

Random walks on Cayley trees as models for relaxation in a hierarchical system

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We consider a nearest-neighbor random walk on a finite Cayley tree with hopping rates depending on a level index in an arbitrary way, and construct an exact formula for the Laplace transform of the first passage time from one point to another. This function determines, by the renewal equation, the propagator and hence all the properties of the solution. Finally, we apply the formalism to a simple case of thermally activated hopping, and show that the time of return to the origin has, to a good degree of approximation, a stable distribution.

The concept of a hierarchically organized set of states seems, in one way or another, to have relevance for a number of physical systems, ranging from relaxation processes in glasses¹ to the equilibrium properties of spin glasses.² Several groups³⁻⁵ have recently studied different versions of random walks on this kind of ultrametric space, all arriving at the conclusion that, for thermally activated hopping, at long times the propagator decays algebraically with a temperature-dependent exponent $\beta(T)$. There is, however, no agreement on the form of $\beta(T)$, suggesting that a closer look at the mathematics might be desirable. The second and main motivation for the present work is to calculate the first-passage time densities from one point to another—which has never been done previously—and check whether their Laplace transform is a stretched exponential, $\exp(-s^{a(T)})$. This dependence characterizes the so-called stable distributions,⁶ which were invoked by the authors on the basis of heuristic self-similarity arguments, in order to explain the experimental form of the spin-glass susceptibility. A peak in $a(T)$ [$a(T) \neq \beta(T)$] would explain the observed features of the susceptibility.

Due to their interesting scaling properties,⁶⁻⁸ stable distributions have a wide range of other possible applications in physics, since they arise naturally in the description of self-similarity.⁹ So far they have been used in the theory of spin-glass relaxation⁷ and in turbulence.⁸ A possible theoretical relevance for the renormalization-group theory was pointed out by Jona-Lasinio¹⁰ some years ago. More recently (the late) Montroll and Bendler¹¹ noticed that the spectral density of lifetimes $g(\lambda, \alpha)$ associated with the Kohlraush law by the formula $\exp[-(t/\tau)^\alpha] = \int_0^\infty g(\lambda, \alpha) e^{-\lambda t} d\lambda$, is (almost) a stable distribution. This establishes a contact between mathematical theories and the heuristics proposed by Palmer, Stein, Abrahams, and Anderson¹ in order to explain glassy relaxation. The mathematical community has long been puzzled by the lack of explicit representation of the stable densities, which are essentially the Fourier transform of the Kohlraush law, and thus have a direct bearing on the response function in amorphous structures. This problem has now been solved by Schneider,¹² who expresses them in terms of Fox functions.

In spite of the fact that stable distributions could play a unifying role in the description of complex systems, their

use is not yet widespread. Perhaps this is due to the fact that no “microscopic” model (at the level of a random-walk description) has ever been shown to give rise to these distributions. This work is an attempt to change the situation by displaying such a model and thereby clarifying the physical picture behind the mathematics.

In order to implement this program, we consider a large but finite Cayley tree obtained by successive branchings from an “ancestor” state, and let a random walker hop between neighboring points at given rates. These, and the branching ratios, may depend on the level index in an arbitrary, even random, fashion. This is a generalization of (the basic ingredients of) the model by Grossman, Wegner, and Hoffmann⁴ who find the eigenvalues. Our approach is complementary, since we focus on the Laplace transform of the propagator which we show has a very simple and beautiful continued fraction structure. Our conclusions regarding the form of $\beta(T)$ generalize the form previously found by Huberman and Kersberg.³ The model differs from that of Refs. 3 and 5 in that each node of the tree represents a possible state of the system, rather than just the bottom nodes. This is unimportant for low temperatures.

Our starting point is the well-known formula^{13,14}

$$\tilde{G}(n, s | n_0) = [1 - \tilde{F}(n, s | n)]^{-1} \times [\delta_{n, n_0} + (1 - \delta_{n, n_0}) \tilde{F}(n, s | n_0)] , \quad (1)$$

connecting the Laplace transforms (denoted by a tilde) of the probability of being at site n at time t , $G(n, t | n_0)$ with the probability of a first visit at n at time t . In both cases, n_0 is the starting point, and the n 's denote the states of a generic Markov process. The above relation shows that F and G contain exactly the same information, although F can be much easier to calculate.

In a Cayley tree, the path from n_0 to n_1 necessarily goes through their unique closest common ancestor. Hence, due to the Markovian character of the process

$$\tilde{F}(n_1, s | n_0) = \tilde{R}_{n_1, n_2}(s) \tilde{R}_{n_2, n_0}(s) , \quad (2)$$

where the R 's are the same quantity as F except that the initial and final states have an ancestor-descendant relationship. Since all the levels between n_0 and n_2 must also

be visited, we also have

$$\tilde{R}_{l',l}(s) = \prod_{r=l}^{l'-1} \tilde{R}_{r+1,r}(s), \quad l \leq l' - 1 \quad (3a)$$

and

$$\tilde{R}_{l',l}(s) = \prod_{r=l'+1}^l \tilde{R}_{r-l,r}(s), \quad l \geq l' + 1. \quad (3b)$$

We therefore only need to consider neighbor levels. It is not necessary to specify the coordinates of a node within a level, since it is equivalent to all its siblings.

In the following, we write “distribution” rather than “Laplace transform of the distribution,” and omit tildes for the sake of brevity. We also let $P^u_l(s)$ and $P^d_l(s)$ be the distribution of waiting times at node l joint to hopping up and down, respectively. They are given by

$$P^u_l(s) = k^u_l / (s + k^u_l z + k^d_l), \quad (4a)$$

$$P^d_l(s) = k^d_l z / (s + k^u_l z + k^d_l), \quad (4b)$$

where k^u_l and k^d_l are the rates for hopping up and down starting at level l . Obviously, $k^d_N = k^u_N = 0$, where N is the highest level in the tree. A possible dependence of z on the level has been omitted for notational simplicity, and does not change the formulas in any essential way.

In order to calculate $R_{l+1,l}(s)$, we note that if the first jump is upwards, the walk terminates, and otherwise the level l must be revisited. Therefore,

$$R_{l+1,l}(s) = P^u_l(s) / [1 - P^d_l(s) R_{l,l-1}(s)] \quad (5a)$$

$$1 \leq l \leq N - 1,$$

with

$$R_{l,0}(s) = P^u_0(s). \quad (5b)$$

Similarly, starting at level l we can reach one of the z level $l-1$ siblings at the first move. If not, we have either jumped down to the “wrong” sibling, or we have jumped up. In the latter two cases, the starting point must be revisited, while otherwise the random walk stops. Hence, for $1 \leq l \leq N-1$ we have

$$R_{l-1,l}(s) = P^d_l(s) / [z - (z-1) R_{l,l-1}(s) P^d_l(s) - z P^u_l(s) R_{l,l+1}(s)] \quad (6a)$$

and, for $l = N$,

$$R_{N-1,N}(s) = P^d_N(s) / [z - (z-1) R_{N,N-1}(s) P^d_N(s)]. \quad (6b)$$

Equations (1)–(3), (5), and (6) give an explicit continued fraction representation for the propagator in the widest possible class of random walks on finite Cayley trees. We finally note for later convenience that the probability of being at the origin, which we take as a zero-level node, can be obtained by the inverse Laplace transformation from

$$G_{0,0}(s) = 1 / [1 - P^u_0(s) R_{0,1}(s)]. \quad (7)$$

As an application of the present formalism, we assign energies $E_l = \Delta \times l$ to the l level nodes, and postulate a thermally activated process with $k^d_l = 1$ and $k^u_l = k = \exp(-\Delta/T)$. Note, however, that as emphasized in Ref.

5, other forms of E_l are also of interest. From now on we take $\Delta = 1$ for simplicity. In terms of the reduced variables $\sigma = s/k$ and $\zeta = z/k$, Eqs. (5) and (6) read

$$R_{l+1,l}(\sigma) = 1 / [\zeta + \sigma + 1 - \zeta R_{l,l-1}(\sigma)], \quad (8a)$$

$$R_{l-2,l-1}(\sigma) = k^{-1} / [\zeta + \sigma + 1 - (\zeta - k^{-1}) R_{l-1,l-2}(\sigma) - R_{l-1,l}(\sigma)], \quad (8b)$$

with

$$R_{1,0}(\sigma) = 1 / (1 + \sigma), \quad (9a)$$

$$R_{N-1,N}(\sigma) = k^{-1} / [\zeta + \sigma - (\zeta - k^{-1}) R_{N,N-1}(\sigma)]. \quad (9b)$$

Equation (8a) is a simple functional iteration with fixed point

$$x(\sigma) = \{1 + \zeta + \sigma - [(1 + \zeta + \sigma)^2 - 4\zeta]^{1/2}\} / (2\zeta), \quad (10)$$

$$\sigma > 0.$$

By considering the recursion relation for $[R_{l+1,l} - x(\sigma)]$, which is easily obtained from Eq. (8), we obtain after some algebra,

$$R_{l+1,l}(\sigma) = x + \frac{(\zeta x^2)^l (R_{1,0} - x)(1 - \zeta x^2)}{1 - \zeta x^2 - \zeta x (R_{1,0} - x)[1 - (\zeta x^2)^l]}. \quad (11)$$

When $\sigma \ll 1$, the weak dependence of x on σ can be neglected, i.e., we put $x(\sigma) = \zeta^{-1}$. We also discard terms of order ζ^{-l} and σ compared to one, and get

$$R_{l+1,l} = \zeta^{-1} + (1 - \zeta^{-1}) h_l / (\sigma + h_l), \quad (12a)$$

with

$$h_l = \zeta^{-l} (1 - \zeta^{-1}). \quad (12b)$$

The pole structure following from Eq. (8a) is quite different, since $R_{l-2,l-1}$ has a pole close to each pole of $R_{l-1,l}$ plus a pole close to the singularity of $R_{l-1,l-2}(\sigma)$. Since now $R_{N-1,N}(\sigma)$ has one pole, it follows by induction that, close to $\sigma = 0$, $R_{l-1,l}$ has $N-l+1$ poles at $\sigma = -p^l_j$, where $\sigma < p^l_1 < \dots < p^l_{N-l+1} < h_{l-1}$. We denote the corresponding residues by r^l_j .

These quantities are most simply determined by constructing from Eq. (8) the recursion relation for $y_{l-1,l} \equiv (1 - R_{l-1,l})^{-1}$. Putting $R_{l-1,l-2} = 1$, a good approximation for $\sigma \ll h_{l-2}$, i.e., z , and/or T large, we obtain for $l \leq N-1$,

$$y_{l-2,l-1}(\sigma) = 1 + k^{-1} y_{l-1,l}(\sigma) / [1 + \sigma y_{l-1,l}(\sigma)], \quad (13)$$

where, by definition, $y_{l-1,l}(\sigma = -p^l_j) = 0$ for each p^l_j . Expanding to first order around the zeros, we get

$$p^{l-1}_j = p^l_j + k / y'_{l-1,l}(-p^l_j)$$

and

$$y'_{l-2,l-1}(-p^{l-1}_j) = k^{-1} y'_{l-1,l}(-p^l_j).$$

Recognizing that $r^l_j = -1 / y'_{l-1,l}(-p^l_j)$, we finally obtain for $1 \leq j \leq N-l+1$,

$$p^{l-1}_j = p^l_j - k r^l_j, \quad (14a)$$

$$r^{l-1}_j = k r^l_j. \quad (14b)$$

The above derivation does not depend in any essential way on the temperature being low, but rather on the fact that $R_{l-1,l-2}(\sigma)$ is not singular in a small interval close to each of the $-p_j^l$'s, and hence can be approximated by a constant. If this constant is not unity, Eq. (13) changes slightly, but the final result is the same.

The last singularity of $R_{l-2,l-1}$ is found by using the full expression (12a) for $R_{l-1,l-2}$ and is given by

$$p_{N-l+2}^{l-1} \propto r_{N-l+2}^{l-1} \propto h_{l-2}, \quad (15)$$

where the precise value of the proportionality constant does not affect the exponent of the decay. Equation (15) holds also for $l = N - 1$, where it describes the only pole of $R_{N-1,N}(\sigma)$. From Eqs. (14) and (15) we find the following for $l = 1$: $r_j^1 \propto k^{2(N-j)} z^{-(N-j)}$ and $p_j^1 \propto k^{(N-j)} z^{-(N-j)}$ ($1 + \text{corrections}$). The corrections are unimportant because they decay faster and moreover, are $O(k^3)$, i.e., small at low T . The final result is

$$R_{0,1}(t) = \sum_{q=1}^{N-1} r_q^1 \exp(-tp_q^1) \sim t^{-\alpha},$$

where

$$\alpha = (\ln z + 2/T) / (\ln z + 1/T), \quad (16)$$

and the result has been obtained by transforming the sum into an integral. The decay law for $G_{0,0}$ is found by using Eq. (7), with $P_0(\sigma) \sim 1$, and the fact that $1 - R_{0,1}(\sigma) \propto \sigma^{\alpha-1}$ for small σ . In the time domain we get $G_{0,0}(t) \sim t^{-\beta}$ with

$$\beta = 2 - \alpha = T \ln z / (T \ln z + 1). \quad (17)$$

This has the same form as found in Ref. 3. The very-low-temperature behavior corresponds to the decay law derived by Ogielski and Stein⁵ who, however, considered a model which is different and much simpler in structure than the present one. Which model is relevant to what, is open to question. Equation (17) has the same structure as Eq. (6.7) of Grossmann *et al.*⁴ In applying their expression to thermally activated hopping, these authors choose a negative coefficient to the temperature, thereby introducing a pole which we do not find.

In order to check the formulas, we have calculated $\tilde{R}_{0,1}(\sigma)$ exactly, according to Eqs. (8) and (9), through 30 orders of magnitude of σ , and plotted in Fig. 1 $\ln(-\ln R)$ vs $\ln \sigma$ for different T values. As expected the result is, for $\sigma < 1$, a straight line with slope $\alpha(T) - 1$, if one disregards a superimposed oscillation of small amplitude. In practice we have calculated $\alpha(T)$ from the average slope over the first 11 decades, and the results coincide very well (within 2%) with Eq. (16) and therefore with Eq. (17) in the whole range of temperatures of interest [for $T < 0.1$, numerical problems render Eqs. (8) and (9) inaccurate]. More information can be gained from Fig. 1, i.e., that the short σ (long-time) asymptotic regime extends up to $\sigma \sim 1$ and that in this whole region the distribution is well approximated by a stretched exponential, as anticipated. There is also a second regime, for $\sigma > 1$, with the same functional dependence but another exponent. It describes times shorter than the inverse attempt frequency of the

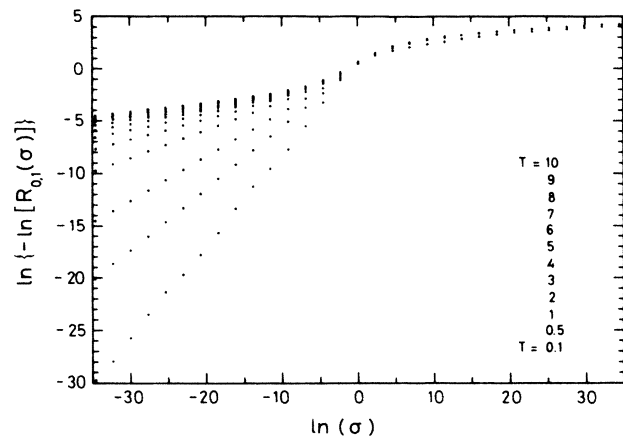


FIG. 1. The graphs show $\ln\{-\ln[\tilde{R}_{0,1}(\sigma)]\}$ vs $\ln(\sigma)$ for different temperatures as indicated above. The branching ratio of the tree is $z = 4$, and $\sigma = s \exp(1/T)$, where s is the Laplace transform variable. The distance between two points is a decade. The function was calculated analytically according to Eqs. (8) and (9). Note that the graphs have zero curvature except in a small region close to $\sigma = 1$. The number of levels used was $N = 100$.

random walker, for $T \geq 1$. It may be physically relevant if decay effects are present due to a process which is independent of the spatial motion of the walker. The Laplace transform variable is then shifted by an amount proportional to the inverse lifetime,¹⁵ and the "large σ " region can be entered even at not-too-low temperatures.

The present formalism can be given a physical content along the lines proposed by Palmer *et al.*¹ of a hierarchy of degrees of freedom. Quite generally, if only the most favorable route between any two points in the phase space of a system is considered, the connection graph of the dynamics has no loops, i.e., is a Cayley tree. This low-temperature approximation is, of course, not sufficient to specify the details of the model. However, one might loosely identify the waiting time for crossing the tree, which has the same qualitative features as Fig. 1, with the waiting time for some event to happen which requires the successful completion of many subtasks, each involving more elementary degrees of freedom. The example we have in mind is given in Ref. 7, where the flipping of a "cluster" is described by a stable distribution of waiting times, with an exponent $\alpha(T)$ having a peak at the glass transition temperature. The idea that stable distributions should come into play seems justified. Recalling that $\sigma = \exp(1/T)s$, we see that for fixed s (or frequency), the main effect of a change in temperature is a change of σ . If the time scales are such that this would imply a shift from one regime to the other of Fig. 1, one might obtain a peak in the observed α .

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