Phase transitions in a disordered granular superconductor near percolation

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The properties of a disordered granular superconductor consisting of superconducting grains of size comparable to the zero-temperature bulk superconducting coherence length embedded in a non-superconducting host are studied by means of a randomly diluted Josephson tunnel junction model near a percolation threshold p_c. A replicated ($n \to 0$) continuum Landau-Ginzburg field theory describing the macroscopic properties of this material is derived from first principles. The mean-field phase diagram as a function of temperature T, applied magnetic field H, and grain concentration p exhibits a Meissner phase, an Abrikosov vortex lattice, and a spin-glass phase all arising from the low-temperature phase coherence of the condensate wave function among the grains. For $H=0$, in the superconducting phase, the macroscopic superfluid density $\rho_s \sim (p-p_c)^t$ as $p \to p_c$ where $t=3$ in mean-field theory. The spin-glass phase arises from frustration among loops of the percolating network in the presence of an applied magnetic field. Here $\rho_s=0$, leading to complete flux penetration on average but with a frozen-in distribution of randomly oriented tunneling supercurrents leading to power-law decaying $\sim x^{-(d-2)}$ local fluctuations in the B field. In the low-temperature limit, vortices are shown to consist of spin-glass cores in a superconducting background.

I. INTRODUCTION

In some recent papers[1-3] it has been suggested that superconductivity in a granular material or otherwise suitably disordered superconducting-nonsuperconducting composite may exhibit novel properties including the analog of a spin-glass phase in the presence of an externally applied magnetic field. Using a randomly diluted Josephson tunnel junction model for this system, we demonstrated[1] the occurrence of a phase transition from a state of macroscopic superconductivity to a state of spin-glass-like order for a disordered system near percolation. The thermally and configurationally averaged condensate wave function $[\langle\psi\rangle_T]_c$ is zero whereas the Edwards-Anderson average $[\,|\langle\psi\rangle_T|^2]_c$ is nonzero. This possibility has also been suggested by Shih, Ebner, and Stroud based on numerical simulations.[2] In this paper we present a detailed derivation of the mean-field phase diagram for the disordered granular superconductor exhibiting a Meissner phase, an Abrikosov vortex phase, and a spin-glass phase (Fig. 1). This is done from first principles by means of an $n \to 0$ replica field theory. The macroscopic properties of these phases are derived by considering solutions to the associated Landau-Ginzburg equations in replica space. The qualitative equilibrium properties of the glass phase are well described by considering replica-symmetric solutions to these saddle-point equations. In a forthcoming publication[4] the results of this mean-field theory will be supplemented by an ϵ expansion about six dimensions describing the critical behavior near the percolation multicritical point and by a more detailed treatment of diamagnetic properties of finite clusters.

The effects of disorder on superconductivity have been discussed by a variety of authors[5-10] in various regimes of interest, and it is important to distinguish the different

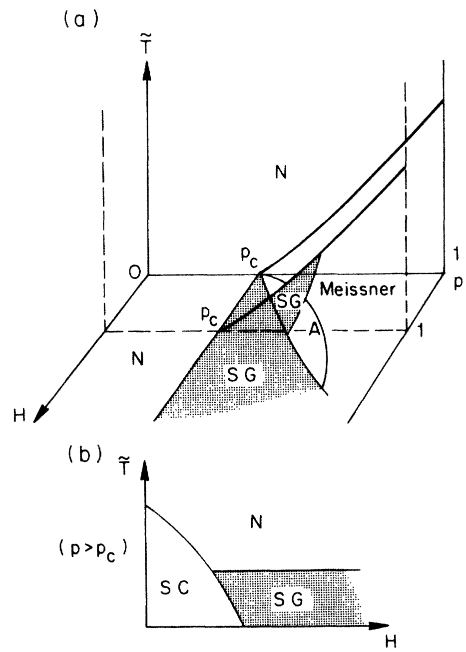

FIG. 1. (a) Mean-field phase diagram as a function of temperature $\tilde{T}=k_BT/K$ (where K is a typical Josephson coupling energy), applied magnetic field H, and Josephson bond occupation probability p near percolation threshold p_c exhibiting normal (N), Meissner, spin-glass (SG) and Abrikosov (A) vortex lattice phases. Dashed lines depict a slice of this phase diagram in the (T,p) plane for fixed $H>0$. (b) Slice of phase diagram in the (T,H) plane for a fixed concentration of superconducting grains above percolation threshold. Shown are the normal (N), superconducting (SC), and spin glass (SG) phases and the multicritical point where they meet.

physical effects pertaining to these regimes. The macroscopic superconducting properties of interest in this paper arise from the low-temperature phase coherence of the condensate wave function among superconducting grains coupled by Josephson tunneling through an insulating host, by superconducting microbridges, or by proximity effect in a normal-metal host [Fig. 2(a)]. When the intergrain coupling energy is small compared to the Bardeen-Cooper-Schrieffer (BCS) gap energy of the grains, long-range superconducting order due to phase coherence is manifest at a transition temperature T_c lower than the bulk transition temperature T_{BCS} of the individual grains.

In a randomly diluted network of Josephson weak links of this type, the transition temperature for long-range phase coherence decreases continuously to zero as the percolation threshold p_c is approached from above. This is illustrated in Fig. 1(a). In the opposite limit of Josephson coupling comparable to $k_B T_{\mathrm{BCS}}$, it is difficult to distinguish the phase coherence transition from the mean-field superconducting transition of the infinite cluster. In such a strongly coupled system the resistive transition of the material is dominated by the first connected superconducting path (short circuit) which appears as the percolation threshold p_c is exceeded.[11] This leads to a very sharp rise in the resistive transition temperature from zero to T_{BCS} in the vicinity of p_c (see Fig. 3).

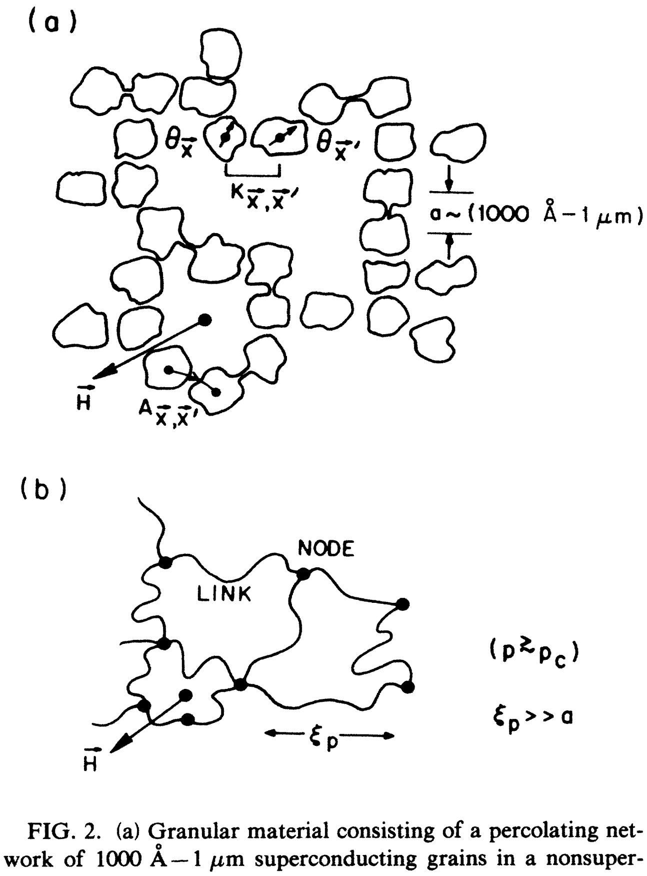

FIG. 2. (a) Granular material consisting of a percolating network of 1000 Å − 1 μm superconducting grains in a nonsuperconducting background. The coupling energy $K_{x,x'}$ between the phases θ_x of the condensate wave function on neighboring grains arises from Josephson tunneling, proximity effect, or superconducting microbridges. In the presence of an applied field H, the energy is minimized when the phase difference is $A_{x,x'}$ which equals $2\pi/\phi_0$ times the line integral of the vector potential between adjacent grains. (b) Skal—Shklovskii—de Gennes node-link picture of a percolating network. A superconducting to spin glass transition occurs when approximately one quantum of applied flux penetrates the typical loop of linear dimension ξ_p of this network.

For present considerations the grains ideally have a size of the order of the mean separation of electrons in a Cooper pair or the zero-temperature bulk coherence length ξ_s, typically a few thousand angstroms. For grains which are significantly smaller than this scale, the intergrain coupling is dominated by quantum fluctuations which tend to destroy phase coherence[12]. Quantum mechanically, the phase θ of the condensate wave function is conjugate to the number N of Cooper pairs which have condensed in a given grain, and fluctuations in these quantities are related by the Heisenberg uncertainty principle $\langle(\Delta\theta)^2\rangle\langle(\Delta N)^2\rangle \gtrsim 1$. Fluctuations in the number density correspond to charge fluctuations ΔQ and the description of macroscopic superconductivity in terms of classical phase coherence (XY spins) among grains is only valid when the charging energy $\langle(\Delta Q)^2\rangle/2C$ is small compared to the typical Josephson coupling energy between nearest-neighbor grains. Here, C is the intergrain capacitance. The small grain limit (~ 100 Å) has been discussed by Deutscher *et al.*[5] and Alexander.[6] Here electronic transport is dominated by diffusion among clusters of grains which leads to a decrease of the effective superconducting coherence length $\bar{\xi}_s$ in Landau Ginzburg theory. When $p > p_c$ and this coherence length is large compared to the percolation coherence length ξ_p, the system is suitably described as a "dirty" superconductor in which the diffusion coefficient is given by its macroscopic value regarding the medium as a homogeneous composite. Closer to the percolation threshold, ξ_p becomes large compared to $\bar{\xi}_s$ and the inhomogeneous nature of the network, in particular the loop structure, becomes important as shown by measurements of the critical magnetic field H_{c2}

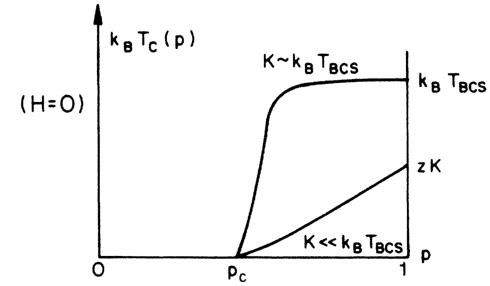

FIG. 3. Resistive transition temperature T_c in zero magnetic field as a function of concentration of grains. For Josephson coupling energy K comparable to $k_B T_{\mathrm{BCS}}$ of the individual grains, the first percolating path shorts out the sample. For $K \ll k_B T_{\mathrm{BCS}}$, there is a more continuous rise of T_c due to the onset of long-range phase coherence to a maximum value of order zK, where z is the coordination number of the network.

for the destruction of long-range superconducting order. For a single loop which is large compared to the coherence length $\bar{\xi}_s$ there are Little-Parks oscillations in the induced supercurrent as a function of applied magnetic field, and superconductivity is only destroyed at the critical field of the wire constituting the loop.[13] In the percolation system of small strongly coupled grains considered by Deutscher, this critical field in turn depends on the internal structure of these wires [links in the Skal—Shklovskii—de Gennes picture[14] as shown in Fig. 2(b)] and in particular the nature of electron diffusion along these links.

The effects of diffusive electron transport on superconductivity have also been studied for bulk systems disordered by impurities in which the elastic electron mean free path becomes short compared to the bulk coherence length ξ_s for the pure system.[15] The slow diffusive transport of electrons leads to an enhancement of the effective electron-electron repulsion in the Cooper pair leading to a suppression of the superconducting transition temperature. The charging effects associated with small grain granular systems may be thought of as an extreme case of the impurity scattering effect. Recently, Jacobs *et al.*[16] have suggested the possibility of a reentrant normal phase in granular films dominated by such quantum fluctuations.

Experiments on larger grain systems in which long-range order may be described by classical XY phase coherence have focused largely on ordered two-dimensional arrays.[17,18] In thin granular films of this type, the London penetration depth is long compared to the superconducting coherence length so that in the presence of an applied field, the resistive behavior is dominated by phase slip in the associated Abrikosov vortex lattice. Boysel *et al.*,[19] however, have studied a disordered three dimensional granular system consisting of micrometer-size Pb grains immersed in a normal-metal host exhibiting-proximity-effect induced superconductivity. Here flux exclusion has been observed to occur in their samples at temperatures below that of the observed resistive transition and also differences have been reported between field-cooled and zero-field-cooled samples. It is the aim of this paper to derive the superconducting properties of disordered granular systems in such an intermediate regime consisting of grains sufficiently large that charging effects may be neglected yet sufficiently small that long-range phase coherence is distinct from the bulk superconducting transition of the individual grains.

The spin-glass phase which occurs in our model has a physical interpretation in terms of the structure of loops of the percolation network. At zero temperature, there is a transition from an Abrikosov vortex lattice phase to the glass phase when approximately one quantum of applied magnetic flux penetrates an area ξ_p^2. Near the percolation threshold p_c, the effective coherence length $\bar{\xi}_s$ is short compared to the loop perimeter so that individual loops respond to the applied field by generating tunneling supercurrents of magnitude inversely proportional to the loop perimeter as in the Little-Parks experiments.[13] Consequently the large fluctuations in the sizes of contiguous loops near percolation lead to a freezing in of the tunneling supercurrents along common links. There is complete flux penetration on average but with strong local fluctuations which decay as a power law with distance. Alternatively, the glass phase may be thought of as the set of frozen-in randomly oriented magnetic moments arising from this random distribution of current loops. We now describe how this picture emerges from a systematic analysis.

II. THE MODEL

We consider a model for a granular superconductor in which the Cooper pair wave function at site x is described by the tight-binding Schrödinger equation

$$i\hbar\frac{d\psi_{\mathbf{x}}}{dt}=-\Delta_{\mathbf{x}}\psi_{\mathbf{x}}-\sum_{\langle \mathbf{x},\mathbf{x}'\rangle}K_{\mathbf{x},\mathbf{x}'}\psi_{\mathbf{x}'}\ . \tag{2.1}$$

Here $\Delta_{\mathbf{x}}$ is the BCS gap energy of the grain at site $\mathbf{x}$ and $K_{\mathbf{x},\mathbf{x}'}$ is a hopping matrix element of the Cooper pair to a nearest-neighbor grain at $\mathbf{x}'$. Charging effects due to quantum fluctuations are neglected by writing

$$\psi_{\mathbf{x}}=\rho^{1/2}e^{i\theta_{\mathbf{x}}} \tag{2.2}$$

where the density of Cooper pairs on each grain is a fixed constant ρ and only the phase degree of freedom $\theta_{\mathbf{x}}$ is allowed to fluctuate. In the absence of voltage differences between the identical grains (static equilibrium) it is convenient to choose the zero of energy by setting $\Delta_{\mathbf{x}}=0$. The expectation value of the Hamiltonian operator [Eq. (2.1)] with respect to the state vector [Eq. (2.2)] yields the well-known Josephson tunneling model.[20] In the presence of a static applied magnetic field, the hopping matrix elements are multiplied by the phase factor $\exp(iA_{\mathbf{x},\mathbf{x}'})$ involving the line integral

$$A_{\mathbf{x},\mathbf{x}'}\equiv\frac{2\pi}{\phi_0}\int_{\mathbf{x}}^{\mathbf{x}'}\mathbf{A}\cdot d\mathbf{l} \tag{2.3}$$

of the vector potential $\mathbf{A}$ between nearest-neighbor grains in units of the flux quantum $\phi_0=hc/2e$. The resulting Hamiltonian (quantum expectation value)

$$H_\alpha=-\sum_{\langle \mathbf{x},\mathbf{x}'\rangle}K_{\mathbf{x},\mathbf{x}'}\cos\delta^\alpha_{\mathbf{x},\mathbf{x}'} \tag{2.4}$$

where

$$\delta^\alpha_{\mathbf{x},\mathbf{x}'}\equiv\theta^\alpha_{\mathbf{x}}-\theta^\alpha_{\mathbf{x}'}-A^\alpha_{\mathbf{x},\mathbf{x}'} \tag{2.5}$$

is similar to that of a classical XY spin system in which the gauge field acts as a source of frustration to phase alignment. Here, a replica index $\alpha=1,\ldots,n$ has been added to the phase angles to facilitate averaging over the disorder. For convenience we place the grains on a d-dimensional cubic lattice of lattice constant a and introduce a bond percolation model for the disorder: each $K_{\mathbf{x},\mathbf{x}'}$ is independently equal to K with probability p and zero with probability $1-p$. A further randomness in bond lengths is then introduced to better simulate a granular material (see Fig.4). In the infinite London penetration depth limit (as might be realized in an infinitesimally thin granular film) the vector potential entering Eq. (2.3) is precisely that of the applied field. More generally, diamagnetic screening currents induced by the applied field generate additional fields so that the true vector potential must be determined self-consistently. Elementary quantum mechanics yields

$$J_{\mathbf{x},\mathbf{x}'} = -\hbar^{-1} K_{\mathbf{x},\mathbf{x}'} \sin\delta_{\mathbf{x},\mathbf{x}'} \tag{2.6}$$

for the tunneling supercurrent between adjacent grains. The addition of a replica index to $A_{\mathbf{x},\mathbf{x}'}$ expresses the fact that this may vary from one realization of the disorder to another. We will refer to the models with and without electromagnetic fluctuations as model I and model II.

The thermodynamic properties of model I are governed by the configuration averaged Helmholtz free energy $F = -k_B T[\ln Z_A]_c$, where the square brackets denote a quenched average over all possible realizations of the intergrain Josephson coupling. The configuration average $[\ ]_c$ may be carried out using the replica procedure:

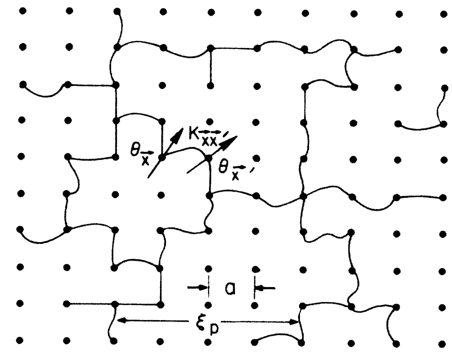

FIG. 4. Bond percolation model for the granular material. Grains are placed on d-dimensional hypercubic grid with lattice constant a, and the Josephson coupling energy $K_{\mathbf{x},\mathbf{x}'}$ is K or zero with probabilities p and $1-p$, respectively. A further randomness in the lengths of occupied bonds is depicted by means of arcs with random curvatures so as to remove any spurious periodicities arising from the underlying lattice.

$$[\ln Z_A]_c = \lim_{n\to 0} \frac{1}{n} \ln[Z_A^n]_c \ . \tag{2.7}$$

Physical quantities such as the expectation value of the tunneling current (2.6) may then be obtained by differentiating (2.7) with respect to one replica of the vector potential and then setting $\mathbf{A}^\alpha$ to be the same in each replica. Taking derivatives in two different replicas yields the associated current correlation functions:

$$-\frac{k_B T}{\hbar^2} \frac{\delta^2}{\delta A^\alpha_{\mathbf{x},\mathbf{x}'} \delta A^\beta_{\mathbf{x},\mathbf{x}'}} \ln[Z_A^n]_c = \delta_{\alpha\beta} \left[\frac{E_c}{\hbar^2} - \frac{1}{k_B T}([\langle J^2_{\mathbf{x},\mathbf{x}'}\rangle_T]_c - [\langle J_{\mathbf{x},\mathbf{x}'}\rangle_T]_c^2) \right] - (1-\delta_{\alpha\beta}) \frac{1}{k_B T}([\langle J_{\mathbf{x},\mathbf{x}'}\rangle_T^2]_c - [\langle J_{\mathbf{x},\mathbf{x}'}\rangle_T]_c^2) \ . \tag{2.8}$$

Here $\langle\ \rangle_T$ denotes a thermal average, and the condensation energy entering the replica diagonal term is

$$E_c \equiv [\langle K_{\mathbf{x},\mathbf{x}'} \cos(\theta_{\mathbf{x}} - \theta_{\mathbf{x}'} - A_{\mathbf{x},\mathbf{x}'})\rangle_T]_c \ . \tag{2.9}$$

III. FIELD THEORY

The disorder average is readily performed to yield an effective Hamiltonian

$$H_{\text{eff}} = \sum_{\langle \mathbf{x},\mathbf{x}'\rangle} H_{\text{int}}(\delta^\alpha_{\mathbf{x},\mathbf{x}'}) \tag{3.1}$$

involving interactions between phase variables in different replicas. Here, $\delta^\alpha_{\mathbf{x},\mathbf{x}'}$ is the replicated gauge-invariant phase difference Eq. (2.5) between nearest-neighbor grains. Then,

$$[Z_A^n]_c = \text{const} \times \int D\theta^\alpha_{\mathbf{x}} e^{-H_{\text{eff}}} \tag{3.2}$$

where

$$H_{\text{int}}(\delta^\alpha) = -\ln\left[1 + v\exp\left[\widetilde{T}^{-1} \sum_{\alpha=1}^{n} \cos\delta^\alpha\right]\right] . \tag{3.3}$$

Here, $v = p/(1-p)$ and $\widetilde{T} = k_B T/K$.

Following the method introduced by Stephen[21] to treat the random resistor network, we decompose this interaction Hamiltonian into its Fourier components $\phi_{\mathbf{k}}(\mathbf{x}) \equiv e^{i\mathbf{k}\cdot\boldsymbol{\theta}_{\mathbf{x}}}$ where $\mathbf{k} = (k_1, k_2, \ldots, k_n)$ is an n-component vector in replica space with integer components conjugate to the replicated phase $\boldsymbol{\theta} = (\theta^1, \theta^2, \ldots, \theta^n)$:

$$H_{\text{int}} = \sum_{\mathbf{k}\neq 0} B_{\mathbf{k}} e^{-ik_\alpha A^\alpha_{\mathbf{x},\mathbf{x}'}} \phi_{-\mathbf{k}}(\mathbf{x}) \phi_{\mathbf{k}}(\mathbf{x}') \ , \tag{3.4}$$

where

$$B_{\mathbf{k}} = \prod_{\alpha=1}^{n} \left[\int_{-\pi}^{\pi} \frac{d\delta^\alpha}{2\pi} \right] e^{-ik_\alpha \delta^\alpha} H_{\text{int}}(\delta^\alpha) \ . \tag{3.5}$$

Expanding the logarithm in Eq. (3.3) it is possible to reexpress the Fourier amplitude Eq. (3.5) as

$$B_{\mathbf{k}} = \sum_{l=1}^{\infty} \frac{(-1)^l}{l} v^l \prod_{\alpha=1}^{n} F_l(k_\alpha) \ , \tag{3.6}$$

where

$$F_l(k_\alpha) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{ik_\alpha\theta + l\cos\theta/\widetilde{T}} \ . \tag{3.7}$$

In the low temperature limit the above integral may be evaluated by steepest descents. To leading order in $\widetilde{T}$ and in the limit $n \to 0$,

$$B_{\mathbf{k}} = \sum_{l=1}^{\infty} \frac{(-1)^l}{l} v^l \exp\left[-\frac{\widetilde{T}k^2}{2l}\right] \tag{3.8}$$

is a function only of the rotationally invariant quantity $\mathbf{k}^2=\sum_{\alpha=1}^{n}k_\alpha^2$. As in the diluted XY model considered by Harris and Lubensky,[22] this symmetry is broken by higher-order terms in $\widetilde{T}$ involving cubic invariants of $\mathbf{k}$ such as $\sum_{\alpha=1}^{n}k_\alpha^4$.

It is convenient to regard the nearest-neighbor sum appearing in Eq. (3.1) to correspond to superconducting grains placed at random near the lattice points of a d-dimensional hypercubic grid of lattice constant a. If identical grains are placed at precisely such a set of points, then the system possesses a fundamental periodicity as a function of applied magnetic field even when the bonds of the lattice are randomly removed. This occurs since all closed loops in the percolating network (projected onto one of the axes of the cubic lattice) have an area which is an integer multiple of the fundamental area a^2. It follows that it is possible for an integer number of flux quanta of applied field H to pierce all loops of the network whenever Ha^2/ϕ_0 is an integer. This periodicity may be removed by allowing the location of the grains to fluctuate randomly about the actual lattice points of a periodic grid as would be appropriate in a disordered granular material. This periodicity is evident in the term $\exp(ik_\alpha A_{\mathbf{x},\mathbf{x}'}^\alpha)$ appearing in the interaction Eq. (3.4). Setting $\nabla\times\mathbf{A}^\alpha=\mathbf{H}$ in each replica, it follows that this interaction is invariant whenever the flux fraction $f\equiv Ha^2/\phi_0$ is increased by an integer. If on the other hand the link variable $A_{\mathbf{x},\mathbf{x}'}$ is given a small randomly fluctuating part and a further average is performed, this periodicity is removed. This is depicted in Fig. 4 by means of a diluted lattice of bonds drawn with random curvature. It provides an interpretation of the continuum field theory limit which we now proceed to consider.

In the continuum limit, the nearest-neighbor sum Eq. (3.1) on the d-dimensional hypercubic lattice may be replaced by an integral. For lattice constant a, the leading term in the gradient expansion for the interaction yields

$$H_{\text{eff}}=\int_{\mathbf{x}}\phi_{-\mathbf{k}}(\mathbf{x})B_{\text{op}}\phi_{\mathbf{k}}(\mathbf{x}) \tag{3.9}$$

where we have used the notation:

$$\int_{\mathbf{x}}\equiv\int\frac{d^dx}{a^d}\ . \tag{3.10}$$

Here, the differential operator

$$d^{-1}B_{\text{op}}\equiv B_{\mathbf{k}}\left[1+\frac{a^2}{2d}\left[\nabla-\frac{2\pi i}{\phi_0}\mathbf{A}^\alpha k_\alpha\right]^2+\cdots\right]. \tag{3.11}$$

A field theory may now be obtained by the Hubbard-Stratonovitch transformation

$$e^{-H_{\text{eff}}}=\int D\psi_{\mathbf{k}}(\mathbf{x})\times\exp\left[-\sum_{\mathbf{k}\neq0}\int_{\mathbf{x}}(\tfrac{1}{4}\psi_{-\mathbf{k}}B_{\text{op}}^{-1}\psi_{\mathbf{k}}+\psi_{-\mathbf{k}}\phi_{\mathbf{k}})\right]. \tag{3.12}$$

Since the operator B_{op} is Hermitian, the range of functional integration here is the entire complex plane and $\psi_{-\mathbf{k}}=\psi_{\mathbf{k}}^*$. By introducing a source term it is straightforward to verify that the expectation value of the order-parameter field $\psi_{\mathbf{k}}$ is related to various moments of the XY order parameter thermally and configurationally averaged:

$$\langle\psi_{\mathbf{k}}(\mathbf{x})\rangle=\langle B_{\text{op}}\phi_{\mathbf{k}}(\mathbf{x})\rangle\approx dB_{\mathbf{k}}\left[\prod_{\alpha=1}^{n}\langle e^{ik_\alpha\theta(\mathbf{x})}\rangle_T\right]_c\ . \tag{3.13}$$

The second form follows from neglecting corrections associated with the gradient expansion of B_{op}. Here, $\langle\ \rangle$ denotes an expectation value with respect to the partition function Eq. (3.2), whereas $\langle\ \rangle_T$ and $[\ ]_c$ denote thermal and configuration averages with respect to the original unreplicated Hamiltonian. For example, $\langle\psi_{1,0,0,\ldots,0}\rangle$ is the XY order parameter and $\langle\psi_{1,-1,0,\ldots,0}\rangle$ is the Edwards-Anderson order parameter.[23] An expansion of the replicated free energy in terms of the fields $\psi_{\mathbf{k}}$ follows by integration over the replicated variables $\theta^\alpha(\mathbf{x})$. This is facilitated by defining the Fourier transform

$$\widetilde{\psi}_\theta\equiv\sum_{\mathbf{k}}\psi_{\mathbf{k}}e^{i\mathbf{k}\cdot\theta}\ , \tag{3.14}$$

$$\psi_{\mathbf{k}}=\int_\theta\widetilde{\psi}_\theta e^{-i\mathbf{k}\cdot\theta}\ , \tag{3.15}$$

where we have again used an abbreviated notation $\int_\theta\equiv\int d^n\theta/(2\pi)^n$. The summation in Eq. (3.14) is over the entire n-dimensional hypercubic lattice in k space with integer components and integration in Eq. (3.15) is over the associated n-dimensional hypercubic "Brillouin zone" defined by $-\pi<\theta^\alpha<\pi$, $\alpha=1,\ldots,n$. It follows that

$$\ln\left[\int D\theta^\alpha(\mathbf{x})\exp\left[-\sum_{\mathbf{k}\neq0}\int_{\mathbf{x}}\psi_{-\mathbf{k}}\phi_{\mathbf{k}}\right]\right]\equiv\int_{\mathbf{x}}H(\widetilde{\psi}_\theta)\ , \tag{3.16}$$

where

$$H(\widetilde{\psi}_\theta)=\ln\int_\theta\exp(-\widetilde{\psi}_\theta)\ . \tag{3.17}$$

Here the absence of the $k=0$ term in the summation yields the condition on the Fourier transform

$$\int_\theta\widetilde{\psi}_\theta=0\ . \tag{3.18}$$

The critical behavior in the vicinity of the percolation multicritical point, as will be described shortly, is determined by expansion of Eq. (3.17). To fourth order in the order-parameter fields this becomes

$$H'(\widetilde{\psi}_\theta)=\int_\theta\left[\frac{1}{2!}\widetilde{\psi}_\theta^2-\frac{1}{3!}\widetilde{\psi}_\theta^3+\frac{1}{4!}\widetilde{\psi}_\theta^4+\cdots\right]-\frac{1}{2}\left[\int_\theta\frac{1}{2!}\widetilde{\psi}_\theta^2\right]^2+\cdots\ . \tag{3.19}$$

Combining Eqs. (3.19), (3.16), and (3.12) yields the required continuum Landau-Ginzburg field theory for the granular superconductor:

$$[Z_A^n]=\int D\psi_{\mathbf{k}}e^{-H_r}\ , \tag{3.20a}$$

$$H_r=\tfrac{1}{2}\sum_{\mathbf{k}}{}'\,\psi_{\mathbf{k}}(\mathbf{x})\left[r_{\mathbf{k}}-c_1\left[\nabla-\frac{2\pi i}{\phi_0}\mathbf{A}^\alpha k_\alpha\right]^2\right]\psi_{-\mathbf{k}}$$
$$+\tfrac{1}{6}\sum_{\mathbf{k}_1,\mathbf{k}_2}{}'\int_{\mathbf{x}}\psi_{\mathbf{k}_1}(\mathbf{x})\psi_{\mathbf{k}_2}(\mathbf{x})\psi_{-\mathbf{k}_1-\mathbf{k}_2}(\mathbf{x})+\cdots\ . \tag{3.20b}$$

Here, the prime on the summation indicates that summation is only over fields with nonzero $\mathbf{k}$: $\psi_{\mathbf{k}=0}=0$. The bare mass of the field $\psi_{\mathbf{k}}$ is given by $r_{\mathbf{k}}=(2dB_{\mathbf{k}})^{-1}-1$ and $c_1=a^2/4d$. The instability to an ordered phase in the absence of an applied magnetic field is determined by the locus of points for which $r_{\mathbf{k}}$ becomes negative. In the low-temperature limit

$$r_{\mathbf{k}}=(p_c-p)+b\tilde{T}\mathbf{k}^2+O(\tilde{T}^2)\ . \tag{3.21}$$

In mean field theory the percolation threshold $p_c=1-\exp(-1/2d)$ and b is a positive constant which depends on p and d.

The tunneling supercurrent expressed by Eq. (2.6) for the lattice model has a thermally- and configuration-averaged expectation value $[\langle J_i(\mathbf{x})\rangle_T]_c$ in the continuum version (model I) given by

$$\langle J_i^\alpha(\mathbf{x})\rangle=-\frac{k_BT}{c_2}\frac{\delta}{\delta A_i^\alpha(\mathbf{x})}\ln[Z_A^n]_c,\quad c_2=\frac{2\pi a\hbar}{\phi_0}\ . \tag{3.22}$$

Current correlation functions may likewise be obtained by differentiating the replicated partition function with respect to different replicas of the vector potential and afterward setting $\mathbf{A}^\alpha=\mathbf{A}$, the physical value, in each replica. Introducing the spatial Fourier transforms $J_i^\alpha(\mathbf{q})$ and $A_i^\alpha(\mathbf{q})$ of the current and vector potential respectively, we define a replicated wave-vector-dependent helicity modulus

$$\gamma_{ij}^{\alpha\beta}\equiv c_2^{-1}\frac{\delta\langle J_i^\alpha(\mathbf{q})\rangle}{\delta A_j^\beta(-\mathbf{q})}\ . \tag{3.23}$$

Comparison with the discrete version Eq. (2.8) yields

$$\gamma_{ij}^{\alpha\beta}(\mathbf{q})=\gamma_{ij}^{(1)}(\mathbf{q})\delta_{\alpha\beta}+\gamma_{ij}^{(2)}(\mathbf{q})\ , \tag{3.24}$$

where

$$\gamma_{ij}^{(1)}(\mathbf{q})\equiv\delta_{ij}E_c/\hbar^2-(k_bT)^{-1}[\langle J_i(\mathbf{q})J_j(-\mathbf{q})\rangle_T-\langle J_i(\mathbf{q})\rangle_T\langle J_j(-\mathbf{q})\rangle_T]_c\ , \tag{3.25}$$

and

$$\gamma_{ij}^{(2)}\equiv(k_BT)^{-1}\{[\langle J_i(\mathbf{q})\rangle_T\langle J_j(-\mathbf{q})\rangle_T]_c-[\langle J_i(\mathbf{q})\rangle_T]_c[\langle J_j(-\mathbf{q})\rangle_T]_c\}\ . \tag{3.26}$$

Associated with each configuration of occupied bonds, there is a physical helicity modulus:

$$\gamma_{ij}(\mathbf{q})\equiv\frac{\delta\langle J_i(\mathbf{q})\rangle_T}{\delta A_j(-\mathbf{q})}\ . \tag{3.27}$$

In the language of replicas, the average of γ_{ij} over bond configurations measures the response of one replica of the current to a replica-independent vector potential

$$[\gamma_{ij}(\mathbf{q})]_c=\frac{\delta\langle J_i^\alpha(\mathbf{q})\rangle}{\delta A_j(-\mathbf{q})}=c_2\sum_{\beta=1}^{n}\gamma_{ij}^{\alpha\beta}\ . \tag{3.28}$$

It follows that $[\gamma_{ij}(\mathbf{q})]_c$ is simply $c_2\gamma_{ij}^{(1)}(\mathbf{q})$ since the contributions from $\gamma_{ij}^{(2)}$ vanishes in the $n\to 0$ limit. The response to a uniform vector potential $Y_{ij}\equiv\lim_{\mathbf{q}\to 0}[\gamma_{ij}(\mathbf{q})]_c$ is the quantity studied previously by others[2] in numerical simulations.

Fluctuations in the magnetic field associated with induced currents (model II) may be incorporated by the addition of a magnetic energy term and an integration over $\mathbf{A}^\alpha(\mathbf{x})$:

$$[Z^n]_c=\int D\mathbf{A}^\alpha[Z_A^n]_c\exp\left[-g\int_{\mathbf{x}}(\nabla\times\mathbf{A}^\alpha)^2\right]\ , \tag{3.29}$$

where $g\equiv(8\pi\mu_0k_BT)^{-1}$ and μ_0 is the bare magnetic permeability of the composite. In this model, $\gamma_{ij}^{\alpha\beta}(\mathbf{q})$ becomes

$$c_2^{-1}\delta\langle J_i^\alpha(\mathbf{q})\rangle/\delta<A_j^\beta(-\mathbf{q})\rangle$$

and contains the usual local-field corrections to Eq. (3.23).

It determines the mass of the replicated photon field (Higg's mechanism):

$$D_{ij}^{\alpha\beta}\equiv\langle\delta A_i^\alpha(\mathbf{q})\delta A_j^\beta(-\mathbf{q})\rangle$$
$$=k_BT[(4\pi\mu_0)^{-1}q^2\delta_{\alpha\beta}\delta_{ij}+c_2^2\gamma_{ij}^{\alpha\beta}(\mathbf{q})]^{-1}\ , \tag{3.30}$$

where $\delta A_i^\alpha(\mathbf{q})=A_i^\alpha(\mathbf{q})-\langle A_i^\alpha(\mathbf{q})\rangle$and the right-hand side is the inverse in both the Cartesian ij and replica $\alpha\beta$ indices.

IV. MEAN-FIELD PHASE DIAGRAM

An approximate mean-field phase diagram as a function of temperature T, concentration of grains p, and applied magnetic field H may be obtained by considering the locus of points for which the quadratic part of H_r develops zero eigenvalues. Setting $\nabla\times\mathbf{A}^\alpha=\mu_0\mathbf{H}$ in each replica, the spectrum consists of a set of Landau levels in which the effective charge or gauge coupling is proportional to

$$s_{\mathbf{k}}\equiv\sum_{\alpha=1}^{n}k_\alpha\ . \tag{4.1}$$

At small finite temperature the lowest eigenvalue (Landau level) for a given k is

$$e_{\mathbf{k}}=p_c-p+b\tilde{T}\mathbf{k}^2+\frac{\pi\mu_0Ha^2}{2d\phi_0}|s_{\mathbf{k}}|\ . \tag{4.2}$$

For $\widetilde{T}>0$ and finite applied field, the lowest eigenvalue is associated with modes for which $\mathbf{k}^2$ and $|s_\mathbf{k}|$ are least, namely $\mathbf{k}=(1,0,0,\ldots,0)$ and $\mathbf{k}=(1,-1,0,\ldots,0)$ as well as the associated replica permutations. At $\widetilde{T}=0$, $e_{1,0,\ldots,0}$ is degenerate with all modes for which $|s_\mathbf{k}|=1$, and $e_{1,-1,0,\ldots,0}$ is degenerate with all modes for which $s_\mathbf{k}=0$. In this limit the XY order parameter $\psi_{1,0,\ldots,0}$ exhibits long range order for applied fields lower than a critical field H_g corresponding to the penetration of approximately one quantum of flux through the typical loop of the percolating network:

$$\mu_0 H_g \xi_p^2 = 2d\phi_0/\pi \quad (p>p_c;\ \widetilde{T}=0)\ . \tag{4.3}$$

Here ξ_p is the percolation correlation length which in mean-field theory is given by $\xi_p/a=|p-p_c|^{-\nu}$ with $\nu=1/2$. However, the charge neutral fields ($s_\mathbf{k}=0$) of which the Edwards-Anderson $\psi_{1,-1,0,\ldots,0}$ or $[\langle e^{i\theta}\rangle_T \langle e^{-i\theta}\rangle_T]_c$ dominates at finite temperature are unaffected by the applied field $\mathbf{H}$. This suggests the identification of Eq. (4.3) with a transition from macroscopic superconductivity to spin-glass order. Higher order terms in the Landau expansion Eq. (3.20), however, introduce couplings between the glass ($s_\mathbf{k}=0$) fields and fields for which $s_\mathbf{k}\neq 0$. These interaction terms all have the property that the sum of the replica vectors $\mathbf{k}$ labeling the order-parameter fields $\psi_\mathbf{k}$ is zero. It follows that in any interaction of order m, if $m-1$ fields have the property $s_\mathbf{k}=0$, then the remaining field must also satisfy this property. Since there is no linear coupling of the glass fields to a mode with nonzero $s_\mathbf{k}$, it follows that for H larger than the critical field H_g, the existence of Edwards-Anderson order does not induce order in modes for which $s_\mathbf{k}\neq 0$ but only those for which $s_\mathbf{k}=0$. The interaction terms affect only the precise location of the superconducting to spin-glass phase phase boundary. More generally, at finite temperature and fixed $H>0$, the surfaces $e_{1,0,\ldots,0}=0$ and $e_{1,-1,0,\ldots,0}=0$ determine, respectively, the transitions from the normal to the Abrikosov and spin-glass phases, whereas the intersection of these two surfaces defines a line of multicritical points where the superconducting, spin-glass, and normal phases meet [Figs. 1(a) and 1(b)].

V. LANDAU-GINZBURG EQUATIONS

The detailed properties of these phases may be obtained in mean field theory by saddle-point evaluation of the functional integral Eq. (3.29). Minimization of the action with respect to $\psi_\mathbf{k}(\mathbf{x})$ and $\mathbf{A}^\alpha(\mathbf{x})$ leads to the analog of the continuum Landau-Ginzburg equations in replica space for the granular system:

$$\left[r_\mathbf{k} - c_1 \left[\nabla - \frac{2\pi i}{\phi_0}\mathbf{A}s_\mathbf{k} \right]^2 \right] \psi_\mathbf{k}(\mathbf{x}) + \tfrac{1}{4} \sum_{\mathbf{q}\neq 0,\mathbf{k}} \psi_{\mathbf{k}-\mathbf{q}}(\mathbf{x})\psi_\mathbf{q}(\mathbf{x}) = 0\ . \tag{5.1}$$

As a first approximation we consider solutions to Eq. (5.1) which preserve replica symmetry, i.e., $\mathbf{A}^\alpha=\mathbf{A}$. This vector potential is in turn determined self-consistently by

$$(4\pi\mu_0)^{-1}\nabla\times(\nabla\times\mathbf{A}) = (\mu_0 c)^{-1}\mathbf{J} = -ic_3 \frac{\widetilde{T}}{n} \sum_{\mathbf{k}\neq 0} s_\mathbf{k}\psi_{-\mathbf{k}} \left[\nabla - \frac{2\pi i}{\phi_0}\mathbf{A}s_\mathbf{k} \right] \psi_\mathbf{k}\ . \tag{5.2}$$

Here, c is the speed of light, $c_3 \equiv \pi a^{2-d}/d\phi_0$, and $\psi_\mathbf{k}$ and $\mathbf{A}$ are understood to represent equilibrium expectation values of the respective fields. We have also made use of Maxwell's equation to identify Eq. (5.2) with the macroscopic tunneling supercurrent $\mathbf{J}$. We now discuss three distinct types of solutions to these equations apart from the trivial one $\psi_\mathbf{k}=0$, $\nabla\times\mathbf{A}=\mu_0\mathbf{H}$ describing the normal phase.

(i) For sufficiently weak applied fields ($H<H_{c1}$) as will be clarified shortly, the energetically favorable solution is that for which $\mathbf{A}=0$ (Meissner effect in the London gauge) and $\psi_\mathbf{k}(\mathbf{x})$ is spatially uniform and precisely that of a randomly diluted XY ferromagnet. As in usual superconductors, the magnetic field is screened from the interior of the sample by surface supercurrents which decay exponentially from the sample boundary on the scale of the London penetration depth λ. Writing

$$\psi_\mathbf{k} = |\psi_\mathbf{k}| \exp[is_\mathbf{k}\phi(\mathbf{x})]\ ,$$

it follows that the superfluid velocity

$$\mathbf{v}_s = \nabla\phi - \frac{2\pi}{\phi_0}\mathbf{A}\ . \tag{5.3}$$

In the London gauge ($\nabla\cdot\mathbf{A}=0$), the phase ϕ remains uniform and from Eq. (5.2) it follows that the supercurrent is given by

$$\mathbf{J} = -\rho_s \frac{e^2}{mc}\mathbf{A}\ , \tag{5.4}$$

where

$$\rho_s = c_4 \lim_{n\to 0} \mu_0 \frac{\widetilde{T}}{n} \sum_{\mathbf{k}\neq 0} s_\mathbf{k}^2 |\psi_\mathbf{k}|^2\ . \tag{5.5}$$

Here $c_4=(da^d)^{-1}2ma^2/\hbar^2$, and ρ_s is the macroscopic superfluid density. The penetration of the vector potential on a scale λ leads to suppression of modes $\psi_\mathbf{k}$ for which $s_\mathbf{k}\neq 0$ as may be seen from the London form of Eq. (5.1) in which the quantity in square brackets becomes $[r_\mathbf{k}+c_1(2\pi\phi_0^{-1})^2 s_\mathbf{k}^2 \mathbf{A}^2]$. In the absence of $\mathbf{A}$, the solution $\psi_\mathbf{k}$ is symmetric under reflections $k_\alpha \to -k_\alpha$. For small $\mathbf{A}$, the breaking of this symmetry may be treated in perturbation theory and so

$$\rho_s = c_4\mu_0 \lim_{n\to 0} \frac{\widetilde{T}}{n} \sum_{\mathbf{k}\neq 0} \mathbf{k}^2 |\psi_\mathbf{k}{}^0|^2 - O(A^2)\ . \tag{5.6}$$

The leading term here, involving the reflection symmetric XY ferromagnet solutions $\psi_{\mathbf{k}}{}^0$, describes the superfluid density in the interior of the sample whereas higher-order terms describe the suppression of ρ_s near the boundary. The critical current for the destruction of long-range superconducting order may likewise be determined by maximizing Eq. (5.4) with respect to $\mathbf{A}$ using the expansion (5.6). In the weak field limit ($\mathbf{A}\to\mathbf{0}$), the implicit dependence of $\psi_{\mathbf{k}}$ on $\mathbf{A}$ may be neglected, and the divergence of the London penetration depth $\lambda\sim\rho_s^{-1/2}$ is governed by the scaling behavior of the zero-field ($H=0$) solution:

$$\psi_{\mathbf{k}}{}^0\sim(p-p_c)^{\beta}f(\widetilde{T}k^2/(p-p_c)^{\phi_T}) , \qquad (5.7)$$

where the scaling function $f(x)$ is of order unity for $x<1$ and vanishes for larger x. In mean-field theory, the thermal crossover exponent $\phi_T=1$, and the presence of a cubic term in the Landau expansion [Eq. (3.20)] yields $\beta=1$. For $H=0$, an ϵ expansion[22] about six dimensions yields $\phi_T=1+\epsilon/42$. In the low-temperature limit ($\widetilde{T}\to 0$) the sum in Eq. (5.6) may be replaced by an integral:

$$\rho_s\sim c_1(p-p_c)^{2\beta}\lim_{n\to 0}\frac{1}{n}\int d^n\mathbf{k}\widetilde{T}k^2f^2(\widetilde{T}k^2/(p-p_c)^{\phi_T}) . \qquad (5.8)$$

A rescaling of the integral yields $\lambda^{-2}\sim(p-p_c)^t$ where $t=2\beta+\phi_T=3$ is identical to the mean-field conductivity exponent for the random resistor network. Below the upper critical dimension $d_c=6$, $c_1\sim|p-p_c|^{-\eta\nu}$ so that the exponent for ρ_s is $(d-2)\nu+\phi_T$ which reduces to 3 at $d=6$. A more detailed treatment of this and other scaling results will appear in a forthcoming paper. The effective superconducting coherence length ξ_s, determined by the mass of the field $\psi_{1,0,\ldots,0}$, is simply ξ_p in the zero-temperature limit. It follows that the ratio $\lambda^2/\xi_s^2\sim|p-p_c|^{-(t-2\nu)}$ diverges near threshold for $d\geq 3$ ($t=3$ and $\nu=1/2$ in mean field theory and $t=1.85$ and $\nu=0.85$ in three dimensions).[24] Thus sufficiently close to p_c, the granular medium behaves like a type-II superconductor. As in a usual superconductor there is a critical field

$$H_{c1}=\frac{\phi_0}{4\pi\lambda^2}\ln\lambda/\xi_p\sim\mu^0|p-p_c|^t \qquad (5.9)$$

above which the Meissner phase becomes unstable to the formation of an Abrikosov flux lattice [Fig. 1(a)].

(ii) In the zero-temperature limit, the structure of an isolated vortex may be obtained by considering solutions to the Landau-Ginzburg equations of the form:

$$\psi_{\mathbf{k}}=\xi_p^{-2}f_{\mathbf{k}}(r)e^{im_{\mathbf{k}}\theta} , \qquad (5.10)$$

where r and θ are cylindrical coordinates ($d=3$) measured from the vortex core and $m_{\mathbf{k}}=m_{-\mathbf{k}}$ is an integer winding number for a single-valued function $\psi_{\mathbf{k}}$. As in a usual vortex, we take $\mathbf{A}=A(r)\hat{\theta}$ with $A(r)\sim rh(0)/2$ for small r and $h(0)$ the peak magnetic field in the core. Equation (5.1) then becomes

$$f_{\mathbf{k}}+\xi_p^2\left[\frac{1}{r}\frac{d}{dr}\left[r\frac{df_{\mathbf{k}}}{dr}\right]-\left[\frac{m_{\mathbf{k}}}{r}-\frac{2\pi A(r)}{\phi_0}s_{\mathbf{k}}\right]^2 f_{\mathbf{k}}\right]-\sum_{\mathbf{q}\neq 0,\mathbf{k}}f_{\mathbf{k}-\mathbf{q}}(r)f_{\mathbf{q}}(r)e^{i(m_{\mathbf{k}-\mathbf{q}}+m_{\mathbf{q}}-m_{\mathbf{k}})\theta}=0 . \qquad (5.11)$$

A necessary condition for a solution is clearly that $m_{\mathbf{k}+\mathbf{q}}=m_{\mathbf{k}}+m_{\mathbf{q}}$ for all $\mathbf{k}$ and $\mathbf{q}$. Defining $m_\alpha\equiv m_{0,\ldots,0,1,0,\ldots,0}$ with "1" appearing in the αth entry, it follows that $m_{\mathbf{k}}=\sum_{\alpha=1}^{n}k_\alpha m_\alpha$. The only replica symmetric solutions are of the form $m_{\mathbf{k}}=ms_{\mathbf{k}}$ where the integer m corresponds to the number of flux quanta contained in the vortex core. A straightforward power-series solution for the radial function yields for small r:

$$f_{\mathbf{k}}(r)\sim r^{m|s_{\mathbf{k}}|}[1-(r/\xi_{\mathbf{k}})^2+\cdots] , \qquad (5.12)$$

where the healing length of the vortex core

$$\xi_{\mathbf{k}}^2=\xi_p^2\frac{2(2+s_{\mathbf{k}})}{1+\pi h(0)\phi_0^{-1}\xi_p^2 s_{\mathbf{k}}^2}\underset{p\to p_c}{\longrightarrow}\xi_p^2 2(2+s_{\mathbf{k}}) . \qquad (5.13)$$

In obtaining Eq. (5.13) terms which vanish as $n\to 0$ have been neglected, and the latter form follows from the fact that $h(0)\sim H_{c1}\sim|p-p_c|^t$. It is also evident from (5.12) that modes for which $s_{\mathbf{k}}=0$ retain their bulk value for small r, suggesting that vortex cores retain spin-glass type order in the low temperature limit (Fig. 5).

(iii) For applied fields $H>H_g$, the energetically favorable solution to the Landau-Ginzburg equations is that in which complete flux penetration occurs on average ($\nabla\times\mathbf{A}=\mu_0\mathbf{H}$) and $\psi_{\mathbf{k}}=0$ unless $s_{\mathbf{k}}=0$. In the glass phase,

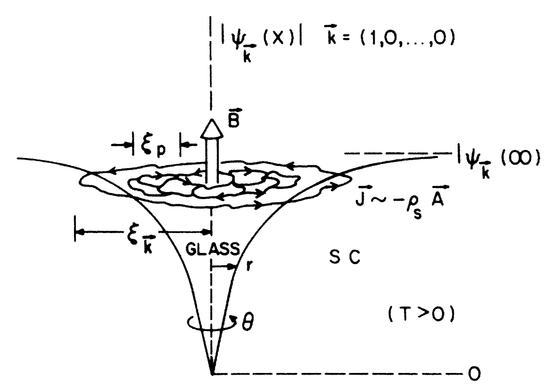

FIG. 5. Structure of an isolated vortex containing one flux quantum in the low-temperature Abrikosov phase. Here r and θ are cylindrical coordinates measured from the vortex core. The magnitude of the XY order parameter $\psi_{\mathbf{k}}$, $\mathbf{k}=(1,0,\ldots,0)$ rises continuously from zero to its bulk value $|\psi_{\mathbf{k}}(\infty)|$ over a healing length $\xi_{\mathbf{k}}$. Inside this core region, there is a frozen-in distribution of randomly oriented tunneling supercurrents characteristic of the spin-glass phase. For $r>\xi_{\mathbf{k}}$ the magnetic field is screened by circulating supercurrents determined by the macroscopic superfluid density ρ_s, and the magnetic field decays exponentially as in more familiar vortices.

the equilibrium macroscopic superfluid density as defined by the London equation [Eqs. (5.4) and (5.5)] is identically zero. The mass tensor of the replicated photon field becomes

$$\gamma_{ij}^{\alpha\beta}=\delta_{ij}\frac{k_BT}{4d}\sum_{s_{\mathbf{k}}=0}k_\alpha k_\beta\,|\psi_{\mathbf{k}}|^2 . \tag{5.14}$$

In the Meissner phase, the reflection symmetry of $\psi_{\mathbf{k}}$ under $k_\alpha\to -k_\alpha$ ensured that this tensor was diagonal in the replica indices $\alpha\beta$. As in the discussion of the superfluid density the penetration of $\mathbf{A}$ breaks this symmetry. If, however, the solution $\psi_{\mathbf{k}}$ remains symmetric under replica permutation, we may replace $k_\alpha k_\beta$ in (5.14) by the symmetrized form

$$n^{-1}\delta_{\alpha\beta}\mathbf{k}^2+(1-\delta_{\alpha\beta})[n(n-1)]^{-1}(s_{\mathbf{k}}^2-\mathbf{k}^2)\underset{n\to 0}{\longrightarrow} n^{-1}\mathbf{k}^2 . \tag{5.15}$$

In obtaining the $n\to 0$ limit above we have used the fact that $s_{\mathbf{k}}=0$ in the summation (5.13). It follows that $\gamma_{ij}^{\alpha\beta}(\mathbf{q})=\delta_{ij}\gamma_g$ is independent of α and β and the glass "stiffness" coefficient

$$\gamma_g=\frac{k_BT}{4d}\frac{1}{n}\sum_{s_{\mathbf{k}}=0}\mathbf{k}^2|\psi_{\mathbf{k}}|^2\underset{n\to 0}{\sim}-[\,|\langle \mathbf{J}(\mathbf{q})\rangle_T|^2]_c . \tag{5.16}$$

The latter identification with the Edwards-Anderson expectation value of the tunneling supercurrent shows that γ_g must be negative. This is most easily seen by considering the vicinity of the spin-glass to normal transition where only the $n(n-1)$ order parameters $\psi_{1,-1,0,\ldots,0}$ dominate the sum (5.16). Analytically continuing $n\to 0$ yields the required physical result. As in the normal and Meissner phases the net macroscopic supercurrent in equilibrium $[\langle \mathbf{J}\rangle_T]_c$ is zero. However, unlike these more familiar phases, the spin glass state is characterized by a randomly oriented distribution of frozen-in tunneling supercurrents as indicated by the nonzero value of γ_g in the $n\to 0$ limit. These frozen supercurrents produce strong local fluctuations in the equilibrium magnetic field despite the fact that their random orientation prevents the occurrence of any global Meissner effect. The familiar Higg's mechanism is modified by the glass condensate which produces interactions between photons in different replicas. The photon propagator Eq. (3.30) for the replica-symmetric condensate becomes

$$\langle \delta A_i^\alpha(\mathbf{q})\delta A_j^\beta(-\mathbf{q})\rangle=\delta_{ij}k_BTM_{\alpha\beta}^{-1} , \tag{5.17}$$

where the matrix

$$M_{\alpha\beta}\equiv(4\pi\mu_0)^{-1}q^2\delta_{\alpha\beta}+\bar\gamma_g,\quad \bar\gamma_g\equiv\gamma_g c_2^2 \tag{5.18}$$

now acquires off-diagonal elements independent of α and β. Clearly $M_{\alpha\beta}$ has one eigenvector $n^{-1/2}(1,1,\ldots,1)$ with eigenvalue $q^2/(4\pi\mu_0)+n\bar\gamma_g$ and a degenerate set of $(n-1)$ eigenvectors orthogonal to this direction in replica space with eigenvalues $q^2/(4\pi\mu_0)$. It follows that

$$M_{\alpha\beta}^{-1}=\frac{4\pi\mu_0}{q^2}\left[\delta_{\alpha\beta}+\frac{1}{n}\left[\frac{1}{1+n(4\pi\mu_0\bar\gamma_g)q^{-2}}-1\right]\right] . \tag{5.19}$$

In the $n\to 0$ limit, all replica photons become massless and

$$\langle \delta A_i^\alpha(\mathbf{q})\delta A_j^\beta(-\mathbf{q})\rangle\underset{n\to 0}{=}\delta_{ij}\frac{4\pi\mu_0k_BT}{q^2}\times\left[\delta_{\alpha\beta}+\frac{4\pi\mu_0}{q^2}|\bar\gamma_g|\right] . \tag{5.20}$$

Here we have made use of the fact that $\bar\gamma_g$ is negative in the $n\to 0$ continuation. In terms of the fluctuating part of the magnetic field $\delta\mathbf{B}(\mathbf{q})=i\mathbf{q}\times\delta\mathbf{A}(\mathbf{q})$, we obtain

$$\begin{aligned}\langle \delta\mathbf{B}^\alpha(\mathbf{q})\cdot\delta\mathbf{B}^\beta(-\mathbf{q})\rangle &\equiv\delta_{\alpha\beta}[\langle\,|\delta\mathbf{B}(\mathbf{q})|^2\rangle_T]_c+(1-\delta_{\alpha\beta})[\,|\langle\delta\mathbf{B}(\mathbf{q})\rangle_T|^2]_c ,\\ &=4\pi\mu_0k_BT[\delta_{\alpha\beta}+4\pi\mu_0|\bar\gamma_g|/q^2] .\end{aligned} \tag{5.21}$$

Thus in the glass phase, there is flux penetration with a uniform nonzero $[\langle\mathbf{B}\rangle_T]_c$ everywhere but with strong local fluctuations of the amplitude proportional to $|\bar\gamma_g|$ leading to power-law $(x^{-(d-2)})$ decay in both $[\langle\delta\mathbf{B}(\mathbf{x})\cdot\delta\mathbf{B}(0)\rangle_T]_c$ and $[\langle\delta\mathbf{B}(\mathbf{x})\rangle_T\cdot\langle\delta\mathbf{B}(0)\rangle_T]_c$.

VI. BEHAVIOR NEAR THE THERMAL TRANSITION

In the vicinity of the thermal transition, the expansion of the replica Hamiltonian H_r [Eq. (3.20b)] may be truncated by retaining only those modes $\psi_{\mathbf{k}}$ which have appreciable amplitude. From Eq. (4.2) it is evident that these are modes for which $\tilde T\mathbf{k}^2$ and $|s_{\mathbf{k}}|$ (in the presence of an applied magnetic field) take on the smallest values. Using the notation $\psi_\alpha(\mathbf{x})\equiv\psi_{(0,\ldots,0,1,0,\ldots,0)}$ where 1 appears in the αth entry and $Q_{\alpha\beta}\equiv\psi_{(1,0,\ldots,-1,0,\ldots,0)}$ where 1 and -1 appear in the αth and βth entries respectively, the replica Hamiltonian may be expanded to third order as

$$\begin{aligned}H_r=&\sum_{\alpha=1}^{n}\int_{\mathbf{x}}\psi_\alpha(\mathbf{x})\left[p_c-p+b\tilde T-c_1\left[\nabla-\frac{2\pi i}{\phi_0}\mathbf{A}^\alpha(\mathbf{x})\right]^2\right]\psi_\alpha{}^*(\mathbf{x})\\&+\sum_{\alpha\neq\beta}\int_{\mathbf{x}}Q_{\alpha\beta}(\mathbf{x})\left[p_c-p+2b\tilde T-c_1\left[\nabla-\frac{2\pi i}{\phi_0}[\mathbf{A}^\alpha(\mathbf{x})-\mathbf{A}^\beta(\mathbf{x})]\right]^2\right]Q_{\beta\alpha}(\mathbf{x})\\&+\sum_{\alpha\neq\beta}\int_{\mathbf{x}}\psi_\alpha(\mathbf{x})Q_{\alpha\beta}(\mathbf{x})\psi_\beta^*(\mathbf{x})+\tfrac{1}{6}\int_{\mathbf{x}}\mathrm{Tr}\underline{Q}^3(\mathbf{x})+\cdots .\end{aligned} \tag{6.1}$$

In the vicinity of the macroscopic superconducting to normal phase transition, cubic and higher-order terms in the spin-glass fields $Q_{\alpha\beta}=Q_{\beta\alpha}{}^*$ may be neglected. In this Gaussian approximation, explicit functional integration over the $n(n-1)$ massive $Q_{\alpha\beta}$ fields produces a quartic coupling among the fields ψ_α. In the long wavelength limit this becomes

$$H_r\{\psi_\alpha\}\equiv-\ln\int DQ_{\alpha\beta}e^{-H_r}$$
$$\approx\int_{\mathbf{x}}\left\{\sum_{\alpha=1}^{n}\psi_\alpha\left[b(\widetilde{T}-\widetilde{T}_c(p))-c_1\left[\nabla-\frac{2\pi i}{\phi_0}\mathbf{A}^\alpha\right]^2\right]\psi_\alpha{}^*+\frac{1}{4b\widetilde{T}}\left[\sum_{\alpha=1}^{n}|\psi_\alpha|^4-\left[\sum_{\alpha=1}^{n}|\psi_\alpha|^2\right]^2\right]+\cdots\right\}. \quad (6.2)$$

Here $T_c(p)\equiv(p-p_c)/b$ is the mean-field transition temperature in the absence of an applied field. This is precisely the form of the free energy that would be obtained from a replicated Landau-Ginzburg model for a disordered bulk superconductor with a local T_c which fluctuates as a Gaussian random variable from point to point in space.[25] Averaging over the random transition temperature yields a term of the form $-(\sum_{\alpha=1}^{n}|\psi_\alpha|^2)^2$. Additional terms of this type arise also from higher-order terms in the expansion (3.19). It follows from (2.7) that the physical free energy F (at the zero loop level) is related to the replica Hamiltonian by

$$F=\lim_{n\to 0}\frac{1}{n}H_r\{\langle\psi_\alpha\rangle\}\ , \quad (6.3)$$

where the field variable ψ_α has been replaced by its equilibrium expectation value. For a replica-symmetric solution $\langle\mathbf{A}^\alpha\rangle=\mathbf{A}(\mathbf{x})$ and $\langle\psi_\alpha\rangle=\psi(\mathbf{x})$, this free energy takes precisely the form of a bulk Landau-Ginzburg superconductor since the last term in Eq. (6.2) is $O(n^2)$ and does not contribute as $n\to 0$. Physically, such an identification arises since near the thermal transition, the effective superconducting coherence length $\bar{\xi}_s$ is very long compared to the scale of inhomogeneities ξ_p of the granular material.

For sufficiently large applied magnetic field (and replica-symmetric vector potential), it is the $Q_{\alpha\beta}$ field which becomes critical at the thermal transition whereas the ψ_α fields remain massive. In this case, the terms involving ψ_α may be neglected in comparison to those involving $Q_{\alpha\beta}$, and the expansion (6.1) becomes

$$H_r\approx\sum_{\alpha\neq\beta}\int_{\mathbf{x}}Q_{\alpha\beta}(p_c-p+2b\widetilde{T}-c_1\nabla^2)Q_{\beta\alpha}$$
$$+\int_{\mathbf{x}}[\tfrac{1}{6}\mathrm{Tr}\underline{Q}^3+\tfrac{1}{8}(\mathrm{Tr}\underline{Q}^2)^2-\tfrac{1}{24}\mathrm{Tr}\underline{Q}^4+\cdots] \quad (6.4)$$

describing the thermal transition from the normal to XY—spin glass phase. Here the $n\times n$ matrix order parameter $Q_{\alpha\beta}$ is the analog of that identified by Parisi[26] as measuring the overlap between two states in different replicas. For a replica symmetric solution this reduces to the familiar Edwards-Anderson order parameter $[\langle e^{i\theta}\rangle_T\langle e^{-i\theta}\rangle_T]_c$. The relevance of replica-symmetry breaking solutions in the present context remains an open question for future investigation. We note, however, that the qualitative features of the spin-glass phase which we have presented within the framework of the replica-symmetric ansatz remain valid in general. Physically, the replica-symmetric ansatz corresponds to the assumption that all possible local minima of the free energy in the space of spin-glass states have the same macroscopic properties. Broken replica symmetry in the present context would correspond[27] to the existence of a spectrum of local minima with different values of the stiffness coefficient $|\gamma_g|\sim[|\langle\mathbf{J}\rangle_T|^2]_c$ measuring the amplitude of frozen-in tunneling supercurrents. Also, the fluctuating part of the magnetic field is related to the induced tunneling current by

$$\mathbf{q}\times\delta\mathbf{B}(\mathbf{q})=(4\pi)/c\mathbf{J}(\mathbf{q})\ .$$

It follows that

$$[|\langle\delta\mathbf{B}(\mathbf{q})\rangle_T|^2]_c=q^{-2}(4\pi/c)^2[\langle\mathbf{J}(\mathbf{q})\rangle_T|^2]_c \quad (6.5)$$

leading as before to power-law-decaying spatial fluctuations in the magnetic field with a spectrum of amplitudes determined by the magnitude of frozen supercurrents in the various spin-glass local minima.

VII. DISCUSSION

In summary, we have shown that a transition from macroscopic superconducting to spin-glass order occurs in a randomly diluted Josephson tunnel junction network by the application of a magnetic field corresponding to approximately one quantum of flux per typical loop of the system in the node-link picture of percolation. This may be experimentally realizable in a granular material of intermediate grain sizes and coupling strengths: The grains must be sufficiently large and strongly coupled that quantum fluctuations may be neglected yet sufficiently small and weakly coupled that the XY phase coherence between grains is distinct from the BCS transition of the grains themselves. The glass phase was obtained in an expansion about the percolation threshold p_c. As $p\to p_c$ from above, both the perimeter of the loops and the fluctuation in their areas becomes very large. Consequently the glass phase in our model occurs for applied fields sufficiently weak that the gauge-invariant phase difference between adjacent grains is very small. In this limit it is sufficient to retain only the leading term in the gradient expansion for the phase difference when passing to the continuum limit. We conjecture, however, that the glass phase is more fundamental in nature and that the percolation limit simply provides an elementary mathematical realization of a system of coupled loops of widely fluctuating areas. Recently, Ebner and Stroud[3] have suggested the possibility of spin-glass behavior in random clusters for which the typical loop of the random network has an area comparable to a^2 where a is the intergrain spacing. In this limit a gradient expansion would be inappropriate. Nevertheless, the condition for the onset of glassy behavior is that $Ha^2 \sim \phi_0$ at zero temperature. It should be emphasized, however, that disorder plays an essential role in the formation of the glass phase which is characterized by an essentially white-noise spectrum of tunneling currents: $[\,|\langle \mathbf{J}(\mathbf{q})\rangle_T|^2]_c \sim |\gamma_g|$ independent of $\mathbf{q}$ for arbitrarily small q leading to power-law decay of spatial fluctuations in the magnetic field. This is distinct from recent studies[28] on ordered Josephson junction arrays in the presence of an irrational or high-order rational number of applied flux quanta per fundamental plaquette which may appear glasslike upon rapid quenching of small samples.

We have shown that in the glass phase the equilibrium macroscopic superfluid density which measures the current response to a zero-frequency vector potential (London gauge) is identically zero. As in more traditional magnetic spin-glasses, as discussed by Sompolinski *et al.*[29] this may correspond to the existence of a finite lifetime of spin-wave excitations due to scattering or tunneling into nearby local minima in the spin-glass configuration space which are separated by small energy barriers. Dynamical properties such as the ac conductivity in the glass phase would provide a valuable probe of the nature of barriers separating metastable states, the associated distribution of relaxation times, and the possible appearance of macroscopic superconducting behavior on time scales short compared to these relaxation times. It has recently been suggested[3] that such relaxation processes would follow an Arhenius law with a thermal attempt frequency $\sim 10^{12}$ sec^{-1} and the scale of typical barrier heights being set by the Josephson coupling energy. The analog of remanence in magnetic spin glasses is also likely to appear for the spin-glass superconductors. For example, if the applied magnetic field is suddenly turned off the frozen-in tunneling currents associated with the glass may decay slowly with time at sufficiently low temperatures.

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[1]Sajeev John and T.C. Lubensky, Phys. Rev. Lett. **55**, 1014 (1985).

[2]W.Y. Shih, C. Ebner, and D. Stroud, Phys. Rev. B **30**, 134 (1984); C. Ebner and D. Stroud, *ibid.* **29**, 5053 (1983).

[3]C. Ebner and D. Stroud, Phys. Rev. B **31**, 165 (1985).

[4]Sajeev John and T.C. Lubensky (unpublished).

[5]G. Deutscher, I. Grave, and S. Alexander, Phys. Rev. Lett. **48**, 1497 (1982).

[6]S. Alexander, Physica (Amsterdam) **126B**, 294 (1984); Phys. Rev. B **27**, 1541 (1983); S. Alexander and E. Halevi, J. Phys. (Paris) **44**, 805 (1983).

[7]P.G. de Gennes and C.R. Seances, Acad. Sci. Ser. B **292**, 279 (1981).

[8]R. Rammal, G. Toulouse, and T.C. Lubensky, J. Phys. (Paris) Lett. **44**, L65 (1983).

[9]P.E. Lindelof, Rep. Prog. Phys. **44**, 949 (1981).

[10]*Percolation, Localization and Superconductivity,* edited by A.M. Goldman and S.A. Wolf (Plenum, New York, 1984).

[11]P. Chaikin (private communication).

[12]B. Abeles, Phys. Rev. B **15**, 2828 (1977); P.W. Anderson, in *Lectures on the Many-Body Problem,* edited by E.R. Caianiello (Academic, New York, 1964), Vol. 2.

[13]M. Tinkham, *Introduction to Superconductivity* (Krieger, Huntington, NY, 1980).

[14]A.S. Skal and B.I. Shkolovskii, Fiz. Tekh. Poluprovodn. **8**, 1586 (1976) [Sov. Phys.—Semicond. **8**, 1029 (1975)]; P.G. de Gennes, J. Phys. (Paris) Lett. **37**, 21 (1976).

[15]See, for instance, L.N. Bulaevskii and M.V. Sadovskii, J. Low Temp. Phys. **59**, 89 (1985).

[16]L. Jacobs, J.V. Jose, and M.A. Novotny, Phys. Rev. Lett. **53**, 2177 (1984).

[17]R.A. Webb, R.F. Voss, G. Grinstein, and P.M. Horn, Phys. Rev. Lett. **51**, 690 (1983).

[18]M. Tinkham, D.W. Abraham, and C.J. Lobb, Phys. Rev. B **28**, 6578 (1983).

[19]R.M. Boysel, A.D. Caplin, M.N.B. Dalimin, and C.N. Guy, Phys. Rev. B **27**, 554 (1983).

[20]R.P. Feynman, *The Feynman Lectures on Physics* (Addison-Wesley, Boston, 1965), Vol. III, Chap. 21.

[21]M.J. Stephen, Phys. Rev. B **17**, 4444 (1978).

[22]A.B. Harris and T.C. Lubensky, J. Phys. A **17**, L609 (1984).

[23]S.F. Edwards and P.W. Anderson, J. Phys. F **7**, 695 (1975).

[24]For a recent compilation of percolation exponents, see A.B. Harris, Phys. Rev. B **28**, 2614 (1983).

[25]D. Boyanovsky and J.L. Cardy, Phys. Rev. B **25**, 7058 (1982).

[26]G. Parisi, Phys. Rev. Lett. **43**, 1754 (1979).

[27]*Heidelberg Colloquium on Spin Glass,* edited by J.L. Van Hemmen and I. Morgenstern (Springer-Verlag, Heidelberg, 1983).

[28]T.C. Halsey, Phys. Rev. Lett. **55**, 1018 (1985).

[29]H. Sompolinsky, G. Kotliar, and A. Zippelius, Phys. Rev. Lett. **52**, 392 (1984).

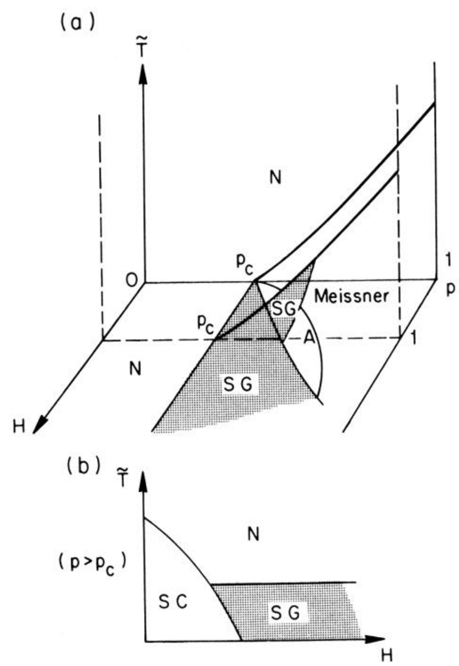

FIG. 1. (a) Mean-field phase diagram as a function of temperature $\widetilde{T}=k_B T/K$ (where K is a typical Josephson coupling energy), applied magnetic field H, and Josephson bond occupation probability p near percolation threshold p_c exhibiting normal (N), Meissner, spin-glass (SG) and Abrikosov (A) vortex lattice phases. Dashed lines depict a slice of this phase diagram in the (T,p) plane for fixed $H>0$. (b) Slice of phase diagram in the (T,H) plane for a fixed concentration of superconducting grains above percolation threshold. Shown are the normal (N), superconducting (SC), and spin glass (SG) phases and the multicritical point where they meet.