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Eigenfunctions of the small oscillations about the double-sine-Gordon kink

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We investigate the spectrum of the small oscillations about the 4π double-sine-Gordon kink. Our method makes use of the supersymmetry which naturally arises in any one-dimensional Schrödinger equation for which the ground-state wave function is exactly known. We provide for what we believe to be the first time analytical expressions for the unnormalized eigenfunctions of the internal and continuum modes. Our result improves previous numerical investigations. Its relevance for the dynamics and thermodynamics of kinks is discussed.

One of the most attractive ideas in modern physics is that of supersymmetry.[1] Besides being a possible vehicle for making realistic phenomenological models in particle and condensed-matter physics, this symmetry has also been exploited as a mathematical tool to solve double-well Schrödinger potentials.[2] Supersymmetric quantum mechanics,[3] in fact, provides, among other things, an intuitive and appealing version of the old factorization and Darboux transform methods.[4]

In this Rapid Communication, we make use of supersymmetry to find the unnormalized eigenfunctions of a one-dimensional Schrödinger equation whose potential is changed from a single-well to a double-well shape as a parameter R is varied between zero and infinity; namely, we look for the solution of the eigenvalue problem defined by

$$Hu \equiv \left[-\frac{d^2}{dx^2} + U(R,x) \right] u = \omega^2 u \tag{1a}$$

with

$$U(R,x) = \tanh^2 R \left[2 \left(\frac{\cosh^2 R - \sinh^2 x}{\cosh^2 R + \sinh^2 x} \right)^2 - 1 \right] - \operatorname{sech}^2 R \frac{\cosh^2 R - \sinh^2 x}{\cosh^2 R + \sinh^2 x} . \tag{1b}$$

Equation (1) arises in the study of the small oscillations about the 4π kink of the double sine-Gordon (DSG) model.[5] This theory has been shown to be relevant in the study of antiferromagnetic chains,[6] in describing experimentally accessible systems such as $CsNiF_3$,[7] $(CH_3)_4$- $NMnCl_3$ (TMMC),[8] or CoBP,[4] and could reveal very interesting features in the study of domain walls.[9] Recently, it has been successfully applied to study the diffraction satellites on the reconstructed Au(111) surfaces.[10] In all of the above applications the parameter R is obviously finite and depends upon the value of external parameters such as pressure or applied and induced fields. In relativistic field theory, when coupled with fermions, the DSG theory provides an extended hadron model[11] which somewhat unifies the views of the MIT and SLAC bag theories.[12,13] In this application, the value of the parameter R is self-consistently determined from the quark pressure.

The relevance of these applications prompted earlier[14] investigations of Eq. (1). These studies showed the existence of a bound state whose eigenvalue depends upon the parameter R and investigated the relevance of this mode on the dynamics[5] and thermodynamics[15] of the kink sector of the DSG model. However, analytical expressions for the eigenvalue and the eigenfunction of the bound state are available only in the Wentzel-Kramers-Brillouin approximation which is valid for large values of R. Furthermore, no explicit form of the continuum eigenfunctions —which are needed to describe the interaction of the radiation field with the DSG kink—has been provided so far. Knowledge of both continuum and bound modes is, however necessary for all values of R in order to achieve a proper theoretical understanding of the remarkable dynamical and thermodynamical properties of the DSG model.

It is the aim of this paper to help in bridging this gap by providing for what we believe to be the first time approximate analytical expressions for the unnormalized eigenfunctions of Eq. (1) for all values of the parameter R.

Our method makes use of the ground-state wave-function representation[16] to associate with the double-well Hamiltonian (1) a supersymmetric partner H_+. Supersymmetry then guarantees that the two Hamiltonians are isospectral and induces an orthogonality preserving[17] one-to-one mapping between their eigenfunctions. Although H_+ describes a simple single-well Schrödinger problem, it is not amenable to an easy exact solution. Consequently, we have replaced H_+ by a symmetric Pöschl-Teller Hamiltonian,[18,19] H_{PT}, for which a complete set of eigenfunctions is known. The area and the depth of the Pöschl-Teller potential are chosen to be equal, for any value of R, to the area and the depth of the potential in H_+. This requirement determines the values of the parameters U_0 and α appearing in Eq. (10). The eigenfunctions of Eq. (1) are then obtained—via supersymmetry—from the eigenfunctions of the Pöschl-Teller Hamiltonian. At this stage, the only justification for the replacement lies in the remarkable agreement between the output of our analytical computation and the result of previous investigations[5,14] which relied on the use of numerical methods to obtain the eigenvalue spectrum and the eigenfunctions of Eq. (1), for all values of R.

Our starting point is the observation[5] that Eq. (1) admits for any finite R an exact nondegenerate nodeless ground state whose unnormalized eigenfunction is given by

$$\psi_0 = \frac{4\cosh x \cosh R}{\cosh^2 R + \sinh^2 x} . \tag{2}$$

Following a standard procedure,[16] we define

$$\psi_0 = e^{-V} . \tag{3}$$

Inserting (3) into (1) we have ($E_0 = 0\ \forall R$)

$$U(R,x) = \left(\frac{\partial V}{\partial x}\right)^2 - \frac{\partial^2 V}{\partial x^2} . \tag{4}$$

This is basically another way to write the potential (1) once the exact ground-state eigenfunction is known; this is a very useful form of Eq. (1) since it allows the use of supersymmetry as a tool to reduce the complexity of the original Schrödinger problem. Namely, we introduce two Hamiltonians $H_\pm$ ($H_- \equiv H$)

$$H_\pm \equiv \left[-\frac{d^2}{dx^2} + \left(\frac{\partial V}{\partial x}\right)^2 \pm \frac{\partial^2 V}{\partial x^2} \right] \tag{5}$$

which are supersymmetric partners.[3] The eigenfunctions of H_- are obtained by applying the supersymmetric "raising operator"

$$Q = \left(-\frac{d}{dx} + \frac{\partial V}{\partial x} \right) \tag{6}$$

to the set of eigenfunctions of H_+.[3]

In our case, use of Eqs. (2) and (3) leads to

$$\frac{\partial V}{\partial x} = \tanh(x+R) + \tanh(x-R) - \tanh x , \tag{7}$$

which in turn implies that

$$H_- = -\frac{d^2}{dx^2} + 1 - \frac{2}{\cosh^2(x-R)} - \frac{2}{\cosh^2(x+R)} + \frac{2}{\cosh^2 R + \sinh^2 x} , \tag{8a}$$

and

$$H_+ = -\frac{d^2}{dx^2} + 1 - \frac{2}{\cosh^2 x} + \frac{2}{\sinh^2 R + \cosh^2 x} . \tag{8b}$$

We now replace (8b) by a symmetric Pöschl-Teller Hamiltonian of the form

$$H_{\rm PT} = -\frac{d^2}{dx^2} + 1 - \frac{U_0}{\cosh^2(\alpha x)} , \tag{9}$$

with

$$U_0 = 2\tanh^2 R , \tag{10a}$$

and

$$\alpha = \tanh^2 R \frac{\sinh(2R)}{\sinh(2R) - 2R} . \tag{10b}$$

Then, we solve the eigenvalue problem

$$H_{\rm PT} v = \omega^2 v .$$

We find it most convenient to use Landau's notation,[20] which allows us to write the eigenvalue problem for $H_{\rm PT}$ in the form

$$\left(\frac{d^2}{dx^2} \pm K^2 + \frac{\alpha^2 s(s+1)}{\cosh^2(\alpha x)} \right) v = 0 , \tag{11}$$

where

$$s = \tfrac{1}{2}\left(-1 + \sqrt{1 + (4U_0/\alpha^2)} \right) , \tag{12a}$$

and

$$\pm K^2 = \omega^2 - 1 \tag{12b}$$

with the $\pm$ sign referring to continuum- and bound-state solutions, respectively.

The spectrum of (11) consists of a continuum starting at $\omega^2 = 1$ whose unnormalized eigenfunctions are[18]

$$v_c = A\phi_e + B\phi_0 , \tag{13}$$

with

$$\phi_e = \cosh^s(\alpha x)\, {}_2F_1(a,b,\tfrac{1}{2}; -\sinh^2(\alpha x)) , \tag{14a}$$

$$\phi_0 = \cosh^s(\alpha x) \sinh(\alpha x) \times {}_2F_1(a+\tfrac{1}{2}, b+\tfrac{1}{2}, \tfrac{3}{2}; -\cosh^2(\alpha x)) . \tag{14b}$$

In Eqs. (14) a and b are defined as

$$a = \frac{1}{2}\left(s + i\frac{K}{\alpha} \right) , \tag{15a}$$

$$b = \frac{1}{2}\left(s - i\frac{K}{\alpha} \right) , \tag{15b}$$

and ${}_2F_1$ are hypergeometric functions.[18,21] The above equations show explicitly the dependence of the continuum eigenfunction on the parameter R. Turning our attention to the discrete spectrum, we recall[20] that the number of bound states is determined by the conditions

$$K < 0 , \tag{16a}$$

$$n < s , \tag{16b}$$

where n is an integer. Since the function $s(R)$ has values in the range [0,1], only the bound state corresponding to $n=0$ belongs to the spectrum for any value of R. Its eigenvalue is given by

$$\omega^2 = 1 - s^2 , \tag{17}$$

and the unnormalized eigenfunction is

$$v_B = \left(\frac{1}{\cosh(\alpha x)} \right)^s . \tag{18}$$

An orthogonal set of solutions of H_- is then given by

$$u_C = \left(-\frac{d}{dx} + \frac{\partial V}{\partial x} \right) v_C , \tag{19a}$$

$$u_B = \left(-\frac{d}{dx} + \frac{\partial V}{\partial x} \right) v_B . \tag{19b}$$

The explicit form of the shape mode[5] is

$$u_B = as\left[\frac{1}{\cosh(ax)}\right]^s \tanh(ax) + [\tanh(x+R)+\tanh(x-R)-\tanh x]\left[\frac{1}{\cosh(ax)}\right]^s \tag{20}$$

and smoothly interpolates between $u_B=\tanh x$ at $R=0$ and

$$u_B = \left[\frac{1}{\cosh(x-R)} - \frac{1}{\cosh(x+R)}\right]$$

as R becomes large. This behavior agrees well with the numerical computation of Ref. 5.

In Fig. 1 we plot the frequency eigenvalue of the shape mode given by Eq. (17). In Figs. 2(a)–2(c) we plot the analytical expression of the shape-mode eigenfunction (solid line) for several relevant values of R, together with the corresponding results of the numerical computation of Campbell, Peyrard, and Sodano[5] (open circles). Agreement with these results is excellent over a wide range of parameter R. In Fig. 3 we plot the analytical expression of the lowest continuum eigenfunction (solid line) and compare it with the result of the numerical integration (open circles) for the same eigenfunction.

Although not fully rigorous, our analysis provides for the first time, and for all values of R, accurate and simple analytical expressions for the eigenfunctions of Eq. (1) which will be useful in explicit computations of the dynamics and thermodynamics of the DSG kink sector.[5,15,22]

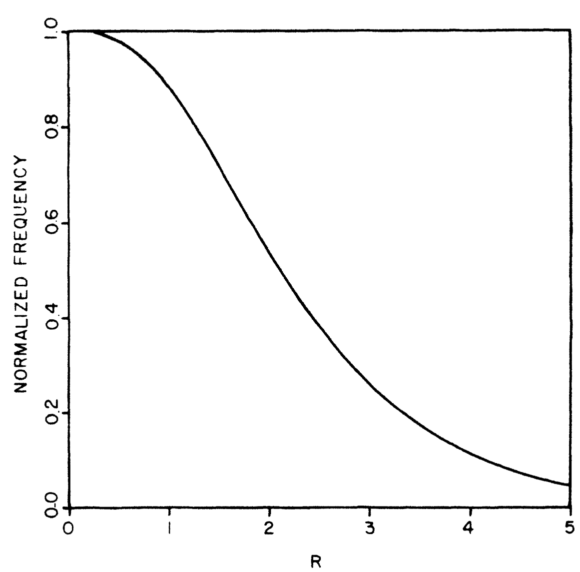

FIG. 1. Frequency of the shape mode as a function of the parameter R. The continuum starts at $\omega=1.0$.

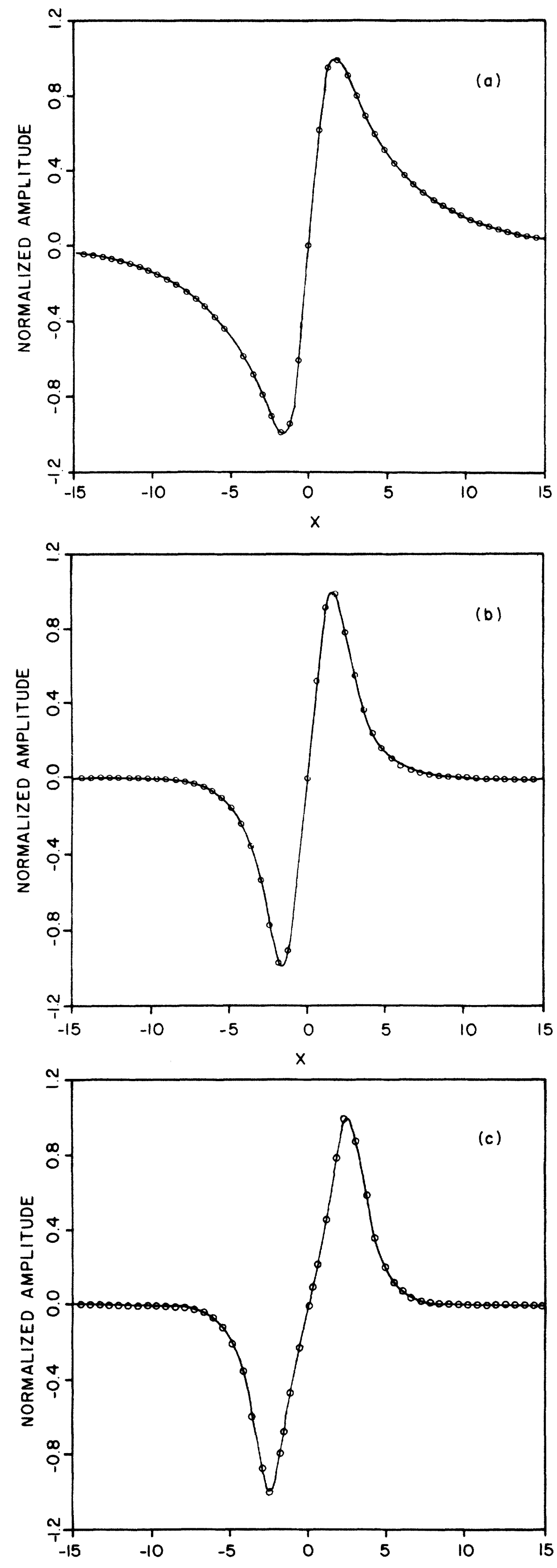

FIG. 2. The shape-mode eigenfunction for $R=$ (a) 0.5, (b) 1.2, and (c) 2.4. The solid curves are the analytic expressions of the present paper, while the open circles correspond to the numerical computations of Campbell *et al.*

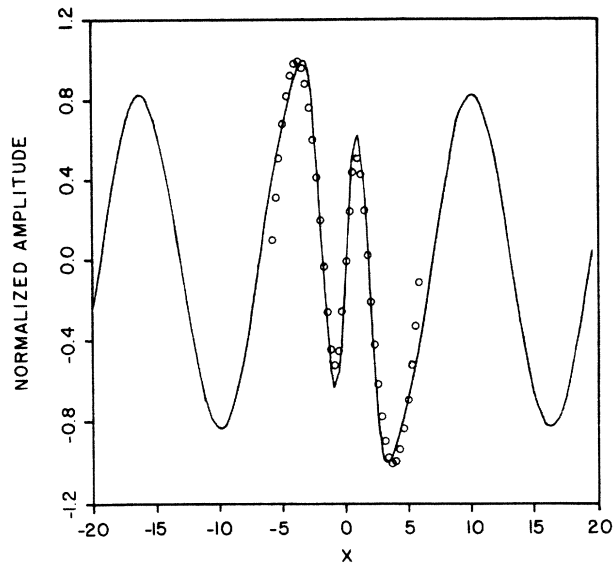

FIG. 3. The lowest continuum odd eigenfunctions for $R=3.0$. The solid curve is the analytic expression, while the open circles correspond to the numerical results of Campbell *et al.*

Questions such as the normalization of the continuum eigenfunctions and the explicit resolution of the completeness condition for the set of eigenfunctions of Eq. (1) remain untouched by this investigation. Perhaps, use of the inverse scattering[23] method together with supersymmetry could provide an answer to those questions as well as the exact solution to this eigenvalue problem.

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