given by Eq. (3) is

$$[p_n/(p_{n+1})]^{1/2} f_B^{(0)}(p_n)/f_F(p_n+1) .$$

This number is about 1.8 for $p_n=2$ if we adopt the numerical result[16] of the five-particle boson system, $f_B^{(0)}(2) \approx 0.22$.

Furthermore, the size of the quasiparticle may be approximately characterized by the magnetic length of its parent state $l_{q'}$. The finite-size effect depends on a dimensionless parameter $\beta = 2\pi l_{q'}^2 n_q$, with $n_q$ the density of the quasiparticles. One finds $\beta(\nu) \le \beta([1;p_n])$. Therefore, Eq. (3) could perhaps be regarded as lower bounds.

It should be noted that at the moment there does not exist accurate results for the quasiparticle-quasihole energy of the high-level state one can critically compare with. In Ref. 7, the hierarchy scheme within the fractional statistics was applied to estimate the energy. The point-charged Coulomb interaction of the excitations was assumed as it is assumed in the present work. Although the quasihole proper energy was obtained using an approximate formula fitted from the $\nu=1$, $\frac{1}{3}$, and $\frac{1}{5}$ electron system calculation, the linear approximation for the proper energy of the quasiparticle proportional to that of quasihole with a constant coefficient independent of the hierarchical level and the filling is *ad hoc,* and it was made only for the purpose of the illustration in the article. Our result of Eq. (3) for hole-type daughter state $\nu=\frac{2}{7}$ is close to the estimate in Ref. 7, while they are very different for the particle-type daughter state $\nu=\frac{2}{5}$. The estimate in Ref. 7 is about 2.5 times of the value given by Eq. (3) here, and is around the estimated upper bound given by Eq. (13) in the present paper. This discrepancy could be either due to the uncertainty between the proper energies of quasihole and quasiparticle made in Ref. 7, or due to approximation in the replacement of the excitation structure in deriving Eq. (3). It is likely due to both. We also note that the structure of quasiparticle is quite different from that of quasihole as shown in the work of Haldane and Rezayi.[13] Therefore the excitation energy given by Eq. (3) for particle-type daughter state may be less accurate since a quasihole-type structure of state $[1;p]$ is applied.

The excitation energies for $\nu=\frac{2}{5}$ and $\frac{2}{7}$ were also calculated by MacDonald and Murray[17] using trial wave functions constructed for several electrons. They found that the excitation energy for the $\nu=\frac{2}{7}$ state is much smaller than that for the $\nu=\frac{2}{5}$ state. Their result at $\nu=\frac{2}{7}$ is comparable with our result of Eq. (3), while the value at $\nu=\frac{2}{5}$ is larger than that of the present work.

The finite-size effect on the excitation energy reduction due to impurities may be discussed in a similar manner. But the effect should be much smaller, because the quasiparticle size is at least one order smaller than the average impurity separation in the FQHE situation.

Experimentally, FQHE has been observed at fillings $\nu=\frac{1}{3}$, $\frac{2}{5}$, $\frac{3}{7}$, and their particle-hole conjugates clearly.[15] There are some indications[14,15] in $\rho_{xx}$ at fillings $\nu=\frac{2}{7}$ and $\frac{4}{9}$. The results of Eq. (9) listed in the table are roughly consistent with the experimental observations with the exception of state $\nu=\frac{2}{7}$. Note that the $\frac{2}{9}$ and $\frac{2}{11}$ states are the daughter states of $\nu=\frac{1}{5}$. The FQHE has not been shown clearly at the $\frac{1}{5}$ state. This explains why FQHE does not show at $\nu=\frac{2}{9}$ and $\frac{2}{11}$. The discrepancy between the theory and the experiments at $\nu=\frac{2}{7}$ is still open. The excitation energy at $\nu=\frac{2}{7}$ is about 10% of that at $\nu=\frac{1}{3}$ in all three different approaches: Ref. 7, Ref. 17, and the present approach. It should be at least in the similar value of $\nu=\frac{3}{7}$ and would have been observed for samples[15] with mobility one order higher in magnitude than the sample[2] showing $\nu=\frac{1}{3}$ effect only. A possible explanation is that it might be due to localization effects similar to the situation at $\nu=\frac{1}{5}$ proposed by Chang *et al.*[15]

In conclusion, the independent impurity model we have considered in this paper should be a good approximation for the dilute impurity system. Modulation-doped GaAs-$Ga_{1-x}Al_xAs$ samples are highly pure samples. Our model may be appropriate for the real systems. Using this model, we have found the scaling behavior of the impurity effect in FQHE. The experimental findings of the termination of the hierarchy of fractional quantum Hall states may be explained as a result of the increasing rescaling of the disorder. A simple algebra of the hierarchy scheme enables us to approximately relate the excitation energy and the critical condition for the presence of FQHE at a high-order Laughlin state to those of the $1/m$ state. Although the finite structure of the quasiparticles will give the correction to the approximation, the present study provides a semiquantitative understanding of how the hierarchy of fractional quantum Hall states can be destroyed by disorder.

The real samples have finite-layer thickness in the direction perpendicular to the magnetic fields. This will produce another length scale, and may be incorporated in the analysis we describe in the paper. We wish to point out that finite-layer corrections[18] are less important for higher-level states than those at $\frac{1}{3}$, because the magnetic length for the quasiparticle system is rescaled by a factor of $(Q_n/p_n)^{1/2}$. The localization problem is not considered in the present approach. The assumption of the existence of the extended quasiexcitation states has been implied in the derivation of the critical conditions for FQHE.

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