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Analytical model for the dielectric response of brine-saturated rocks

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Rocks whose pore space is saturated with a conducting brine have a dielectric permittivity which increases significantly with decreasing frequency in the range 1—1000 MHz, and a conductivity which shows a corresponding decrease in the same range. Here a simple model is developed which accounts for this effect within the framework of an analytical representation proposed by Bergman. The model postulates a particular analytic form for the density of resonances characterizing the geometry of the pore structure. The parameters of the model are completely determined by information about the dc conductivity of the rock, plus two exact sum rules satisfied by any composite density function and an inequality which may be taken as an equality in the limit when the contact area between rock grains is small in comparison to the surface area of the grains. The resulting dielectric permittivity of the composite varies as ω^{-b} at low frequencies, where b can be calculated in the model from measurements of the static conductivity of the rock. No microscopic derivation is given for the resulting composite dielectric permittivity in terms of any geometric model of the rock. Nevertheless, the model is shown to agree well with measurements of the permittivity and the conductivity of a variety of brine-saturated rocks over a broad range of frequencies.

I. INTRODUCTION

The dielectric response of rocks whose pore (or void) space is saturated with brine and/or oil is of great interest in oil exploration.[1-3] In these applications, the practical goal is to infer the relative quantities of oil and brine in the pore space of the rock from a knowledge of the complex dielectric permittivity and the porosity of the rock. This task is greatly facilitated by the knowledge of a mixing formula—that is, a relation, ideally of an analytic form, which connects the complex dielectric permittivity of the composite to the permittivities of the individual components, and the relative volume fractions of the components. A variety of such mixing formulas have been proposed by various authors for brine-saturated rocks and for other composite media.[1-5] These account, with varying degrees of success, for some aspects of the dielectric response of rocks. While it is unrealistic to assume the existence of a universal mixing formula appropriate to all rocks (irrespective of their pore structure), it seems reasonable that there may exist certain families of rocks, each characterized by a single mixing formula. Thus, for example, sandstones and shales might be described by different mixing formulas, but the same mixing formula might apply to different types of sandstone.

Besides mixing formulas, the dielectric response of brine-saturated rocks can also be studied in other ways. Lysne,[6] for example, has analyzed this response in terms of a distribution of response times which could be adjusted so as to account for some of the observed data (see also Korringa[7]). Korringa and LaTorraca[8] have made use of exact bounds which can be placed on the complex dielectric permittivity of a composite, in conjunction with known analytic properties of this permittivity, to provide limits within which experimental observations must fall.

In this paper we propose an analytic mixing formula for the limiting case of a rock fully saturated with brine (that is, a two-component composite). The mixing formula is shown to be in agreement with experimental data over a broad range of frequencies for a variety of rock types. The range of frequencies discussed roughly covers those currently of interest in oil exploration.

The basis of the proposed mixing formula is an analytic form for the dielectric permittivity of a two-component composite originally proposed by Bergman.[9-11] This analytic form may be supplemented by two exact sum rules and a number of inequalities which must be obeyed by the complex dielectric permittivity of any such composite. These relations are not, of course, sufficient to determine the mixing formula uniquely. Our postulated form, therefore, makes certain implicit geometric assumptions about the composite. The fact that we obtain good agreement with experiment is then taken as evidence that these assumptions are reasonable. One assumption, which mathematically makes its appearance by taking one of the inequalities as an equality, is equivalent to a "small grain contact area hypothesis," i.e., we assume that the contact area of neighboring grains is small compared to the grain surface area. Thus the mixing formula is only appropriate to such families of rocks. The case of a threecomponent composite in which both oil and brine are present in the pore space will be discussed in a later paper.

The observed dielectric response of rocks[1–3,12,13] has two striking features: an increase or enhancement of the dielectric permittivity as the frequency is decreased, and a corresponding decrease of the conductivity. The mixing formula proposed here can quantitatively reproduce both these features. This suggests that the mixing formula is a reasonable representation of the geometry in a variety of rock types.

We turn now to the body of the paper. Section II presents the basic model of a brine-saturated rock as a classical two-component composite medium, and also reviews the Bergman analytic representation of the composite dielectric permittivity. Several simple approximations to this permittivity, some of them well known in the literature, are discussed in Sec. III, and their deficiencies in the present context outlined. In Sec. IV we develop the mixing formula proposed here, based on the formalism of Sec. II and certain plausible geometric assumptions about the pore structure of rocks. The integral defining the mixing formula is evaluated analytically in Sec. V in several asymptotic regimes which together comprise the entire range of current experimental interest. A detailed comparison with experiment on a variety of rock samples is presented in Sec. VI, following which Sec. VII summarizes and discusses the principal findings.

II. BERGMAN ANALYTIC REPRESENTATION OF A TWO-COMPONENT COMPOSITE

We consider a two-component composite medium, consisting of the rock matrix with complex dielectric constant ϵ_r and the brine with complex dielectric function ϵ_w. In general, ϵ_w will have the form

$$\epsilon_w=\epsilon'_w+\frac{i\sigma_w}{\epsilon_0\omega} \tag{1}$$

comprised of a real part ϵ'_w and an imaginary part dependent on the brine conductivity σ_w, the angular frequency ω, and the permittivity of free space ϵ_0 (we use mks units throughout). The rock matrix will usually have a fixed real dielectric constant

$$\epsilon_r=\epsilon'_r \; . \tag{2}$$

At the frequencies of interest here, i.e., below about 1 GHz, dielectric losses due to dipolar relaxation of the water molecules can be neglected to a first approximation. Thus both σ_w and ϵ'_w, like ϵ'_r, can be treated as constants independent of frequency.

If the wavelengths and attenuation scale lengths of the applied electric field are much greater than the dimensions of the largest grains and pores in the composite, then scattering of electromagnetic radiation is negligible, and we can characterize the saturated rock by an effective complex dielectric constant

$$\epsilon=\epsilon'+\frac{i\sigma}{\epsilon_0\omega} \; , \tag{3}$$

with the real part ϵ' and the conductivity σ both dependent on the frequency of the applied electric field.

A useful general property of the complex dielectric function in the quasistatic limit considered here is that of homogeneity. If both ϵ_r and ϵ_w are multiplied by a constant λ, then so is the resulting effective dielectric constant. This implies

$$\epsilon(\lambda\epsilon_r,\lambda\epsilon_w)=\lambda\epsilon(\epsilon_r,\epsilon_w) \; . \tag{4}$$

Alternatively, one may say that the ratio ϵ/ϵ_r depends on the (complex) ratio ϵ_w/ϵ_r, rather than on ϵ_w and ϵ_r individually. This homogeneity relation is equivalent to the statement that a suitably defined function

$$f=1-\epsilon/\epsilon_r \; , \tag{5}$$

depends only on the variable ϵ_w/ϵ_r. Instead of considering this ratio, however, it proves more convenient to work with a variable

$$s=(1-\epsilon_w/\epsilon_r)^{-1} \tag{6}$$

in terms of which the analytic representation of f is relatively simple.

The analytic properties of $f(s)$ are central to the present discussion, and follow from elementary considerations, as was first shown by Bergman. To clarify the argument, let us suppose that $f(s)$, viewed as an analytic function of the complex variable s, is in fact a rational function of s. Since the composite must dissipate energy whenever both components dissipate energy, we have

$$\operatorname{Im}\epsilon>0 \;\; \text{whenever } \operatorname{Im}\epsilon_r>0 \text{ and } \operatorname{Im}\epsilon_w>0 \; , \tag{7}$$

and this implies, after a small amount of algebraic manipulations, that if $\operatorname{Im}s\neq0$ then

$$\operatorname{Im}s/\operatorname{Im}[f(s)]<0 \; . \tag{8}$$

Hence the change in the sign of $\operatorname{Im}[f(s)]$ which occurs at a pole (or zero) of $f(s)$ must be accompanied by a change in the sign of $\operatorname{Im}s$. This implies that if $f(s)$ has a simple pole at $s=s_i$, then s_i must lie on the real axis. Moreover, such poles must have positive residues, in order to ensure that $\operatorname{Im}[f(s)]/\operatorname{Im}s$ has the correct sign near the poles. Likewise, higher-order poles cannot occur, because near such poles $\operatorname{Im}[f(s)]$ changes sign several times and, therefore, condition (8) is certain to be violated. Therefore, all the poles of $f(s)$ are simple, are located on the real axis, and have positive residues. Thus $f(s)$ has the representation

$$f(s)=\sum_n\frac{B_n}{s_n-s} \; , \;\; \text{with } B_n\geq0 \; , \tag{9}$$

where the magnitudes B_n of the residues and the locations s_n of the poles reflect the pore structure of the rock. From the rigorous requirement that ϵ be real and positive whenever ϵ_r and ϵ_w are real and positive, it follows that the poles s_n are restricted to the interval $0\leq s_n\leq1$, and that $f(1)$ obeys the inequality

$$f(1)\leq1 \; . \tag{10}$$

In any composite with a random geometry, all the poles, except the one at the end point $s=0$, are expected to be broadened into a branch cut along the real axis. The pole at $s=0$ retains its identity and is best treated separately. Hence Eq. (6) needs to be replaced by the integral representation[7,11]

$$f(s)=\frac{A}{s}+\int_0^1 \frac{g(s')}{s-s'}ds' , \tag{11}$$

where the density of resonances, $g(s')$, takes positive real values.

The constant A appearing in Eq. (11) has a simple interpretation.[7] In the limit of very low frequencies, both ϵ and ϵ_w are dominated by their imaginary parts in a typical brine-saturated rock, while ϵ_r remains constant. Hence s is small and $sf(s)$, as given by Eqs. (5) and (6), approaches the ratio σ_0/σ_w of the static (dc) saturated rock conductivity σ_0 to the brine conductivity σ_w. Since in this limit the value of $f(s)$ is dominated by the contribution of the pole at $s=0$, we can, from Eq. (11), identify A with $sf(s)$. Thus A takes the simple form

$$A=\sigma_0/\sigma_w=\phi^m , \tag{12}$$

where the last equality follows from Archie's empirical "law," which states that the static conductivity of a brine-saturated porous rock varies as a power law of the porosity ϕ (ϕ is the volume fraction occupied by the pore structure).[14] Archie's law is known to be approximately valid in many brine-saturated rocks. The exponent m typically has values in the range 1.5–4, depending on rock type.[2,3]

The constraints that the density function $g(s')$ must satisfy are not completely known. When the composite is nearly homogeneous, the electrostatic fields are almost uniform and a perturbation expansion for $f(s)$ in powers of s can be developed. From the results thus obtained by Brown[15] (see also Herring[16]), it follows that $f(s)$ has the expansion

$$f(s)=\frac{\phi}{s}+\frac{\phi(1-\phi)}{3s^2}+O(1/s^3) \tag{13}$$

to second order in $1/s$. Hence by expanding (11) in powers of $1/s$ and equating terms with (13), we obtain the two sum rules[7,11]

$$\int_0^1 g(s')ds'=\phi-A , \tag{14}$$

$$\int_0^1 s'g(s')ds'=\tfrac{1}{3}\phi(1-\phi) . \tag{15}$$

From (10) we also have the inequality constraint

$$\int_0^1 \frac{g(s')}{1-s'}ds'\leq 1-A . \tag{16}$$

Some additional subtle inequalities on the function have been obtained by Schulgasser[17] and by one of us.[18] We choose, however, to ignore these inequalities, because they impose rather mild constraints on $g(s')$. Inequality (16), however, is important and will be included.

We have already noted that low frequencies, $\omega \ll \sigma_w/\epsilon_0\epsilon_r'$, correspond to small values of s. This feature is more clearly illustrated in Fig. 1, where the values of s are plotted for sandstone saturated with two types of brine: one very saline, the other moderately saline (and thus not as conducting). The low-frequency dielectric enhancement in these materials becomes especially important at frequencies below 1000 MHz in the former and below 100 MHz in the latter (see Fig. 5), corresponding in both cases to values of s in the range 0.0–0.1. Thus, measurements of the dielectric enhancement in the range of frequencies below 1000 MHz provide a sensitive tool for probing the local variations of the spectral function $g(s')$ in the vicinity of $s'\cong 0.01$. By contrast, the sum rules (14) and (15) and the inequality constraint (16) only provide global restrictions on the shape of $g(s')$ over the entire range $0\leq s'\leq 1$.

Consequently, in order to determine $g(s')$, we need to make additional assumptions about the form of the density function beyond the integral constraints noted above. Ideally, $g(s')$ would be determined by some microscopic model which incorporates information about the rock geometry to give a composite dielectric function or a mixing law. Rather than seeking such a model, we attempt to produce a form for $g(s')$ which is compatible with the observed dielectric enhancement. We begin by investigating several simple choices of $g(s')$ to gain a qualitative idea of how the shape of this function affects the low-frequency dielectric enhancement. We then offer a form which seems to fit the observed data. Note, however, that the dielectric properties will depend principally on $g(s')$ at relatively small s', and will be insensitive to any assumptions made about $g(s')$ at large values of this argument.

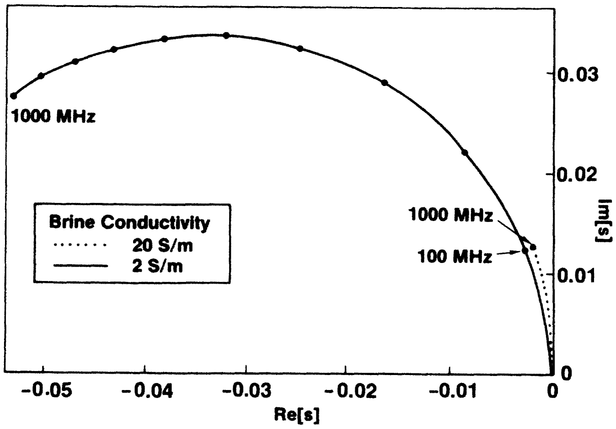

FIG. 1. Trajectories of the variable $s=1/(1-\epsilon_w/\epsilon_r)$ as a function of frequency for a sandstone saturated with different brines in the frequency range 1–1000 MHz. These trajectories show the range of the variable s probed by dielectric measurements.

III. SIMPLE APPROXIMATIONS FOR THE DENSITY FUNCTION

The existence of a low-frequency dielectric enhancement can be shown to imply a nonzero density function $g(s')$ at small values of s'. This result can be readily understood by considering formula (11) at low frequencies where $s\cong i\omega\epsilon_0\epsilon_r'/\sigma_w$; if $g(s')$ vanishes below some small-s' cutoff s_0, it is easily seen that ϵ' must saturate at some maximum value for frequencies corresponding to small s'. At the opposite extreme of large values of s', near $s'=1$, the inequality constraint (16) implies that $g(s')\rightarrow 0$ as

$s'\to 1$. Thus, any choice of $g(s')$ which is to produce dielectric enhancement must exhibit both these features.

To see how the form of $g(s')$ influences the low-frequency dielectric enhancement, we consider first the simple linear choice

$$g(s')=C(1-s')\ , \tag{17}$$

which is nonzero at small values of s' and goes to zero at $s'=1$ as required. The second sum rule (15) implies that the constant C must take the positive value

$$C=2\phi(1-\phi)\ . \tag{18}$$

Substituting this formula back into the first sum rule (14) gives the expression

$$A=\phi^2 \tag{19}$$

for the dc conductivity ratio σ_0/σ_w. Thus the form (17) directly implies Archie's law (12) with an exponent $m=2$ (which is typical of many rocks). However, this feature of the approximation should not be regarded as significant, because it is unlikely that such a simple density of resonances as (17) would be characteristic of real rocks.

Substituting these expressions into the integral representation (11) for $f(s)$ gives

$$f(s)=\frac{\phi^2}{s}+1-(1-s)\ln\left[1-\frac{1}{s}\right]\ , \tag{20}$$

which implies, via Eqs. (5) and (6), that

$$\epsilon=\phi^2\epsilon_w+(1-\phi)^2\epsilon_r+2\phi(1-\phi)\left[\frac{1}{\epsilon_r}-\frac{1}{\epsilon_w}\right]\ln\left[\frac{\epsilon_w}{\epsilon_r}\right] \tag{21}$$

is the complex dielectric function corresponding to the choice (17) of $g(s')$. The formula is clearly invariant under the interchange $\epsilon_r\to\epsilon_w$, $\epsilon_w\to\epsilon_r$, $\phi\to 1-\phi$, implying that the approximation treats the rock matrix and the pore space geometries on a symmetric basis. However, this approximation turns out to be unsatisfactory when compared with experimental data because the low-frequency dielectric enhancement predicted by (21) is too low (see Fig. 2).

Next, we consider an approximation which is found empirically to work remarkably well at high frequencies (near 1000 MHz). This is the so-called complex refractive index method (CRIM), given by

$$\sqrt{\epsilon}=\phi\sqrt{\epsilon_w}+(1-\phi)\sqrt{\epsilon_r}\ , \tag{22}$$

which has been discussed extensively in a variety of contexts.[1,19] Equation (22) has the equivalent form

$$\epsilon=\phi^2\epsilon_w+(1-\phi)^2\epsilon_r+2\phi(1-\phi)\sqrt{\epsilon_r\epsilon_w} \tag{23}$$

obtained by squaring (22). Like the approximation (21), the complex-refractive-index mixing formula treats both rock and pore space geometries on a symmetric basis and gives the expression (19) for the dc conductivity, which is compatible with Archie's law.

To deduce the spectral function $g(s')$ corresponding to this mixing formula, we first consider the more general problem of determining $g(s')$ given any function

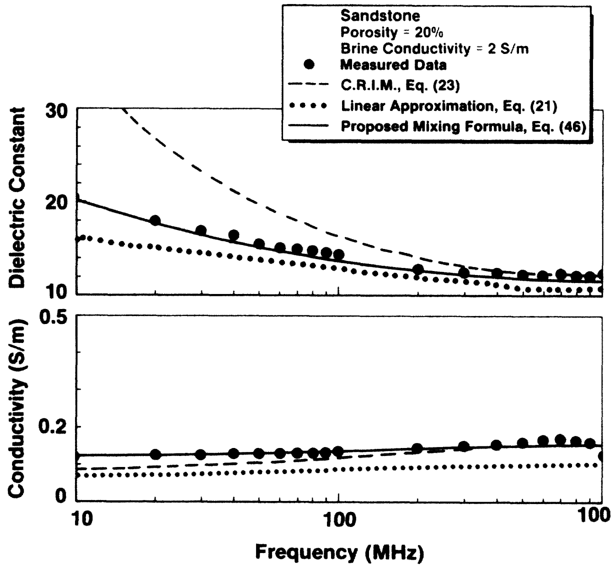

FIG. 2. Measured data on brine-saturated sandstone compared with the results of various theoretical models.

$$h(s)=\int_0^1\frac{g(s')}{s-s'}ds'\ . \tag{24}$$

Following standard analysis, we focus on values of s near the branch cut of $h(s)$. Substituting $s=a+i\delta$, where $0<a<1$ and $\delta\ll 1$, we obtain

$$\mathrm{Im}[h(a+i\delta)]=\mathrm{Im}\left[\int_0^1\frac{(a-i\delta-s')g(s')}{(a-s')^2+\delta^2}ds'\right]\ . \tag{25}$$

Since the integrand is very sharply peaked near $s'=a$, we can replace $g(s')$ by $g(a)$, extend the range of integration over the entire real axis, and change the variable s' to $u=(a-s')/\delta$ to obtain the estimate

$$\mathrm{Im}[h(a+i\delta)]\cong -g(a)\int_{-\infty}^{\infty}\frac{1}{1+u^2}du=-\pi g(a)\ , \tag{26}$$

which becomes asymptotically exact in the limit $\delta\to 0$, giving the Stieltjes inversion formula[20]

$$g(s')=-\frac{1}{\pi}\lim_{\delta\to 0}\mathrm{Im}[h(s'+i\delta)]\ . \tag{27}$$

We next apply Eq. (27) to the present case. From the complex-refractive-index method (23) and the representation (11), together with (19), we have

$$h(s)=2\phi(1-\phi)\left[1-\left[1-\frac{1}{s}\right]^{1/2}\right]\ . \tag{28}$$

Substituting this expression into (27) gives the expression

$$g(s')=C(s')^{-1/2}(1-s')^{1/2} \tag{29}$$

for the density function, where s' is real and

$$C=2\phi(1-\phi)/\pi\ . \tag{30}$$

Thus $g(s')$ diverges as $s'\to 0$, and consequently the low-frequency dielectric enhancement is stronger than that associated with the simple linear form for $g(s')$. In fact, as demonstrated in Fig. 2, the enhancement predicted by the complex-refractive-index mixing law is *too* strong at low frequencies, when compared with experiment.

One other feature of this mixing law is unsatisfactory. Although $g(s')$ given by the above equations satisfies the first sum rule (14), it fails to satisfy the second. Specifically, the moment

$$\int_0^1 s' g(s')ds' = \tfrac{1}{4}\phi(1-\phi) \tag{31}$$

is too small when compared with (15). Clearly, to rectify this discrepancy we need to shift some of the weight of the density function to higher values of s', thereby reducing the low-frequency dielectric enhancement. This is precisely what is needed to improve agreement with experiment.

IV. PROPOSED MIXING FORMULA

It is clear from this formulation that any dielectric enhancement in the composite must come from the function $g(s')$. This function depends on the pore structure in a complicated way which must be difficult to determine from any microscopic theory. The dielectric enhancement must be associated with long relaxation times for the electric currents to equilibrate once a uniform electric field is established across the sample.[6] Such long relaxation times can result from many geometric effects, including the presence of platelike grains, as has been suggested by Sen,[5] or needle-shaped pores as suggested by Lysne.[6] Another possibility is a pore structure which is inhomogeneous on many length scales, giving it a fractal or self-similar character.

Since the CRIM fares well at high ω, it seems natural to consider a form for $g(s')$ with a smooth crossover in analytic behavior at low frequencies from, e.g., a dependence characteristic of the complex-refractive-index method for large s to a linear dependence such as (17) for small s. We have not succeeded in finding such a function for which the integral (11), giving $f(s)$, can be easily approximated at the frequencies of interest. Instead, we propose a simple form for $g(s')$ which gives many of the features of the observed dielectric enhancement and of the frequency dependence of the conductivity, though we have no microscopic justification for the form at present.

Motivated by formula (29) for the density function of the complex-refractive-index method, we propose the following form for $g(s')$:

$$g(s') = C(s')^{-b}(1-s')^{e} , \tag{32}$$

where C, b, and e are parameters which may be determined as follows. By substituting (32) into (14), (15), and (16) and using the well-known identity[21]

$$\int_0^1 (s')^{\alpha-1}(1-s')^{\beta-1}ds' = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta) \tag{33}$$

for expressing the resulting integrals in terms of the gamma function $\Gamma(x)$, we obtain the three equations

$$C\Gamma(1-b)\Gamma(1+e)/\Gamma(2-b+e) = \phi - A , \tag{34}$$

$$C\Gamma(2-b)\Gamma(1+e)/\Gamma(3-b+e) = \phi(1-\phi)/3 , \tag{35}$$

$$C\Gamma(1-b)\Gamma(e)/\Gamma(1-b+e) \leqq 1 - A , \tag{36}$$

which must be satisfied by the three unknowns C, b, and e. The inequality (36) is not useful unless it can be replaced by an equality. Hence we make the additional assumption that (36) can be treated as an equality, or equivalently that $f(s)=1$ when $s=1$. Physically, this assumption implies, via Eqs. (5) and (6), that a "conjugate rock" obtained by interchanging rock and brine in the saturated sample would be poorly conducting. This should be a reasonable approximation if the rock grains have a small contact area in relation to their cross section. (There must, of course, be many such points of contact to give the observed rigidity to the rock.)

By supposing that (36) is an equality, and noting that the gamma function satisfies $\Gamma(1+x) = x\Gamma(x)$, we can take ratios of the preceding equations to obtain the identities

$$(1-b)/(2-b+e) = \tfrac{1}{3}\phi(1-\phi)/(\phi - A) , \tag{37}$$

$$(1-b+e)/e = (1-A)/(\phi - A) , \tag{38}$$

which have the solution

$$b = 1 - \phi(1-\phi)/[2\phi - A(3-\phi)] , \tag{39}$$

$$e = \phi(\phi - A)/[2\phi - A(3-\phi)] , \tag{40}$$

for the exponents b and e. Typical values of these exponents are given in Table I for $A=\phi^2$, corresponding to Archie's law with $m=2$. Note that both the exponents take on reasonable, positive values over the entire range of

TABLE I. Typical values of exponents b and e with $m=2$.

ϕ	e	b	ϕ	e	b
0.05	0.0256	0.4872	0.55	0.3793	0.3103
0.1	0.0526	0.4737	0.6	0.4286	0.2857
0.15	0.0811	0.4595	0.65	0.4815	0.2593
0.2	0.1111	0.4444	0.7	0.5385	0.2308
0.25	0.1429	0.4286	0.75	0.6000	0.2000
0.3	0.1765	0.4118	0.8	0.6667	0.1667
0.35	0.2121	0.3939	0.85	0.7391	0.1304
0.4	0.2500	0.3750	0.9	0.8182	0.0909
0.45	0.2903	0.3548	0.95	0.9048	0.0476
0.5	0.3333	0.3333	1.0	1.00	0.00

porosity. At lower m the exponent b is sometimes negative, and then the model predicts no enhancement in the low-frequency dielectric constant.

Given b and e, the constant C can be computed from any of the formulas (34)–(36). The function $g(s')$ is then completely determined. In Fig. 3 we have compared the resulting density function with that obtained from the complex-refractive-index method. As expected, some of the weight of the spectral function has been shifted to larger values of s'. In Fig. 4 the dependence of $g(s')$ on A is explored by varying Archie's exponent m in Eq. (12).

Digressing a little, if we invoke the view of a pore structure that is inhomogeneous on many length scales, giving it a fractal character, then the real part $\epsilon'(\omega)$ of the complex dielectric permittivity $\epsilon(\omega)$ might be expected to vary as ω^{-b} at sufficiently low frequencies ω, where the constant b is ideally a "critical exponent" taking the same universal value for all rocks in the same broad structural group.[22] Experimental difficulties in obtaining accurate data for $\epsilon'(\omega)$ at low frequencies currently hamper the testing of such universality in the exponent b. In the theory of phase transitions in statistical physics,[23] the approach to the critical point (in this case at $\omega=0$) is first typically characterized by mean-field exponents b having integer or half-integer values. Close to criticality a smooth crossover occurs to a new exponent b, the critical exponent, which is often not a low-order fraction (p/q with q a small integer). This analogy may explain why the complex-refractive-index method works relatively well at high frequencies: it is characterized by a mean-field exponent, $b=\frac{1}{2}$. With this interpretation, the departure from the complex-refractive-index method at lower frequencies signifies the start of the crossover to a new exponent.

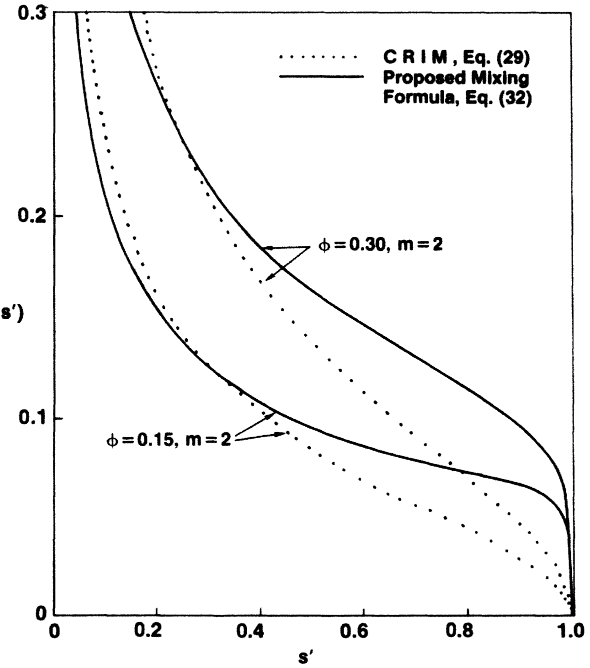

FIG. 3. Density function $g(s')$ for two approximation methods at different values of the porosity.

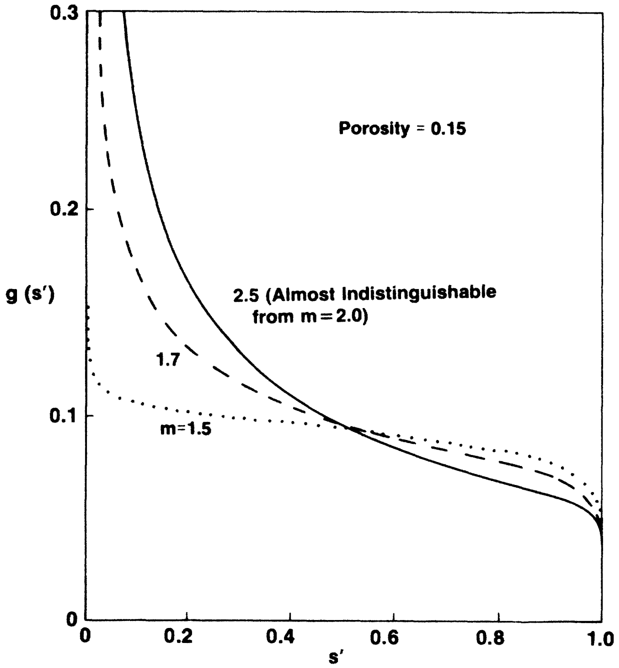

FIG. 4. Dependence of $g(s')$ on Archie's exponent m. Note that the dielectric enhancement at low frequencies is greater for larger m.

V. ASYMPTOTIC BEHAVIOR

To obtain the effective dielectric permittivity corresponding to the new mixing law, we need to evaluate the integral

$$h(s)=\int_0^1 \frac{g(s')}{s-s'}ds' =C\int_0^1 \frac{(s')^{-b}(1-s')^e}{s-s'}ds' , \tag{41}$$

which is, in fact, related to an integral representation of the hypergeometric function $F(a,b,c;d)$; the exact relation is

$$h(s)=C\frac{\Gamma(1-b)\Gamma(1+e)}{s\Gamma(2+e-b)}F(1,1-b,2+e-b;1/s) . \tag{42}$$

A further identity, namely that

$$h(s)=\frac{e}{b}(1-A)F(1,b-e,1+b;s) -C\pi(-s)^{-b}(1-s)^e/\sin(b\pi) , \tag{43}$$

follows from known properties of hypergeometric functions and from Eqs. (34)–(36) relating C to b and e. With the use of a standard series expansion, this becomes

$$h(s)=\frac{e}{b}(1-A)\left[1+\frac{b-e}{1+b}s+\frac{(b-e)(b-e+1)}{(1+b)(2+b)}s^2+\cdots\right] -C\pi(-s)^{-b}(1-s)^e/\sin(b\pi) , \tag{44}$$

which is rapidly convergent for $|s| \ll 1$, and when b is not too close to a negative integer, corresponding to most cases of interest. In fact, it usually suffices to keep only the leading term, and we thereby obtain

$$h(s) \cong \frac{e}{b}(1-A) - C\pi(-s)^{-b}(1-s)^{e}/\sin(b\pi) \; . \tag{45}$$

The effective dielectric constant ϵ of the composite is obtained from either (44) or (45) via the relation

$$\epsilon = A\epsilon_w + (1-A)\epsilon_r - \epsilon_r h(s) \; , \tag{46}$$

where we have used (11) and the definitions (5) and (6). The constants b, e, and C can be computed from (34)–(36), (39), and (40). The constant A can be varied to fit the data, or better still, it can be determined from measurements of the ratio σ_0/σ_w in the low-frequency limit [see Eq. (12)]. This gives a mixing formula valid at finite frequencies entirely in terms of exact sum rules and "zero-frequency" measurements.

Direct numerical evaluations of the integral formula (41) are easy to perform for all frequencies of interest, and provide a check on the approximation formula (45).

It is instructive to examine the asymptotic behavior of the complex dielectric constant. At sufficiently low frequencies, it suffices to write

$$\epsilon_w - \epsilon_r \cong \epsilon_w \cong i\sigma_w/\epsilon_0\omega \; , \tag{47}$$

and to use Eq. (46) with the estimate (45) for $h(s)$. If we now write ϵ in the form (3), ϵ' and σ are found to have the asymptotic behaviors

$$\epsilon' = \epsilon_r^{1-b}(\sigma_w/\epsilon_0)^{b}[C\pi/2\sin(b\pi/2)]\omega^{-b} \; , \tag{48}$$

$$\sigma = A\sigma_w + \epsilon_r^{1-b}\sigma_w^{b}(\epsilon_0\omega)^{1-b} \; , \tag{49}$$

which are typical of brine-saturated rocks at low frequencies.

One other asymptotic limit deserves mentioning. It is well known from the theory of classical dielectrics that the conductivity ratio A satisfies the inequality

$$A \leqq 2\phi/(3-\phi) \; . \tag{50}$$

This inequality is just a special case of a well-known theorem due to Hashin and Shtrikman.[24] The theorem states that the conductivity of an isotropic two-component composite lies between two extreme cases obtained from the Clausius-Mossotti (or Maxwell Garnett) approximation by alternately regarding each of the two components as the host material. In the present case, a very simple result is obtained when the inequality (50) is approached as an equality. Letting $A = 2\phi/(3-\phi) - \delta$, where $\delta \ll 1$, we readily find from Eqs. (39) and (40) that in this limit b becomes large and negative, while e becomes large and positive and the ratio b/e approaches a constant. The density function $g(s')$ therefore becomes very sharply peaked, and may indeed be regarded as a δ function, at

$$s_0 = 1/(1-e/b) = 1-\phi/3 \; . \tag{51}$$

The weight of the δ function may be determined from any of the equations (34)–(36), and the function $f(s)$ [Eq. (11)] finally takes the form

$$f(s) = \frac{A}{s} + \frac{\phi - A}{s - s_0} \quad \text{where } A = 2\phi/(3-\phi) \; , \tag{52}$$

and s_0 is given by (51). The corresponding dielectric constant is readily found to be

$$\epsilon = \epsilon_w \left[1 + \frac{(1-\phi)(1-\epsilon_w/\epsilon_r)}{1-(1-\phi/3)(1-\epsilon_w/\epsilon_r)} \right] , \tag{53}$$

which is just the Clausius-Mossotti (or Maxwell Garnett) expression with the *conductor* (i.e., the brine) regarded as the host and the rock as the embedded material. Thus the proposed new mixing formula reduces to the Clausius-Mossotti expression, as it should, in the limit where the dc conductivity approaches the Clausius-Mossotti limit. Note that since b is large and negative in this regime, the mixing formula predicts that there will be no dielectric enhancement at low frequencies in such composites.

VI. COMPARISON WITH EXPERIMENT

We have compared the predictions from the proposed mixing formula, Eq. (46), with a variety of data on brine-saturated rocks: data on sandstone (range of brine salinities, same porosity) and limestone (range of porosities, same brine salinity) obtained at Chevron by a method described elsewhere,[13] and data on whitestone, disaggregated whitestone, and disaggregated marble obtained by Kenyon.[3] These data are presented in Figs. 5–8, where other relevant parameters are also presented. In general, the accuracy of the measurements degrades at lower frequencies and higher salinities. The error bars in the Kenyon data in Fig. 7 are typical of the expected errors in the data presented.

The dielectric permittivity ϵ_w' for the brine in our theoretical calculations was corrected for the effects of temperature T (=75 °F) and the salinity X (in kppm, or kiloparts per million) using the following empirical relations:[25,26]

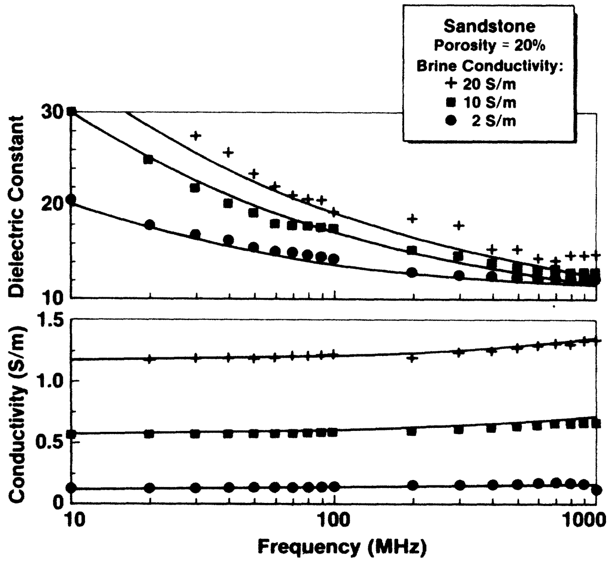

FIG. 5. Data on brine-saturated sandstone for various brine conductivities compared with theoretical predictions (solid lines). The value of ϵ_r' used is 4.65.

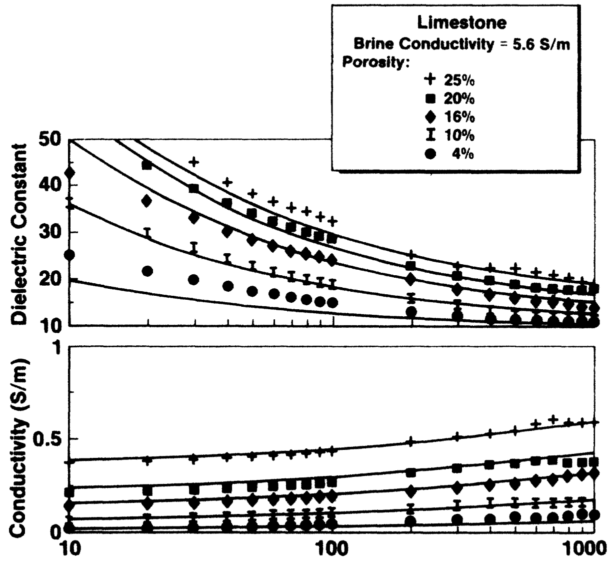

FIG. 6. Data on brine-saturated limestone for various porosities compared with theoretical predictions (solid lines). The value of ϵ_r' used is 9.0.

$$\epsilon_w'(0,T)=94.88-0.2317T+0.000217T^2\,,$$

$$\epsilon_w'(X,T)=\left[\frac{1}{\epsilon_w'(0,T)}+\frac{2.4372X}{58.443(1000-X)}\right]^{-1},$$

with the relationship between the brine conductivity σ_w and the salinity X being also an empirical one,[27]

$$\sigma_w=\frac{T+7}{82}\left[0.0123+\frac{3647.5}{(1000X)^{0.955}}\right]^{-1}. \tag{54}$$

One of the input parameters in the theoretical calculations is the static or dc rock conductivity σ_0. In our calculations we have taken this to be equal to the measured conductivity for each sample at the lowest frequency of measurement.

Figures 5—8 show the results of the theoretical calculations based on Eq. (46). In all cases the series expansion (44) and the direct numerical evaluation of the integral (41) give identical answers.

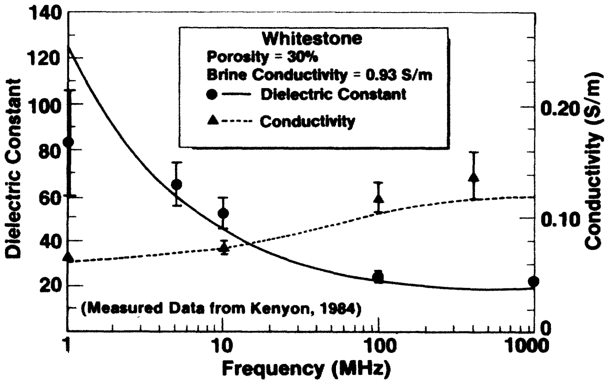

FIG. 7. Kenyon's data on brine-saturated whitestone compared with theoretical predictions (lines) of this paper. The value of ϵ_r' used is 7.5.

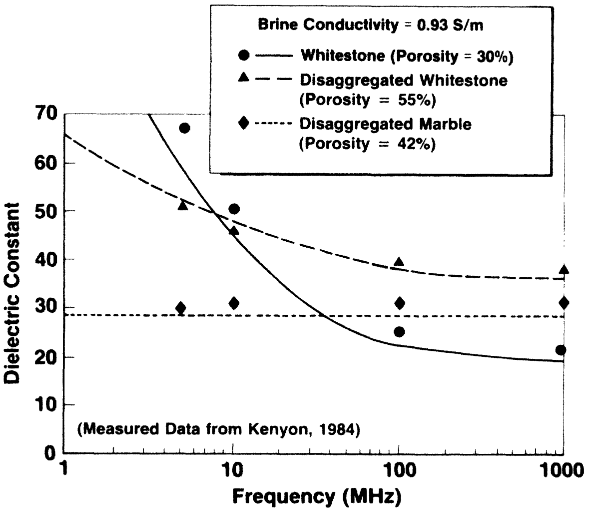

FIG. 8. Kenyon's data on brine-saturated rocks for three different rock types compared with theoretical predictions (lines) of this paper. The value of ϵ_r' used is 7.5.

Several features stand out in all these curves. In every case there is a significant dielectric enhancement at low frequencies, but some rocks show much more enhancement than others. There is also a characteristic frequency dependence of the conductivity, which invariably increases from its low-frequency value to a somewhat higher value in the vicinity of 1 GHz. Overall, both the degree and the frequency dependence of the dielectric enhancement and the frequency dependence of the conductivity are well reproduced by the theoretical calculations. This is particularly satisfying when it is realized that the theory is constructed on the basis of static quantities: The input parameters are the porosity and the static conductivity. Note, for example, that in the Kenyon data the three samples shown exhibit, respectively, strong, modest, and very weak dielectric enhancement—a trend that is predicted by the theory on the basis of static quantities only.

The predicted dielectric enhancement at low frequencies is very sensitive to the assumed static conductivity of the composite. The reason for this sensitivity is that the static conductivity enters into one of the sum rules determining $g(s')$ [see Eq. (14)] and hence has a strong effect on the exponent b which determines the dielectric dispersion at low frequencies. To test this sensitivity, we have recalculated some of the results of the sandstone assuming two slightly different values of σ_0 (with difference of about 10%). This produces a change in the predicted dielectric constants of about 30% at 10 MHz, though much less at higher frequencies.

VII. DISCUSSION

We have presented in this paper a simple mixing formula for calculating the dielectric constant and conductivity of brine-saturated porous rocks as a function of frequency. This mixing formula is derived from purely static inputs—the porosity of the rock, the dielectric properties of both rock and brine, and the dc conductivity of the rock. The mixing formula is designed to satisfy the known analytic constraints and sum rules on the complex dielectric response of a composite medium. In this respect it is superior to the complex-refractive-index method, which works well only in the high-frequency range.

The fact that our mixing formula works well on a wide variety of samples suggests that it includes in some way the relevant feature of the pore geometry. Simply satisfying the analytic constraints on the dielectric function is not enough to ensure agreement with experiment. An explicit microscopic model which starts from the pore geometry and leads to the proposed mixing law is still lacking. While it would be of value to have such a model, its absence does not make the mixing formula any less useful.

Further work is needed in several directions. More detailed tests of the model are still needed. An extension to where both brine and oil are present in the pore space would be the ultimate practical goal, though considerably more difficult than dealing with the present case. This task is now being attempted. And finally, as noted above, it would be of interest to have a microscopic model that would lead to the proposed mixing formula, so that one could have an explicit picture of the geometrical effects which produce the enhancement.

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[1] R. N. Rau and R. P. Wharton, J. Pet. Tech. **34**, 2689 (1982).

[2] P. N. Sen, C. Scala, and M. H. Cohen, Geophysics **46**, 781 (1984).

[3] W. E. Kenyon, J. Appl. Phys. **55**, 3153 (1984).

[4] W. R. Berry II, M. P. Head, and M. L. Mougne, Aust. Pet. Assoc. J. **19**, 142 (1979).

[5] P. N. Sen, Geophysics **49**, 586 (1984).

[6] P. C. Lysne, Geophysics **48**, 775 (1983).

[7] J. Korringa, Geophysics **49**, 1760 (1984).

[8] J. Korringa and G. A. LaTorraca, J. Appl. Phys. (to be published).

[9] D. J. Bergman, in *Electrical Transport and Optical Properties of Inhomogeneous Media,* AIP Conf. Proc. No. 40, edited by J. C. Garland and D. B. Tanner (AIP, New York, 1978), p. 46.

[10] D. J. Bergman, Phys. Rev. C **43**, 377 (1978).

[11] D. J. Bergman, Ann. Phys. (N.Y.) **138**, 78 (1982).

[12] J. P. Poley, J. J. Nooteboom, and P. J. deWaal, Log Anal. 8 (1978).

[13] B. R. De, Log Anal. **27**, 63 (1986).

[14] G. E. Archie, Trans. AIME **146**, 54 (1942). See also the discussion in Ref. 2.

[15] W. F. Brown, J. Chem. Phys. **23**, 1514 (1955).

[16] C. Herring, J. Appl. Phys. **31**, 1939 (1960).

[17] K. Schulgasser, J. Math. Phys. **17**, 378 (1976).

[18] G. W. Milton, in *Physics and Chemistry of Porous Media,* AIP Conf. Proc. No. 107, edited by D. L. Johnson and P. N. Sen (AIP, New York, 1984), p. 66.

[19] J. R. Birchak, C. G. Gardner, J. E. Hipp, and J. M. Victor, Proc. IEEE **62**, 93 (1974).

[20] H. S. Wall, *Analytic Theory of Continued Fractions* (Van Nostrand, New York, 1948), p. 250.

[21] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1980).

[22] For a general discussion of percolation in other contexts, see, e.g., S. Kirkpatrick, Rev. Mod. Phys. **45**, 574 (1973).

[23] See, e.g., H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford University Press, New York, 1971).

[24] Z. Hashin and S. Shtrikman, J. Appl. Phys. **33**, 3125 (1962).

[25] R. P. Wharton, G. A. Hazen, R. N. Rau, and D. L. Best, Soc. Pet. Eng. Paper No. 9267 (1981) (unpublished).

[26] H. C. Helgeson, D. H. Kirkham, and G. C. Flowers, Am. J. Sci. **281**, 1298 (1981).

[27] *Log Interpretation Charts* (Dresser Atlas, Houston, 1983).