RAPID COMMUNICATIONS

PHYSICAL REVIEW B VOLUME 34, NUMBER 6 15 SEPTEMBER 1986

Probability distributions in the scaling theory of localization

Boris Shapiro*

Department of Physics, Princeton University, Princeton, New Jersey 08544

(Received 1 May 1986)

Probability distributions in disordered electronic systems are studied by a real-space scaling transformation. It is shown that the resistance distribution at the mobility edge is very broad. The resistivity distribution at the metallic side of the transition is also broad, so that the metallic resistivity is not a self-averaging quantity. The calculation is consistent with a single-parameter scaling for the distributions. Near the mobility edge the transformation works better when the dimensionality d is close to 2, i.e., $d=2+\varepsilon$ ($\varepsilon \ll 1$). The results are then extrapolated to $d=3$.

Some time ago Abrahams, Anderson, Licciardello, and Ramakrishnan formulated a one-parameter scaling theory for the electronic transport in disordered systems.[1] The scaling parameter in their theory is a dimensionless zero-temperature dc conductance $g(L)$ for a system of size L. It is claimed that $g(L)$ scales according to $d\ln g/d\ln L = \beta(g)$, where $\beta(g)$ is a universal scaling function, depending on dimensionality d only (in the absence of magnetic, spin-orbit, or electron-electron interactions).

Since, however, the system is disordered, its conductance (for a given L) will depend on the specific realization of the random potential. It is therefore appropriate to define a statistical ensemble of all possible realizations and study the distribution function $W_L(g)$ for conductances, at a scale L. In terms of the distribution the single-parameter scaling of Ref. 1 means that—in spite of the fact that the "bare," microscopic distribution is model dependent and can be any function of g—the distribution at a macroscopic scale assumes some universal shape. Only one parameter is then needed to specify the distribution and its scaling properties (see, e.g., Refs. 2 and 3 for a somewhat more detailed discussion). In particular, for a system at the mobility edge one should expect a fixed, scale-independent distribution with no parameters at all. The purpose of the present work is to study the distribution $W_L(g)$, or rather the distribution $P_L(\rho)$ for resistances ($\rho \equiv 1/g$), at the mobility edge and away from it. It will be shown that the distributions are rather broad. This is, perhaps, not surprising since it has been known that in disordered systems various transport-related quantities exhibit broad distributions,[4] or strong fluctuations,[5] even in a large system. Even in the metallic regime the conductance displays universal, size-independent fluctuations.[6] There have also been indications, numerical[7] as well as analytical,[8] that the distribution at the mobility edge is rather broad. Reference 8, which appeared when the present work was in progress, uses the Migdal-Kadanoff scaling approach.[9] This approach, which for the localization problem was developed in Refs. 3 and 10, is also used in the present work. However, the results and conclusions of this work are quite different from those of Ref. 8 (see below). Recent work of Imry[6] also suggests a broad distribution at the mobility edge.

The essence of the approach[10] is a scaling transformation done in two steps: one starts with a d-dimensional cube of a size L and first combines b such cubes in series, according to the one-dimensional quantum rule.[11] Next, b^{d-1} chains (of b cubes each) are combined in parallel, according to the classical Ohm's law, to form a cube of size bL. One thus neglects quantum interference in transverse directions, while performing the transformation. This would be, clearly, unacceptable if done for a large sample at once (one would always end up with one-dimensional localization). In the Migdal-Kadanoff approach, however, the approximation is combined with a scaling transformation, with only an infinitesimal increase in scale at each step. An important, and fortunate, feature of the transformation is that it gives a qualitatively correct description of the weak scattering regime. Indeed, in this regime the one-dimensional β function for a typical conductance, as defined in Ref. 11, is equal to $-1-(\frac{1}{2}g)$. Combining then b^{d-1} chains classically gives[3] $\beta(g)=d-2-(\frac{1}{2}g)$, which does have the proper large-g behavior and, for small $\varepsilon \equiv d-2$, correctly describes the transition. It appears, thus, that in the weak scattering regime quantum interference effects are approximately one dimensional in character. In other words, the resistance of a wire in this case scales classically with respect to transverse directions (i.e., under change of the cross section), whereas in the longitudinal direction it scales as for a one-dimensional quantum chain.[12] In fact, the transformation described above is quite similar to the "fan" transformation of Ref. 12.

One thus starts with the one-dimensional quantum rule for combining, in series, two distributions, $P_1(\rho_1)$ and $P_2(\rho_2)$, for resistances ρ_1 and ρ_2, with random phases[13] [see Eq. (A8) of Ref. 11]. Taking a piece of length L as ρ_1 and a small piece ΔL as ρ_2, one can write this equation, with a change of variables, as

$$P^{(1)}_{L+\Delta L}(\rho)=\frac{1}{\pi}\int_{-1}^{1} dy\,(1-y^2)^{-1/2}\int_0^{\infty} d\rho_2 P_{\Delta L}(\rho_2)P_L(u), \tag{1}$$

where $P^{(1)}_{L+\Delta L}(\rho)$ is the combined distribution and the argument u of P_L is

$$u(\rho,\rho_2,y)\equiv\rho+\rho_2+2\rho\rho_2+2y[\rho(1+\rho)\rho_2(1+\rho_2)]^{1/2}.$$

For small enough ΔL, ρ_2 is small. One can then expand

$P_L(u)$ in ρ_2, with the following result:

$$P^{(1)}_{L+\Delta L}(\rho)=P_L(\rho)+\overline{\Delta\rho}\frac{\partial}{\partial\rho}\left[(\rho^2+\rho)\frac{\partial P_L(\rho)}{\partial\rho}\right], \quad (2)$$

where $\overline{\Delta\rho}=\int d\rho_2\rho_2 P_{\Delta L}(\rho_2)$ is the average resistance of a segment of size ΔL. Terms of order $\overline{\Delta\rho^2}$ were omitted in the derivation, whereas terms $\rho\overline{\Delta\rho}$ have been kept (note that terms proportional to $\sqrt{\rho_2}$ disappear after integration). For most ρ's this is justified since, typically, $\rho\gg\Delta\rho$ (L is much larger than ΔL). Nevertheless, the very small-ρ tail of the distribution (smaller than $\overline{\Delta\rho}$) need not obey Eq. (2). Such a tail is totally unimportant for calculating the average resistance $\bar{\rho}$ and its higher moments but might dominate the average conductance $(\overline{1/\rho})$ and its moments. An equation essentially identical to Eq. (2) has been previously derived in Ref. 14, with the following difference: There a particular model, described by a single parameter, was considered, whereas Eq. (2) applies to a large class of models, specified by a distribution $P_{\Delta L}(\rho)$ (i.e., by an infinite number of parameters). This is important, if one wants to demonstrate scaling and universality (see below).

In what follows it will be more convenient to use in Eq. (2) the average resistance $\bar{\rho}_L$ at a scale L, instead of $\overline{\Delta\rho}$. These two quantities are related by[11] $2\bar{\rho}_L=\exp(2\beta L)-1$, where $\beta\equiv\overline{\Delta\rho}/\Delta L$, is the small-scale resistivity. Equation (2) can then be written as

$$P^{(1)}_{L+\Delta L}(\rho)=P_L(\rho)+\tfrac{1}{2}\ln(1+2\bar{\rho}_L)\zeta\frac{\partial}{\partial\rho}\left[(\rho^2+\rho)\frac{\partial P_L(\rho)}{\partial\rho}\right], \quad (3)$$

where $\zeta\equiv\Delta L/L$.

To complete the scaling transformation it is necessary to combine b^{d-1} independent chains ($b=1+\zeta$), with the distribution $P^{(1)}(\rho)$ each, into a single d-dimensional cube of size $L+\Delta L$. Let us, first, make an additional approximation and neglect transverse fluctuations (the extremely anisotropic case of Ref. 8), i.e., assume that all b^{d-1} chains have the same resistance, chosen from the distribution $P^{(1)}$ (it will be shown later that for small ε this assumption is justified). The transformation from the chain distribution $P^{(1)}_{L+\Delta L}(\rho)$ to the cube distribution $P_{L+\Delta L}(\rho)$ amounts then to a trivial rescaling: $P_{L+\Delta L}(\rho)=b^{d-1}P^{(1)}_{L+\Delta L}(b^{d-1}\rho)$. If one now sets $b=1+\zeta$, uses Eq. (3) for $P^{(1)}$, linearizes in ζ, and, finally, replaces the difference $P_{L+\Delta L}(\rho)-P_L(\rho)$ by a differential, one obtains the following differential equation for the distribution $P_L(\rho)$:

$$\frac{\partial P}{\partial\ln L}=\frac{\partial}{\partial\rho}\left[\tfrac{1}{2}\ln(1+2\bar{\rho}_L)(\rho^2+\rho)\frac{\partial P}{\partial\rho}+(d-1)\rho P\right]. \quad (4)$$

This equation describes the evolution of the distribution under change of scale. It contains the average resistance $\bar{\rho}_L$ as a parameter. A closed equation for this parameter is derived by multiplying Eq. (4) by ρ and integrating over ρ:

$$\frac{\partial\bar{\rho}_L}{\partial\ln L}=\tfrac{1}{2}(1+2\bar{\rho}_L)\ln(1+2\bar{\rho}_L)-(d-1)\bar{\rho}_L . \quad (5)$$

I now show that, depending on the initial value of the parameter $\bar{\rho}$ at some arbitrary scale L_0 (this value is denoted below as ρ_0), Eq. (4) displays three distinct limiting distributions.

(i) $\rho_0=\rho_C$, where ρ_C is the root of the equation $(1+2\rho_C)\ln(1+2\rho_C)=2(d-1)\rho_C$. Note that this equation has a real positive solution only for $d>2$. This corresponds to the mobility edge. $\bar{\rho}_L$ is then independent of L and equal to ρ_C. Equation (4) in this case has a fixed, scale-independent solution $P^*(\rho)$, which is obtained by setting the left-hand side to zero and integrating twice with respect to ρ. The first integration constant is zero, whereas the second one is fixed by normalization of the distribution. The result is

$$P^*(\rho)=(\alpha-1)(1+\rho)^{-\alpha}, \quad (6)$$

where $\alpha\equiv 2(d-1)/\ln(1+2\rho_C)$. For $d-2\equiv\varepsilon\ll 1$, ρ_C and α are calculated analytically: $\rho_C=\varepsilon+(\frac{2}{3})\varepsilon^2+O(\varepsilon^3)$, and $\alpha=(1/\varepsilon)+(\frac{4}{3})+O(\varepsilon)$. Extrapolation to $d=3$ gives $\alpha=\frac{7}{3}$. Note that if one solves Eq. (5) directly at $d=3$, numerically, one gets a somewhat different value for α, namely, $\alpha\approx 2.5$. In either case the distribution is so broad that already its second moment diverges [it is important, though, that $\rho^2P^*(\rho)\to 0$ when $\rho\to\infty$, since such boundary terms were set to zero in deriving Eq. (6) (Ref. 15)]. It is not surprising, thus, that the authors of Ref. 8, who did not study the full distribution but only its first two moments (at $d=3$), have arrived at the erroneous conclusion that there is no fixed distribution when transverse fluctuations are neglected.

(ii) $\rho_0<\rho_C$. In this case Eq. (5) tells us that $\bar{\rho}_L$ decreases under increase of L, and for large enough L it scales as $\bar{\rho}_L=AL^{-\varepsilon}$, where A is a constant depending on ρ_0 (this constant diverges when $\rho_0\to\rho_C$). The system thus scales to a metallic behavior. One can neglect then the ρ^2 terms in Eq. (4) and replace $\ln(1+2\bar{\rho}_L)$ by $2AL^{-\varepsilon}$. The resulting equation is solved by $P_L(\rho)=(L^\varepsilon/A)\times\exp(-\rho L^\varepsilon/A)$, which can also be written as $P_L(\rho)=(1/\bar{\rho}_L)\exp(-\rho/\bar{\rho}_L)$. For the resistivity $\lambda\equiv\rho L^\varepsilon$ this translates into a distribution $\tilde{P}(\lambda)=(1/\bar{\lambda})\exp(-\lambda/\bar{\lambda})$, which is a fixed, scale-independent distribution, as should be expected in the metallic regime ($\bar{\lambda}=A$ is the average resistivity). The distribution, however, is broad: the nth moment of λ is $\overline{\lambda^n}=(\bar{\lambda})^n(n!)$, so that, e.g., the standard deviation is not small compared to the mean[16] [moments of the resistance are $\overline{\rho_L^n}=A^n(n!)L^{-n\varepsilon}$].

(iii) $\rho_0>\rho_C$. In this case the right-hand side of Eq. (5) is positive, so that $\bar{\rho}_L$ increases with L and becomes exponentially large, $\ln\bar{\rho}_L=BL$, where B is a constant depending on ρ_0 (it approaches zero when $\rho_0\to\rho_C$). The ρ^2 term in Eq. (4) dominates and the equation reduces to the one-dimensional equation $\partial P/\partial L=B(\partial/\partial\rho)(\rho^2\partial P/\partial\rho)$ in the strongly localized regime (the properties of the d-dimensional sample enter only via the constant B). This equation is satisfied by a Gaussian distribution for the variable $x=\ln\rho$,[14] with a mean $\bar{x}_L=BL$ and variance $\overline{\Delta x_L^2}=2BL$. Thus, again, one can parametrize the distribution by a single parameter, $\bar{x}_L$, or alternatively by $\bar{\rho}_L$.

The present calculation thus suggests that the single-parameter scaling does hold, in the sense that the limiting shape of the distribution $P_L(\rho)$ is universal and can be specified by a single parameter, e.g., $\bar{\rho}_L$. The calculation, however, does not really prove this, since only the existence

of limiting distributions has been demonstrated: It has not been shown that any (or almost any) initial distribution scales towards the limiting solution.

It remains now to be shown that it was justified, at least when ε is small, to neglect transverse fluctuations. To derive the scaling transformation without neglecting those fluctuations it is more convenient to work with conductance, rather than resistance, distributions. The problem of combining chains in parallel amounts then to a standard problem in probability theory, of finding a distribution of a sum of independent random variables (it is not so for resistance). One thus should perform the following steps: (i) Rewrite Eq. (2) in terms of the conductance distributions, $W_L(g)$ and $W^{(1)}_{L+\Delta L}(g)$, instead of the corresponding resistance distributions; (ii) Fourier transform the resulting equation, in order to obtain a relation between characteristic functions, $F^{(1)}_{L+\Delta L}(t)$ and $F_L(t)$, of the distributions (variable t includes an infinitesimal $i\delta$ term to ensure convergence of the integrals and the disappearance of boundary terms). (iii) Raise $F^{(1)}(t)$ to power b^{d-1} $(b=1+\zeta)$ which, with the help of (ii), leads to a recursion (or a differential) equation for $F_L(t)$. It turns out that the resulting equation for $F(t)$ differs from the corresponding equation in the absence of transverse fluctuations only by one term, namely, $F\ln F$ instead of $t\partial F/\partial t$. However, for small ε, the distribution at the mobility edge and in the metallic regime is such that g is typically large, i.e., t is small (of order ε). Since $F(t)=1+it\mu_1+O(t^2)$, where μ_1 is the first moment of the distribution, it follows that the difference between the two terms, $F\ln F$ and $t\partial F/\partial t$, is of order ε^2 (again, the $i\delta$ term in t ensures convergence). Furthermore, in the insulating (strongly localized) regime those terms are negligible anyway. Thus, for small ε, it appears to be justified to neglect transverse fluctuations.

It should be emphasized that, contrary to some recent statements (e.g., Ref. 8), validity of single-parameter scaling does not require that the distribution (in g, $\ln g$, or any other variable) be narrow, but only requires that the large-scale distribution be universal and depend on one parameter. One can chose, for instance, the average resistance $\bar{\rho}_L$ to parametrize the distribution. The scaling equation for this parameter is given by Eq. (5).[17] However, there is nothing special about $\bar{\rho}_L$ and many other quantities can be used to parametrize the distribution. For instance, the median conductance $\tilde{g}_L$ (or resistance $\tilde{\rho}_L = 1/\tilde{g}_L$) represents another possible choice[18] and should also scale according to some universal scaling equation, since it is a universal function of $\bar{\rho}_L$. It appears thus that there are many possible choices of a scaling conductance in Ref. 1 (e.g., $g=1/\bar{\rho}_L$ or $g=\tilde{g}_L$) and correspondingly, many different, although interrelated scaling functions (i.e., β functions). The essential physics (such as localization for small g, metallic transport for large g, a transition at some intermediate g, the corresponding conductivity exponent) should be the same for any meaningful choice of the scaling conductance. "Meaningful choice" means that, first, the quantity should be well defined (i.e., not diverging, as some moments of the distribution do) and, second, it should be controlled by the "bulk" of the distribution, not by some highly unprobable configurations of the random potential (tails of the distribution). In particular, the average conductance $\bar{g}_L$ does not seem to be a meaningful scaling variable. This is certainly so in one dimension, where $\bar{g}_L$ is known to diverge[19] due to transmission resonances.[20] In the present calculation these resonances survive in any dimension, which leads to a divergent $\bar{g}_L$ also for a three-dimensional metal. This clearly contradicts the perturbation theory results for the average conductance and its higher moments (i.e., the conductance fluctuations treated in Ref. 6). The issue of transmission resonances in a three-dimensional metal requires further investigation. In any case, one should keep in mind that the average conductance and its moments might be dominated by the large-g tail of the distribution.[21] If this is indeed so, it can perhaps explain the difficulties which seem to occur in scaling theories based on calculating the average conductance and its moments.[22] The point is that, even if single-parameter scaling does hold, it only means that the bulk of the distribution is universal, whereas the tails need not be. Absence of universal scaling for higher moments of conductance, found in Ref. 22, might reflect just this nonuniversality of the tails.

Finally, the present calculation, as well as that of Ref. 8, demonstrates that the metallic resistivity λ is not a self-averaging quantity: even in the $L\to\infty$ limit λ remains a statistical quantity with a broad, exponential distribution (see above). This implies that for a single realization of the random potential λ fluctuates, as a function of L, at any scale. Alternatively, for an infinite (but fixed) potential λ, generally, will not change in a monotonic way when the mobility edge is approached. Only at finite temperature, even infinitesimal in the $L\to\infty$ limit, the resistivity acquires the self-averaging property. The problem then reduces to a classical resistor network, of cubes of size L_{in} (the inelastic mean free path), with a broad distribution of the cube resistances (in addition, each resistance fluctuates with time). Fluctuation and noise phenomena in classical networks, primarily in the percolation context, have been recently considered in Ref. 23 (and references therein). It would be of interest to study such phenomena in the present context. For the conductivity and its moments, in the weak scattering regime these phenomena were recently studied in Refs. 6 and 24.

I am indebted to P. W. Anderson for many current and past discussions and explanations, and in particular for impressing on me the importance of distributions in the scaling theory. Useful conversations with I. Affleck, P. A. Lee, C. Kuper, A. Pruisken, M. Revzen, and L. Schulman are also gratefully acknowledged. Research at Princeton University was supported by the National Science Foundation under Grant No. DMR 8518163, and at Technion by the U.S.–Israel Binational Science Foundation.

*Permanent address: Department of Physics, Technion-Israel Institute of Technology, 32000 Haifa, Israel.

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