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Nonuniversality of diffusion exponents in percolation systems

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We study diffusion on the incipient infinite percolation cluster in $d=2$ with a power-law distribution of transition rates $P(W)\sim W^{-\alpha}$, $\alpha<1$. Using the exact enumeration method we find that the diffusion exponent $\bar{d}_w(\alpha)$ sticks at its $\alpha=-\infty$ value for $\alpha\leqq 0$. For $\alpha>0$, $\bar{d}_w$ is bounded by $d_f+1/[(1-\alpha)\nu]\leqq\bar{d}_w(\alpha)\leqq\bar{d}_w(-\infty)+\alpha/[(1-\alpha)\nu]$. Specifically, for small α our numerical results are close to the upper bound, while for larger α they are close to the lower bound.

Recently, it has been shown by Halperin, Feng, and Sen[1] that continuum random systems such as the random-void models can be mapped onto random percolation networks[2] with a distribution of bond conductivities σ,

$$P(\sigma)\sim\sigma^{-\alpha},\ \alpha<1\ , \tag{1}$$

where by definition $\sigma\leqq 1$. For the random-void model one has $\alpha=-1$ in $d=2$ and $\alpha=\frac{1}{3}$ in $d=3$.

Discrete percolation systems with a random distribution of conductivities, Eq. (1), have been studied recently by Kogut and Straley,[3] Ben-Mizrahi and Bergman,[4] Straley,[5] and Sen, Roberts, and Halperin.[6] It was found that the conductivity exponent $\bar{\mu}$ is nonuniversal and depends on the parameter α, but different predictions for the α dependence have been made. Using a nodes-link-blob scaling analysis, Halperin *et al.*[1] presented arguments that the predictions of Kogut and Straley[3] and Straley[5] yield upper and lower bounds for $\bar{\mu}$, respectively, i.e.,

$$1+(d-2)\nu+\alpha/(1-\alpha)\leq\bar{\mu}\leq\mu+\alpha/(1-\alpha),\ 0\leq\alpha<1\ , \tag{2}$$

where μ is the standard conductivity exponent for percolation systems with constant-bond conductivities and ν is the correlation-length exponent.[7]

In this note we study how the diffusion exponents change on the incipient infinite percolation cluster (in $d=2$) when the transition rates W between nearest-neighbor sites follow from a power-law distribution

$$P(W)\sim W^{-\alpha},\ \alpha<1\ , \tag{3}$$

and $W\leq 1$, analogous to Eq. (1). The diffusion exponents, $\bar{d}_w$ and $\bar{d}_w^l$, are defined by

$$\langle r^2(t)\rangle\sim t^{2/\bar{d}_w},\ \langle l(t)\rangle\sim t^{1/\bar{d}_w^l}\ , \tag{4}$$

where $\langle r^2(t)\rangle$ is the mean-square displacement and $\langle l(t)\rangle$ is the mean chemical ("topological") distance[8] of a random walker from its starting point. For uniform transition rates ($W=1$, corresponding to $\alpha=-\infty$) it has been shown[9] from the Nernst-Einstein relation that $d_w\equiv\bar{d}_w(-\infty)$ is related to μ by

$$d_w=2+\frac{\mu-\beta}{\nu}\ , \tag{5}$$

where β is the percolation order parameter exponent.[7] Using similar arguments it can be shown that Eq. (5) holds also in the more general case of random transition rates, where d_w and μ are substituted by $\bar{d}_w$ and $\bar{\mu}$. Thus, from (2) we obtain bounds for $\bar{d}_w$,

$$d_f+1/(1-\alpha)\nu\leq\bar{d}_w\leq d_w+\alpha/(1-\alpha)\nu\ , \tag{6}$$

where $d_f=d-\beta/\nu$ is the fractal dimension of the incipient infinite percolation cluster. A similar relation can be obtained for $\bar{d}_w^l$, since both $\bar{d}_w$ and $\bar{d}_w^l$ are related to each other by

$$\bar{d}_w^l/\bar{d}_w=d_l/d_f\ , \tag{7}$$

where d_l is the topological ("chemical") dimension.[8] This yields

$$d_l+1/(1-\alpha)\nu_l\leq\bar{d}_w^l\leq d_w^l+\alpha/(1-\alpha)\nu_l\ , \tag{8}$$

where $\nu_l\equiv\nu d_f/d_l$ is the "chemical correlation exponent" (see Ref. 8). Relations (6) and (8) are easily accessible to a direct numerical test. To this purpose we applied the exact enumeration method (see, e.g., Ref. 10), which allows us to calculate exactly the distribution function $P(\mathbf{r},t)$ of a random walker on a given percolation cluster for a fixed starting point $\mathbf{r}_0$. First we generated a percolation cluster on a square lattice at criticality. The transition rates $W_{\mathbf{r},\mathbf{r}+\delta}$ between neighboring cluster sites $\mathbf{r}$ and $\mathbf{r}+\boldsymbol{\delta}$ are chosen according to

$$W_{\mathbf{r},\mathbf{r}+\delta}=W_{\mathbf{r}+\delta,\mathbf{r}}=\tfrac{1}{4}R^{1/(1-\alpha)}\ , \tag{9}$$

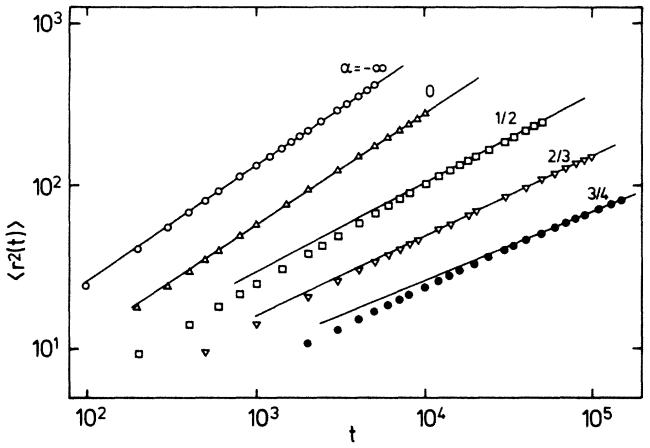

FIG. 1. Plot of $\langle r^2(t)\rangle$ vs t for various α. For our calculations, we generated clusters up to 150 shells (typically 10000 sites) using the Leath algorithm (Ref. 12). To determine $\langle r^2(t)\rangle$, we used the exact enumeration for the "blind ants," Eqs. (9)–(11). For each value of α we averaged over 400 configurations each.

where R is a random number between 0 and 1. By (9), the rate distribution Eq. (3) is generated. Then the time evolution of $P(\mathbf{r},t)$ is calculated as follows: At $t=0$, the walker starts in the origin, i.e., $P(\mathbf{r},0)=\delta_{\mathbf{r},0}$. At $t=1$, the walker steps with probability $W_{0,0+\delta}$ to the neighboring cluster sites δ. Here

$$P(\mathbf{r},1)=\begin{cases} W_{0,\delta} \text{ for } \mathbf{r}=\delta\ , \\ 1-\sum_{\delta} W_{0,\delta} \text{ for } \mathbf{r}=0\ . \end{cases} \tag{10}$$

For $\mathbf{r}\neq 0$ and $\mathbf{r}\neq\delta$, $P(\mathbf{r},1)\equiv 0$. By iterating this procedure we find $P(\mathbf{r},2)$, etc. From $P(\mathbf{r},t)$ we obtain the mean square displacement

$$\langle r^2(t)\rangle=\sum_{\mathbf{r}} r^2 P(\mathbf{r},t) \tag{11}$$

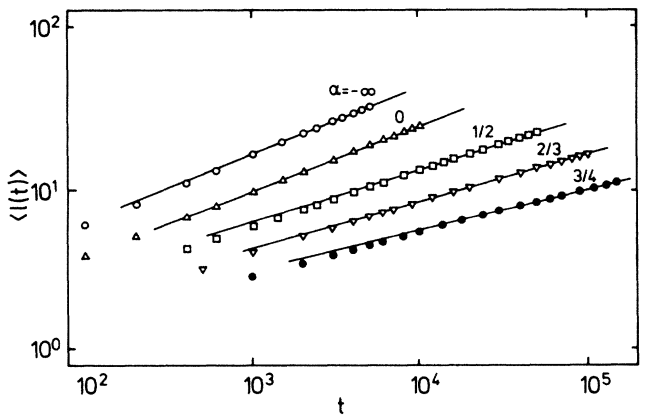

FIG. 2. Plot of $\langle l(t)\rangle$ vs t for various α.

TABLE I. The diffusion exponents $\bar{d}_w$ and $\bar{d}_w^l$ extracted from Figs. 1 and 2 for five values of α, compared with upper and lower bounds from Eq. (6); $\bar{d}_w$ and $\bar{d}_w^l$ are related to each other by Eq. (7), where $d_f=\frac{91}{48}$ and $d_l\simeq 1.64$ in $d=2$ percolation. The error bars were estimated from a least-squares fit of $\langle r^2(t)\rangle$ and $\langle l(t)\rangle$ in the asymptotic regime.

α	$\bar{d}_w$	$\bar{d}_w^l$	Upper bound $\bar{d}_w$	Lower bound $\bar{d}_w$
$-\infty$	2.85 ± 0.04	2.47 ± 0.04	2.87	2.65
0	2.90 ± 0.05	2.50 ± 0.05	2.87	2.65
$\frac{1}{2}$	3.60 ± 0.10	3.13 ± 0.06	3.62	3.40
$\frac{2}{3}$	4.12 ± 0.10	3.51 ± 0.08	4.37	4.15
$\frac{3}{4}$	4.80 ± 0.20	4.00 ± 0.15	5.12	4.90

of the random walker with starting point at the origin for the considered cluster. In order to obtain the corresponding configurational averaged quantities one has to average over many clusters. From $P(\mathbf{r},t)$ it is easy to calculate $P(l,t)$, which is defined as the probability to find the walker in topological distance l from the origin. To each cluster site $\mathbf{r}$ a certain topological distance l is associated and, thus,

$$P(l,t)=\sum_{\mathbf{r}}^{(l)} P(\mathbf{r},t)\ , \tag{12}$$

where $\sum_{\mathbf{r}}^{(l)}$ denotes a sum over all sites with the same topological distance l from the origin. From $P(l,t)$ we obtain $\langle l\rangle$.

For our actual computations, we generated clusters up to 150 shells and averages over 400 configurations have been made.[11] The results for $\langle r^2(t)\rangle$ and $\langle l(t)\rangle$ are shown in Figs. 1 and 2 for various values of α. From the asymptotic slopes[13] we obtained the diffusion exponents $\bar{d}_w$ and $\bar{d}_w^l$, which are presented in Table I. Our results strongly support the inequalities (2) and (6) and thus also support the scaling arguments leading to these relations.

The lower bound has been derived[1] by employing the one-dimensional nature of the backbone of the percolation cluster which consists of single-connected bonds and multiconnected bonds ("blobs"),[14] and neglecting the contribution of the blobs to the resistivity. It is interesting to note that our results are close to the upper bound for small α, $\alpha<\frac{1}{2}$, while they are close to the lower bound for larger α. This result is consistent with very recent findings of Lubensky and Tremblay[15] when using the ε expansion and of Machta, Guyer, and Moore[16] for a hierarchical network.

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