Jump in current at the gap voltage in a superconducting junction

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For many materials not previously considered, we have calculated the jump, at the gap voltage, in the quasiparticle current of a tunnel junction. An empirical relationship between the jump and the effective electron-phonon coupling λ-μ^* previously established is confirmed. Further, a new and equally as accurate correlation is found with the strong coupling index $T_c/\omega_{\ln}$, where T_c is the critical temperature and $\omega_{\ln}$ a specific characteristic phonon energy. A simple formula for the jump which includes a strong-coupling correction is derived and found to fit the observed correlation well. Finally, we study the effect on the jump of unusual values of Coulomb pseudopotential μ^*. Also a δ-function electron-phonon spectral density $\alpha^2F(\omega)$ is used to help in the understanding of the range of values that is possible for the jump when $\alpha^2F(\omega)$ is not restricted to realistic shapes.

I. INTRODUCTION

In a previous paper, Harris, Dynes, and Ginsberg[1] showed that there exists a simple empirical relationship between the jump, at the energy gap voltage, in the quasiparticle current (ΔI_{qp}) of a superconducting tunnel junction and the effective electron-electron coupling parameter λ-μ^*. Here, λ is the electron-phonon mass-renormalization parameter and μ^* the Coulomb repulsion pseudopotential. For many cases μ^* is small compared with λ so that an approximate correlation of ΔI_{qp} with λ is also implied.

In this paper we extend the work of Harris, Dynes, and Ginsberg[1] in several ways. First, we calculate ΔI_{qp}, within Eliashberg theory, for many materials previously unconsidered, including transition metals, $A15$ compounds, and model systems with large values of μ^* or with a δ function spectrum. These last systems make it possible for us to check on the validity of the empirical relationship in extreme situations. All the numerical work is done using the imaginary-axis formulation of the Eliashberg equations[2,3] with Padé approximants[4,5] to determine the real part of the gap and its derivative at real frequencies near the gap edge.

In addition to the observed correlation of ΔI_{qp} with λ-μ^* we also consider a possible correlation with the now familiar strong-coupling parameter $T_c/\omega_{\ln}$. Here, T_c is the critical temperature and $\omega_{\ln}$ is a characteristic phonon frequency first introduced and used by Allen and Dynes[6] within the context of a discussion of approximate formulas for T_c. Since a strong correlation between ΔI_{qp} and $T_c/\omega_{\ln}$ is indeed established, we proceed to derive, from the full Eliashberg equations on the real axis, an approximate expression for ΔI_{qp} which contains, in a crude way, strong-coupling effects through a term of the form

$$b\left[\frac{T_c}{\omega_{\ln}}\right]^2 \ln\left[\frac{\omega_{\ln}}{aT_c}\right],$$

with a and b constants. It is found that a and b can be assumed to be material independent with our approximate formula for ΔI_{qp} giving a good qualitative fit to the exact numerical data at the 11% accuracy level for most of the cases considered.

In the work described so far, we have restricted the value of the electron-phonon spectra density [$\alpha^2F(\omega)$] to observed shapes and strengths. To obtain some information on how widely ΔI_{qp} might range when such restrictions on $\alpha^2F(\omega)$ are removed, we consider model spectra consisting of a single δ function at the frequency Ω_E. In this case, the jump is completely independent of the area under $\alpha^2F(\omega)$, at least for $\mu^*=0.7$ In contrast, the normalized jump $J_R \equiv \Delta_{qp}/\Delta I_{qp}^{WC}$ (where WC refers to weak coupling) is found to range from a value near 1 when $\Omega_E \gg T_c$ to 1.92 at $\Omega_E=0.25$ meV. While the jump was still rising with decreasing Ω_E in this region, the calculations were nevertheless terminated because some numerical difficulties were becoming apparent and 0.25 meV is already unrealistically low.

The paper is divided into five sections and an appendix. Section II deals with the numerical work for observed as well as some model spectra. In Sec. III the two parameters introduced during the derivation of an approximate analytic formula for the jump are fixed by comparison with the exact calculations. The derivation itself is outlined in the Appendix. Section IV deals briefly with our δ function results while conclusions are given in Sec. V.

II. NUMERICAL CALCULATIONS OF THE JUMP

To calculate the jump in the current-voltage characteristics of a tunneling junction at zero temperature which occurs at the gap edge, we use the imaginary axis formulation of the Eliashberg equations. They are[2,3]

$$\Delta(i\omega_n)Z(i\omega_n)=\pi k_B T \sum_{m=-\infty}^{+\infty} [\lambda(n-m)-\mu^*(\omega_c)\Theta(\omega_c-|\omega_m|)]\frac{\Delta(i\omega_m)}{[\omega_m^2+\Delta^2(i\omega_m)]^{1/2}} \quad (1)$$

and

$$\omega_n Z(i\omega_n)=\omega_n+\pi k_B T\sum_{m=-\infty}^{+\infty}\lambda(n-m)\frac{\omega_m}{[\omega_m^2+\Delta^2(i\omega_m)]^{1/2}}\ , \tag{2}$$

where Δ and Z are the Matsubara gap and renormalization function, respectively, evaluated on the imaginary frequency axis at the discrete points

$$i\omega_n=i\pi k_B T(2n-1),\quad n=0,\pm1,\pm2,\ldots .$$

In Eqs. (1) and (2), T is the temperature, k_B the Boltzmann constant, ω_c a cutoff on the Coulomb repulsion needed to get a convergent sum over m, and $\mu^*(\omega_c)$ the Coulomb pseudopotential appropriate to the cutoff ω_c. The electron-phonon function is given by

$$\lambda(n-m)=2\int_0^\infty d\Omega\frac{\Omega\alpha^2F(\Omega)}{\Omega^2+(\omega_n-\omega_m)^2}\ ,$$

where $\alpha^2F(\Omega)$, the electron-phonon spectral density, is known, in a large number of cases, from inversion of tunneling data. We can take $\alpha^2F(\Omega)$ as given.

Equations (1) and (2) are solved numerically for a given $\alpha^2F(\omega)$ with μ^* adjusted to get the measured T_c. Next, the equations are solved at a low temperature, usually $0.1T_c$ and a set of $\Delta(i\omega_n)$ and $Z(i\omega_n)$ obtained. From this information, the gap at zero temperature is obtained for real frequencies by a method of Padé approximants first described by Vidberg and Serene[4] and implemented by Mitrović *et al.*[5] The gap edge Δ_0 is defined as the value of the real part of the gap $\Delta(\omega)$ at $\omega=\Delta_0$ with

$$\Delta_0=\mathrm{Re}\Delta(\omega=\Delta_0)\equiv\Delta_1(\omega=\Delta_0)\ . \tag{3}$$

The imaginary part of Δ is zero at $\omega=\Delta_0$ and the jump in the quasiparticle current ΔI_{qp} at voltage Δ_0 is obtained from the formula[16]

TABLE I. Various parameters characterizing the electron-phonon spectral densities $[\alpha^2F(\omega)]$ used in this work as well as the results of the Eliashberg calculations based on these spectra. The specific quantities considered are the derivative of the real part of the gap at the gap edge $[d\Delta_1(\omega)/d\omega]\,|_{\Delta_0}$ and the derived quantities J_R and $(1+\lambda/\lambda)[d\Delta_1(\omega)/d\omega]|_{\Delta_0}$. J_R is the ratio of the jump, at the gap voltage, in the quasiparticle current of a tunnel junction compared to the BCS value. λ is the electron-phonon coupling parameter.

Material	T_c (K)	$\omega_{\ln}$ (K)	$T_c/\omega_{\ln}$	ω_m (meV)	N	$\mu^*(\omega_c)$	λ	$\frac{d\Delta_1(\omega)}{d\omega}\Big\|_{\Delta_0}$	J_R	$\frac{1+\lambda}{\lambda}\frac{d\Delta_1(\omega)}{d\omega}\Big\|_{\Delta_0}$
V	5.38	171.6	0.0314	33.1	3	0.2025	0.8013	0.0078	1.0078	0.0174
Ta	4.48	132.	0.0339	20.9	3	0.1195	0.6923	0.0074	1.0075	0.0182
V_3Si	10.1	223.6	0.0452	44.4	6	0.1093	0.7494	0.0098	1.0098	0.0228
$In_{0.9}Tl_{0.1}$	3.28	63	0.0521	16.2	6	0.1323	0.8503	0.0162	1.0163	0.0353
Nb	9.26	148.9	0.0622	28.3	3	0.1749	1.009	0.0209	1.0210	0.0416
Nb(scaled)	9.26	148.9	0.0622	28.3	3	0.0	0.6241	0.0152	1.0153	0.0397
Nb(scaled)	9.26	148.9	0.0622	28.3	3	0.75	1.8132	0.0283	1.0285	0.0439
V_3Si	13.8	218.7	0.0631	43.5	6	0.1268	0.9186	0.0193	1.0194	0.0403
V_3Si	15.4	228.7	0.0673	42.2	6	0.1018	0.9017	0.0203	1.0204	0.0428
NbN	14.0	174.6	0.0802	60.65	4	0.3522	1.4731	0.0393	1.0397	0.0660
V_3Si(k)	18.0	200.9	0.0896	44.5	5	1.3	2.2849	0.0489	1.0495	0.0703
$Pb_{0.4}Tl_{0.6}$	4.60	48	0.0958	11.0	6	0.1149	1.1459	0.0376	1.0380	0.0705
$Pb_{0.6}Tl_{0.4}$	5.90	50	0.1180	10.9	6	0.1252	1.3813	0.0537	1.0544	0.0925
Pb	7.19	56.0	0.1284	11.0	10	0.1508	1.5477	0.0621	1.0631	0.1022
Pb	7.19	56.0	0.1284	11.0	6	0.1438	1.5477	0.0623	1.0633	0.1026
Pb(anal)	7.19	56.0	0.1284	11.0	6	0.1438	1.5477	0.0623	1.0632	0.1025
Pb	7.19	56.0	0.1284	11.0	3	0.1308	1.5477	0.0636	1.0646	0.1046
Pb(anal)	7.19	56.0	0.1284	11.0	3	0.1308	1.5477	0.0635	1.0645	0.1045
Pb	7.19	56.0	0.1284	11.0	6	0.0	1.0802	0.0501	1.0507	0.0964
Pb(scaled)	7.19	56.0	0.1284	11.0	6	0.75	2.5748	0.0804	1.0820	0.1116
Nb_3Sn	18.05	124.0	0.1456	28.7	6	0.1576	1.7005	0.0730	1.0743	0.1159
Hg	4.19	28.6	0.1465	14.3	6	0.1244	1.6241	0.0750	1.0764	0.1212
$Pb_{0.6}Bi_{0.2}Tl_{0.2}$	7.26	47.9	0.1516	10.2	6	0.1525	1.8142	0.0811	1.0827	0.1298
$Pb_{0.9}Bi_{0.1}$	7.65	50	0.1530	9.9	6	0.1054	1.6629	0.0775	1.0790	0.1241
Nb_3Ge	20.0	125.4	0.1595	34.4	6	0.0878	1.6000	0.0770	1.0785	0.1251
$Pb_{0.8}Bi_{0.2}$	7.95	46	0.1728	10.97	6	0.1116	1.884	0.0931	1.0952	0.1425
δ function	8.09	46.4	0.1743	11.0	6	0.1	2.0	0.1017	1.1043	0.1526
δ function	12.16	69.6	0.1746	11.0	6	0.1	2.0	0.1025	1.1052	0.1538
δ function	4.07	23.2	0.1753	11.0	6	0.1	2.0	0.1020	1.1046	0.1530
$Pb_{0.7}Bi_{0.3}$	8.45	47	0.1798	10.4	6	0.1095	2.0145	0.1026	1.1053	0.1535
$Pb_{0.65}Bi_{0.5}$	8.95	45	0.1989	10.1	6	0.0913	2.1320	0.1145	1.1178	0.1682

$$\frac{\Delta I_{qp}}{\Delta I_{qp}^{WC}} = \left[1+\frac{1}{2}\frac{d\Delta_1(\omega)}{d\omega}\bigg|_{\omega=\Delta_0}\right]^2 \equiv J_R \ , \tag{4}$$

where we have normalized ΔI_{qp} to its weak-coupling value. We see from (4) that J_R depends only on the first derivative of the real part of the gap evaluated at Δ_0. The required derivative is obtained from a knowledge of $\Delta_1(\omega)$ at three frequencies near Δ_0 through the fitting of a parabola.

In Table I we present our results for a large number of superconductors. References to the source of $\alpha^2F(\omega)$ (electron-phonon spectral density) are given in Refs. 9–19 as well as Mitrović *et al.*[15] and Daams and Carbotte.[2] Along with $d\Delta_1(\omega)/d\omega|_{\omega=\Delta_0}$, from which J_R follows [Eq. (4)], additional useful physical data was given in Table I. They are the critical temperature T_c, the Allen and Dynes[6] phonon energy $\omega_{\ln}$ defined by

$$\omega_{\ln}=\exp\left[\frac{2}{\lambda}\int^{\infty}\frac{d\omega}{\omega}\alpha^2F(\omega)\ln(\omega)\right] , \tag{5}$$

the ratio $T_c/\omega_{\ln}$, the maximum phonon energy ω_m in $\alpha^2F(\omega)$, the cutoff ω_c, in units of ω_m, the corresponding $\mu^*(\omega_c)$ chosen to get the measured critical temperature, the mass renormalization $\lambda=2\int_0^{\infty}[\alpha^2F(\omega)/\omega]d\omega$, and $d\Delta_1(\omega)/d\omega\,|_{\omega=\Delta_0}$. The two final columns of Table I give the right-hand side of (4) denoted by J_R and $(1+\lambda)/\lambda$ times $d\Delta_1(\omega)/d\omega\,|_{\omega=\Delta_0}$. This will be useful later on.

Four of the materials in Table I have also been considered by Harris *et al.*[1] They were chosen to check the analytic continuation technique at low, medium, and high values of $T_c/\omega_{\ln}$. They are $In_{0.9}Tl_{0.1}$, Hg, $Pb_{0.7}Bi_{0.3}$, and Pb. With the exception of Pb, the agreement with Harris *et al.* is within 2% for the derivative $d\Delta_1(\omega)/d\omega\,|_{\omega=\Delta_0}$. This is very satisfactory since our method of calculation is so very different from that of Harris *et al.* While we work on the imaginary axis, they use the real frequency axis Eliashberg equations which yield directly $\Delta(\omega)$ without the intermediate step of using Padé approximants. It should be noted that we cannot expect perfect agreement since μ^* is treated somewhat differently in the two approaches. In both cases, a sharp cutoff is used on ω but a sharp cutoff on the real axis does not correspond to a sharp cutoff on the imaginary axis and vice versa, as described by Leavens and Fenton.[20]

For the case of Pb we have varied both the cutoff and the temperature used for the analytic continuation. Referring to Table I, we note entries for three different cutoffs, namely, $\omega_c=10\omega_m$, $6\omega_m$, and $3\omega_m$. As the cutoff is increased there is a small reduction in J_R going from 1.0646 to 1.0631. This is considerably larger than the 1.056 quoted by Harris *et al.*[1] To make sure that the difference cannot be due to the temperature used in our analytic continuation, we have recalculated everything at a new lower temperature, namely $t=0.05T_c$ rather than $t=0.1T_c$. The results are entered as Pb(anal). It is clear that $t=0.1T_c$ is low enough to get the normalized jump with sufficient accuracy.

Other data entered in Table I that should be mentioned explicitly are the results for NbN and for V_3Si. Note that the entry under $V_3Si(k)$ has a $\mu^*=1.3$, which is enormous, while the value for NbN is more modest but still large at 0.35 compared with most other entries. These data can be used to help us understand the effect of μ^* on the current jump as can the entries under Pb(scaled) and Nb(scaled). In these last two cases the electron-phonon spectrum $\alpha^2F(\omega)$ for Pb and Nb were rescaled without changing their shape so as to retain the same T_c value with $\mu^*=0$ and 0.75, respectively. It is seen that under these circumstances, J_R increases quite substantially with increasing μ^*. For example, for a spectrum having the shape of Pb it can range from 1.051 to 1.082. To make further comparison, it is best to plot the data of Table I.

In view of the work of Harris *et al.* we show, in Fig. 1, J_R versus λ-μ^*. The dashed line is the empirical curve of Harris *et al.* for their data which was read from their figure. It is seen that our new data confirms the established trend, although only few data fall at large values of λ-μ^*. Some deviations from the general trend which are worth mentioning have been labeled explicitly in the figure. It is seen that the case of our scaled spectrum Nb(scaled) and Pb(scaled) with unusual values of μ^*, the trend given by the dashed line is violated. This is also so for the V_3Si data labeled $V_3Si(k)$. This spectrum was derived from tunneling data by Bangert *et al.*[16] but has an unreasonably large Coulomb pseudopotential value $\mu^*=1.3$. It has been shown that it does not lead to a good fit to the observed thermodynamics or to some optical properties. On the other hand, the more conventional spectra for similar samples of V_3Si by Kihlstrom[17] do fall close to the dashed line. We can conclude from all this that the relationship between J_R and λ-μ^* is more complicated in cases when μ^* is unusually large, but that for most of the real material cases known at present the empirical relationship of

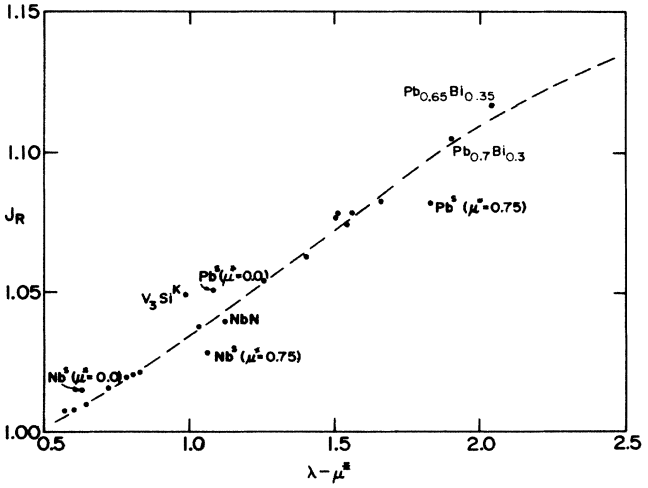

FIG. 1. Plot of the calculated values of J_R, the jump ratio at the gap voltage in the quasiparticle current of a tunnel junction versus the effective electron-phonon coupling λ-μ^*. The dashed line is the empirical curve of Harris *et al.*

Harris *et al.*[1] is reasonably valid and can be used with confidence.

For thermodynamic properties, there exist in the literature approximate analytic formulas[5,9,21–23] derived from the full Eliashberg theory, which contain a rough correction for strong coupling of the form

$$c\left[\frac{T_c}{\omega}\right]^2 \ln\left[\frac{\omega}{T_c d}\right], \tag{6}$$

where c and d are constants and ω is an appropriate phonon frequency. In the original literature,[21–23] c and d are found to take on definite values fixed during the course of the approximations, but the precise nature of ω remains ambiguous and several suggestions have been made in the past as to the most appropriate choice for this parameter. Very recently, complete numerical solutions of the Eliashberg equations have been generated for many materials[10–15] for which $\alpha^2F(\Omega)$ and μ^* are known from tunneling. It has been found that if ω appearing in the form (6) is chosen to be the Allen-Dynes parameter $\omega_{\ln}$ defined in Eq. (5), the exact numerical data for several thermodynamic coefficients can be fit qualitatively by an expression of the form (6) with c and d taken to be material independent and adjusted to give the best overall agreement to the many cases considered. These works on thermodynamics suggest we consider plotting our data on J_R as a function of the strong-coupling parameter $(T_c/\omega_{\ln})$ to see if a new empirical correlation holds between these two quantities. This is shown in Fig. 2 where it is seen that the correlation of J_R with $T_c/\omega_{\ln}$ is as good as with λ-μ^* in most cases and can be better as in the case of V_3Si. The dashed curve through the points was adjusted by eye to give a reasonable fit to the data but does not imply any theory.

III. AN APPROXIMATE ANALYTIC FORMULA FOR THE JUMP

To better understand the approximate correlations found in Fig. 2, we now derive an approximate formula for J_R based on Eliashberg theory. It is most convenient for this work to begin with the Eliashberg equations on the real axis. The derivation, as outlined in the Appendix, follows the method used by Mitrović, Zarate, and Carbotte[5] in deriving an approximate result for $2\Delta_0/k_BT_c$. The result is

$$\frac{1+\lambda}{\lambda}\frac{d\Delta_1(\omega)}{d\omega}\bigg|_{\omega=\Delta_0} = a\left[\frac{T_c}{\omega_{\ln}}\right]^2 \ln\left[\frac{\omega_{\ln}}{bT_c}\right], \tag{A10}$$

where a and b are our fitting parameters.

Recall that in Fig. 2 we plotted J_R against $T_c/\omega_{\ln}$ to see how good a correlation exists between these two quantities. Formula (A10) suggests that we should instead plot

$$\frac{1+\lambda}{\lambda}\frac{d\Delta_1(\omega)}{d\omega}\bigg|_{\omega=\Delta_0}$$

versus $T_c/\omega_{\ln}$. This is done in Fig. 3. Note that the scatter of data points is much less than in Fig. 2. The dashed curve in this figure was chosen to conform with (A10) and the parameters a and b chosen to give the best visual fit to the overall set of points. Of course, no single curve can reproduce the entire data set exactly but that is not the point here. We are willing to sacrifice some precision in order to get material-independent values for a and b. We see that in most cases,

$$\frac{1+\lambda}{\lambda}\frac{d\Delta_1(\omega)}{d\omega}\bigg|_{\omega=\Delta_0} = 4.6\left[\frac{T_c}{\omega_{\ln}}\right]^2 \ln\left[\frac{\omega_{\ln}}{2T_c}\right] \tag{7}$$

will give a good first estimate of the slope of the gap at

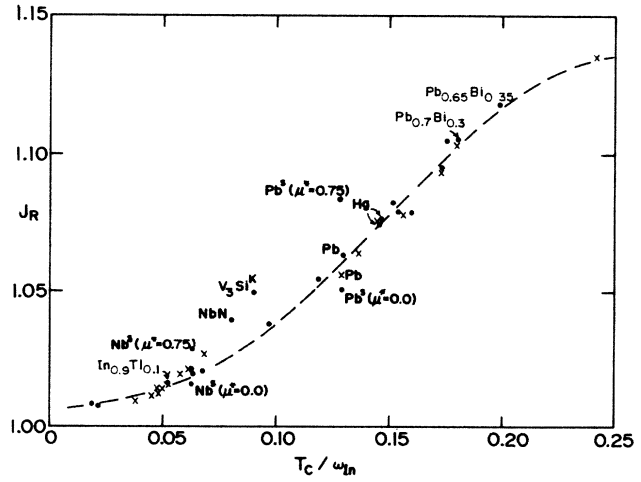

FIG. 2. Plot of the calculated values of J_R (dots) and the data of Harris *et al.* (crosses) plotted versus the strong-coupling parameter $T_c/\omega_{\ln}$. The dashed line is empirical and does not represent any theory.

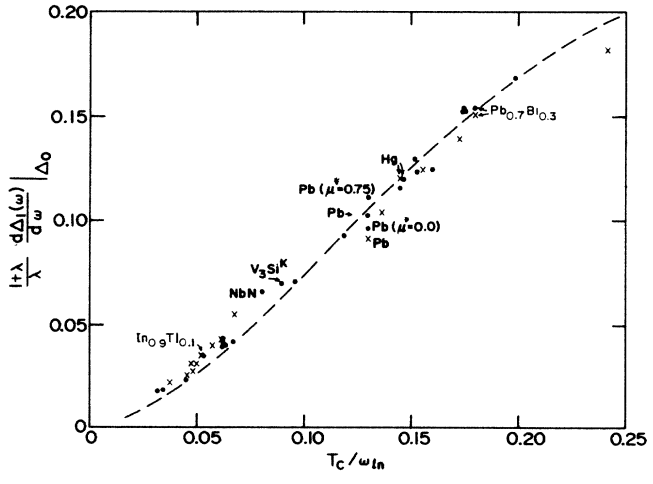

FIG. 3. Plot of the calculated values of $(1+\lambda/\lambda)[d\Delta_1(\omega)/d\omega]|_{\Delta_0}$ solid circles and the values derived from the data of Harris *et al.* (crosses) plotted versus the strong-coupling parameter $T_c/\omega_{\ln}$. The dashed line is that of the approximate formula given in Eq. (7).

the gap edge and therefore of the jump in the current voltage characteristics. To get more precision it is of course necessary to perform numerical work which can be quite tedious. For most applications, formula (7) should be sufficient.

IV. A δ FUNCTION SPECTRUM

So far, we have considered mainly realistic forms for $\alpha^2F(\Omega)$ and values of μ^* that fall within the conventional range of 0 to 0.2. A few large values for μ^* were also discussed. At this point we would like to study a related question, namely: What range of values can we get for J_R when the shape of $\alpha^2F(\Omega)$ is not constrained to be that observed in tunneling experiments? A complete answer to such a question cannot be obtained since we cannot try all possible shapes. This is not necessary for our main purpose here. We can use a δ function,

$$\alpha^2F(\Omega)=A\delta(\Omega-\Omega_E) \tag{8}$$

positioned at $\Omega=\Omega_E$ with area under α^2F given by A. Such an α^2F ranges from weak to very strong coupling as the frequency Ω_E ranges from large to small values. It is to be noted that for the spectrum (18) it can be shown[7] that $\Delta(\omega)$ scales like A as does T_c so that $\Delta(\omega)/A$ is independent of A. This implies that all properties dependent only on the gap will also scale like A. This scaling is exact for $\mu^*=0$ and very nearly so for finite μ^*. The three δ function entries of Table I show that this is indeed true. We find that as A is changed from 2.0 to 6.0 the correction to J_R due to strong coupling varies by only 0.5%. It is sufficient, therefore, to choose a single value for A and change Ω_E. Results for the current jump J_R as a function of Ω_E are presented in Fig. 4. It is seen that as $\omega_{\ln}=\Omega_E$ is decreased, J_R increases radically from its weak-coupling limit of 1 to 1.92 at $\Omega_E=0.25$ meV. The calculations were stopped at this point for two reasons. Firstly, the numerical work was showing signs of breaking down and secondly, Ω_E is already unreasonably small. No real spectrum could be close to a δ function with all its weight around 0.25 meV. Figure 4, however, does

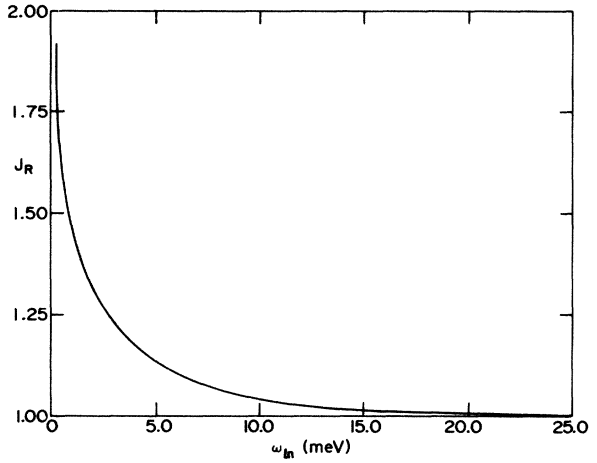

FIG. 4. Plot of J_R calculated for δ function spectra of constant area versus the frequency.

show clearly that J_R can get very large if the shape of $\alpha^2F(\Omega)$ is allowed to vary beyond the expected physical region.

V. CONCLUSIONS

The jump (J_R) in the quasiparticle current at the gap voltage has been calculated numerically from $\alpha^2(\omega)$ data for many materials not previously considered. A correlation previously found between J_R and the coupling parameter λ-μ^* is confirmed. An additional correlation with the much used strong-coupling parameter $T_c/\omega_{\ln}$ is noted and an approximate analytic formula is derived that fits the numerical data at the 11% level well. Using a δ function for $\alpha^2F(\omega)$ with all the weight at a single frequency Ω_E it is found that J_R can become very large for such an unconstrained spectrum compared with the values found for real spectra.

ACKNOWLEDGMENT

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APPENDIX: JUMP APPROXIMATE ANALYTICAL FORMULA DERIVATION

We begin with the Eliashberg equations on the real axis. As a first approximation we ignore the imaginary part of $\Delta(\omega)$. The real part will satisfy the equations,[8]

$$\Delta_1(\omega)Z_1(\omega)=\int_{\Delta_0}^{\infty}d\omega'\mathrm{Re}\left[\frac{\Delta(\omega')}{[(\omega')^2-\Delta^2(\omega')]^{1/2}}\right][\bar{K}_+(\omega,\omega')-\mu^*(\omega_c)\Theta(\omega-\omega_c)]\,, \tag{A1a}$$

$$Z_1(\omega)=1-\frac{1}{\omega}\int_{\Delta_0}^{\infty}d\omega'\mathrm{Re}\left[\frac{\omega'}{[(\omega')^2-\Delta^2(\omega')]^{1/2}}\right]\bar{K}_-(\omega,\omega')\,, \tag{A1b}$$

where

$$\bar{K}_+(\omega,\omega')=2\int_0^{\infty}d\Omega\alpha^2F(\Omega)\frac{(\Omega+\omega')}{(\Omega+\omega')^2-\omega^2}\,, \tag{A2}$$

and

$$\bar{K}_-(\omega,\omega')=-2\int^{\infty} d\Omega\alpha^2F(\Omega)\frac{\omega}{(\Omega+\omega')^2-\omega^2}\ , \tag{A3}$$

with Z_1 the real part of the renormalization function. Taking the derivative of $\Delta_1(\omega)$ with ω and evaluating it at $\omega=\Delta_0$, we get from (A1a) and (A1b) with reference to (A2) and (A3)

$$\left.\frac{d\Delta_1(\omega)}{d\omega}\right|_{\omega=\Delta_0}=\frac{1}{Z(\Omega_0)}\int_{\Delta_0}^{\infty} d\omega' \mathrm{Re}\left[\frac{\Delta(\omega')}{[(\omega')^2-\Delta^2(\omega')]^{1/2}}\right]\left.\frac{\partial\bar{K}_+(\omega,\omega')}{\partial\omega}\right|_{\omega=\Delta_0}$$

$$+\frac{\Delta_1(\Omega_0)}{Z(\Delta_0)}\int_{\Delta_0}^{\infty} d\omega' \mathrm{Re}\left[\frac{\omega'}{[(\omega')^2-\Delta^2(\omega')]^{1/2}}\right]\left.\frac{\partial}{\partial\omega}\left[\frac{\bar{K}_-(\omega,\omega')}{\omega}\right]\right|_{\omega=\Delta_0}. \tag{A4}$$

To proceed further, we will assume that in the integrals $\Delta(\omega')=\Delta_0$ and that all the important phonon frequencies in $\alpha^2F(\Omega)$ are much larger than the gap. With these assumptions, Eq. (A4) takes on the simpler form:

$$\left.\frac{d\Delta_1(\omega)}{d\omega}\right|_{\omega=\Delta_0}=\left[-\frac{4\Delta_0}{Z(\Delta_0)}\int_0^{\infty} d\Omega\alpha^2F(\Omega)\int_{\Delta_0}^{\infty} d\omega'\frac{\Delta_0}{[(\omega')^2-\Delta_0^2]^{1/2}}\frac{1}{(\omega'+\Omega)^3}\right.$$

$$\left.+\frac{4\Delta_0\Delta_1(\Delta_0)}{Z(\Delta_0)}\int_0^{\infty} d\Omega\alpha^2F(\Omega)\int_{\Delta_0}^{\infty} d\omega'\frac{\omega'}{[(\omega')^2-\Delta_0^2]^{1/2}}\frac{1}{(\omega'+\Omega)^4}\right], \tag{A5}$$

which can be further reduced to

$$\left.\frac{d\Delta_1(\omega)}{d\omega}\right|_{\omega=\Delta_0}=-\frac{4\Delta_0^2}{Z(\Delta_0)}\int_0^{\infty} d\Omega\alpha^2F(\Omega)\left[-\frac{3}{2}\frac{1}{\Omega^3}-\frac{1}{\Omega^3}\ln\left[\frac{\Delta_0}{2\Delta}\right]\right]+\frac{4\Delta_0^2}{Z(\Delta_0)}\int_0^{\infty} d\Omega\alpha^2f(\Omega)\frac{13}{6}\frac{1}{\Omega^3}\ . \tag{A6}$$

Following Marsiglio and Carbotte,[9] we approximate the final integrals over $\alpha^2F(\Omega)$ by

$$\int_0^{\infty} d\Omega\frac{2\alpha^2F(\Omega)}{\Omega^3}\ln\left[\frac{\Delta_0}{2\Omega}\right]\cong\alpha_1\frac{\lambda}{\omega_{\ln}^2}\ln\left[\frac{2\Delta_0}{\omega_{\ln}}\right], \tag{A7}$$

and

$$\int_0^{\infty} d\Omega\frac{2\alpha^2F(\Omega)}{\Omega^3}\cong\alpha_2\frac{\lambda}{\omega_{\ln}^2}\ , \tag{A8}$$

with α_1 and α_2 to be treated later as parameters to be fitted to our exact data for J_R.

An expression for $Z(\Delta_0)$ has been worked out in the paper by Mitrović, Zarate, and Carbotte[5] which contains a strong-coupling correction but this is not needed here since in Eq. (A6) each term is already proportional to a strong-coupling correction which we take to be small. To be consistent, we replace $Z(\Delta_0)$ by its approximate value $1+\lambda$. This leads to a final formula of the form

$$\left.\frac{d\Delta_1(\omega)}{d\omega}\right|_{\omega=\Delta_0}=\frac{\lambda}{1+\lambda}\left[2\alpha_2\left[\frac{\Delta_0}{\omega_{\ln}}\right]^2\ln\left[\frac{\omega_{\ln}}{2\Delta_0}\right]\right.$$

$$\left.+\tfrac{22}{3}\alpha_1\left[\frac{\Delta_0}{\omega_{\ln}}\right]^2\right]. \tag{A9}$$

Finally, in Eq. (A9) we change from Δ_0 to T_c using the BCS relation $2\Delta_0/k_BT_c=3.53$ and introduce two new parameters a and b, to get the simple form

$$\frac{1+\lambda}{\lambda}\left.\frac{d\Delta_1(\omega)}{d\omega}\right|_{\omega=\Delta_0}=a\left[\frac{T_c}{\omega_{\ln}}\right]^2\ln\left[\frac{\omega_{\ln}}{bT_c}\right], \tag{A10}$$

which is our final expression.

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